Mechanism of trapped surface formation

D. Valls-Gabaud and T. Zannias
Department of Physics, Queen's University, Kingston, Ontario, Canada K7L 3N6
(Received 16 February 1993; revised manuscript received 6 May 1993)

By a combination of analytical and numerical methods, the density profile of a spherical star momentarily at rest is varied, and the corresponding response in the area of the spherical shells is monitored. It is shown that the inner apparent horizon (if it exists) must lie within or at most on the star's surface while no such restriction is found for the outer apparent horizon. An apparent horizon, however, lying in the vacuum region will always have a nonvanishing area, as long as the ADM mass of the system is nonzero. Furthermore for density profiles not decreasing outwards, it appears that all spherical trapped surfaces lie on a thick spherical shell. Finally for a uniform density star a simple criterion is found, relating the density and proper radius that guarantees the presence or absence of trapped regions.

PACS number(s): 04.20.Cv

There has been lately considerable renewed effort aiming to discover conditions upon a given initial data set that permit us to infer the presence or absence of trapped surfaces [1–5]. Although most of the cited work focuses on characterizing the state of initial data triggering the formation of trapped surfaces, it would be also beneficial to understand the mechanism that leads to their formation as well as any other relevant property. For instance how do spherical trapped surfaces appear and how are they distributed around the center of the star? Where (if it exists) is the location of the outer apparent horizon? What is the response of the apparent horizon under smooth variations of the parameters describing the initial data?

It is the purpose of the present Brief Report to provide, at least for a special class of configurations, answers to the above mentioned questions. For this we shall consider a sequence of instantaneous states describing spherical stars of proper radius R and proper density ρ, just at the onset of their gravitational collapse. (In what follows the term trapped surface stands for an outer trapped surface as defined in Hawking and Ellis.) By examining the proper area of spheres around the center of the star as a function of distance away from the center, the trapped region can be identified: it lies between successive maxima and minima of the area function. We vary the density profile (in essence we are moving from one member of the sequence to another) and monitor the behavior of the area function. Such "variations" of the density cause the appearance or disappearance of trapped surfaces and forces the inner and outer apparent horizons to "move" around in the stars interior or exterior region. The results are presented as a series of plots of the area function against the proper distance.

Recall that initial data on a time-symmetric slice Σ satisfy the Hamiltonian constraint, i.e.,

$$^3\mathcal{R}=16\pi\frac{G}{c^2}\rho \ , \tag{1}$$

where the ρ stands for the non-negative density. The assumption of spherical symmetry allows the introduction of geodesic-type coordinates and thus the line element of Σ takes the form

$$ds^2=dr^2+B(r)(d\theta^2+\sin^2\theta\, d\phi^2) \ , \quad 0\le r<\infty \ . \tag{2}$$

Evaluating the scalar curvature $^3\mathcal{R}$ of the metric (2), Eq. (1) reduces to

$$-\frac{2}{B}\frac{d^2B}{dr^2}+\frac{1}{2B^2}\left[\frac{dB}{dr}\right]^2+\frac{2}{B}=16\pi\frac{G}{c^2}\rho \ . \tag{3}$$

We look for solutions admitting a regular geometry at $r=0$. This demands

$$B(r)\bigg|_{r=0}=0 \ , \quad \frac{dB}{dr}\bigg|_{r=0}=0 \ . \tag{4}$$

For later use note that the amount of mass-energy within each sphere of radius r is given by [6]

$$m(r)=\frac{1}{2}\frac{c^2}{G}B^{1/2}\left[1-\frac{1}{4B}\left[\frac{dB}{dr}\right]^2\right] , \tag{4a}$$

and it obeys, by virtue of (3),

$$\frac{dm(r)}{dr}=2\pi\rho B^{1/2}\frac{dB}{dr} \ . \tag{4b}$$

Our intention is to investigate the behavior of solutions of Eq. (3) subject to (4). A number of conclusions can be drawn by inspection of Eq. (3). At first for $\rho=0$ a solution obeying (4) is given by $B(r)=r^2$, which corresponds to flat three-space. Further if a solution $B=B(r)$ admits inflection points of nonzero area (and thus the area function passes through a saddle point) they must be either interior points, or lie at the star's surface. At such points, one deduces from Eq. (3) that the density ρ and area $B(r)$ obey $B\rho=c^2/8\pi G$.

Let us now consider a solution $B=B(r)$ extending from the interior, i.e., $r\le R$ to the exterior vacuum region. The absence of surface layers at the star's surface requires $B(r)$ to be C^1 at $r=R$, i.e., $B(r)$ and $dB(r)/dr$ should be continuous across the surface (note, however, that d^2B/dr^2 may exhibit a discontinuous behavior at $r=R$). Since, on the other hand, in the vacuum Eq. (3) implies $d^2B/dr^2>0$, therefore, if the surface of the star is not trapped (i.e., $dB/dr>0$, there), the exterior solution will be a monotonically increasing function; i.e., trapped surfaces can never develop in the vacuum region. In the opposite case, i.e., if the surface is trapped, Eq. (3) implies that only a local minimum of $B(r)$ can develop in the vacuum region. This outermost local minimum is the outer apparent horizon. Note, however, in general it may also lie in the interior. In the case where it lies in the vacuum one may naturally ask can it have a zero area? We shall show that as long as the data possess a nonzero Arnowitt-Deser-Misner (ADM) mass, this is not possible. In fact evaluating the right-hand side of (4a) on the outer horizon (of zero area), and utilizing (3) we get

$$m(r)=\frac{1}{2}\frac{c^2}{G}B^{1/2}\left[2-\frac{d^2B}{dr^2}\right]=0\ ,$$

which, in view of (4b) is impossible. (The above argument also rules out vacuum inflection points of zero area.) Let us now shift attention to the behavior of the local maxima of $B(r)$. The first local maximum of $B(r)$ (if it exists) marks the location of the inner apparent horizon. Generically it lies interior to the surface, but one may ask can the inner horizon lie on the surface of the star? The answer depends upon the behavior of the density profile at the star's surface; if $\rho(r)$ is continuous at $r=R$, then Eq. (3) implies that the inner horizon cannot be located at the surface. If it exists it must be an interior point. In the case where $\rho(r)$ is discontinuous at $r=R$, Eq. (3) implies that the left derivative $(d^2B/dr^2)_L$ and surface density ρ satisfy

$$\frac{1}{B}=8\pi\frac{G}{c^2}\rho+\frac{1}{B}\left[\frac{d^2B}{dr^2}\right]_L\ . \tag{5}$$

Therefore arbitrary density profiles allow the possibility that the left second derivative is negative while simultaneously the right-hand side of (5) is positive, implying that the surface of the star may become the location of the inner horizon. Notice in that case the C^1 matching of $B(r)$ across the surface dictates that the second (right) derivative $(d^2B/dr^2)_R$ is given by

$$\frac{1}{B}\left[\frac{d^2B}{dr^2}\right]_R=8\pi\frac{G}{c^2}\rho+\frac{1}{B}\left[\frac{d^2B}{dr^2}\right]_L>0\ , \tag{6}$$

implying that the surface must be an outer apparent horizon as suggested from the exterior geometry. (This situation will be verified below by an explicit exact solution.) In summary, therefore, no inner horizon can lie exterior to the star in contradistinction to the outer apparent horizon.

How many distinct local maxima and minima are admitted by an arbitrary solution of (3) and (4) matched in a C^1 fashion across the surface? The answer to that question is not obvious. The nonlinear structure of (3) does not allow us to get further insight into the nature of solutions of (3) and (4) and, thus, we have to search for exact solutions. For an arbitrary $\rho=\rho(r)$ exact solutions of (3) and (4) appear difficult to be found. Therefore we shall use numerical techniques to construct solutions of the system. Note that one case where the system is soluble corresponds to the uniform density star, i.e., $\rho(r)=$const [7]. Before we comment on the constant density solution let us rewrite the systems (3) and (4) in a slightly different form more suitable for numerical computations. It is convenient to introduce the variables

$$y=r/R\ ,\quad \beta=B/R^2\ , \tag{7}$$

where R is the physical (proper) radius of the star, so that Eqs. (3) and (4) take the dimensionless form

$$-\frac{2}{\beta}\frac{d^2\beta}{dy^2}+\frac{1}{2\beta^2}\left[\frac{d\beta}{dy}\right]^2+\frac{2}{\beta}=16\pi\Upsilon\ , \tag{8}$$

$$\beta\bigg|_{y=0}=\frac{d\beta}{dy}\bigg|_{y=0}=0\ , \tag{9}$$

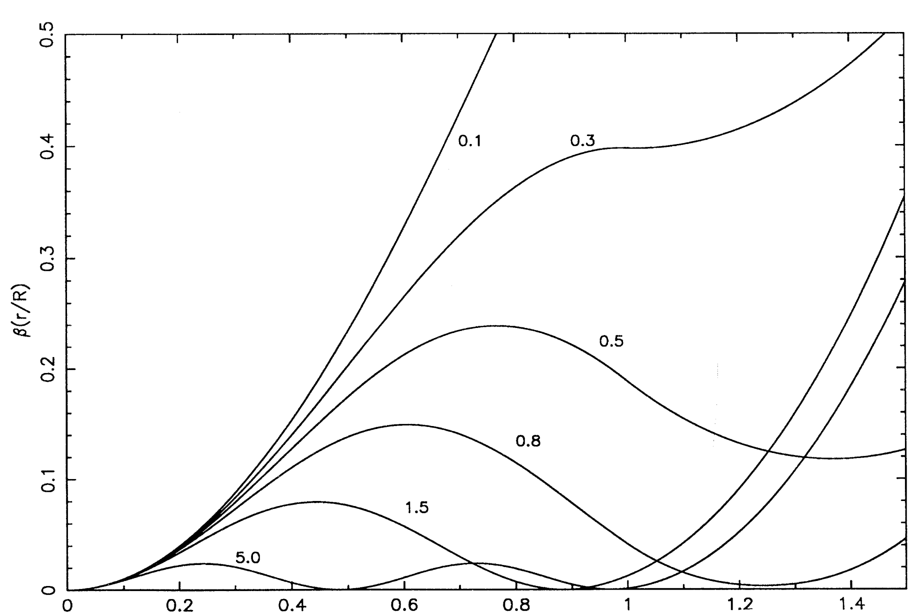

FIG. 1. Dimensionless area β as a function of the dimensionless proper distance $y=r/R$ for a uniform density distribution. The values of the dimensionless parameter Υ_0 are indicated.

where

$$\Upsilon(y)=\frac{G}{c^2}R^2\rho(r)\ . \tag{10}$$

In the above notation, we shall examine two dimensionless density profiles corresponding, respectively, to a uniform configuration and a Gaussian one:

$$\Upsilon_U(y)=\begin{cases}\Upsilon_0, & 0\le y\le 1\ ,\\ 0, & y>1\ ,\end{cases} \tag{11}$$

$$\Upsilon_G(y)=\begin{cases}\Upsilon_0 e^{\alpha(1-y^2)}\ , & 0\le y\le 1\ ,\\ 0, & y>1\ ,\end{cases} \tag{12}$$

where Υ_0 and the compactness parameter α characterize the profiles. The solution of (8) and (9) with the source given by (11) has, in the interior region, the form

$$\beta(y)=\frac{3}{8\pi\Upsilon_0}\sin^2\left[\left[\frac{8\pi\Upsilon_0}{3}\right]^{1/2}y\right]\ .$$

In the exterior region Eq. (8) is numerically integrated with boundary conditions dictated by the C^1 matching of the solution on the surface. In Fig. 1 solution curves for various values of the dimensionless parameter Υ_0 are shown. It is worth mentioning briefly a few features of the solution relevant to our discussion. One may easily verify that as long as $\Upsilon_0<3\pi/32$, $\beta(y)$ increases monotonically. At precisely $\Upsilon_0=3\pi/32$, $\beta(y)$, and consequently $B(r)$, develops the first local maximum taking place at the surface of the star. Note because of the discontinuity of the density across the surface, this critical point is a local minimum according to the exterior geometry. Thus for this particular value of Υ_0 there exists only one trapped surface which is simultaneously the location of the inner and outer apparent horizon. As Υ_0 keeps increasing the degeneracy is lifted and consequently inner and outer apparent horizons start "moving" in opposite directions (see also Fig. 1). Upon further increase the inner horizon moves moderately towards the center while the outer horizon moves towards the surface and simul-

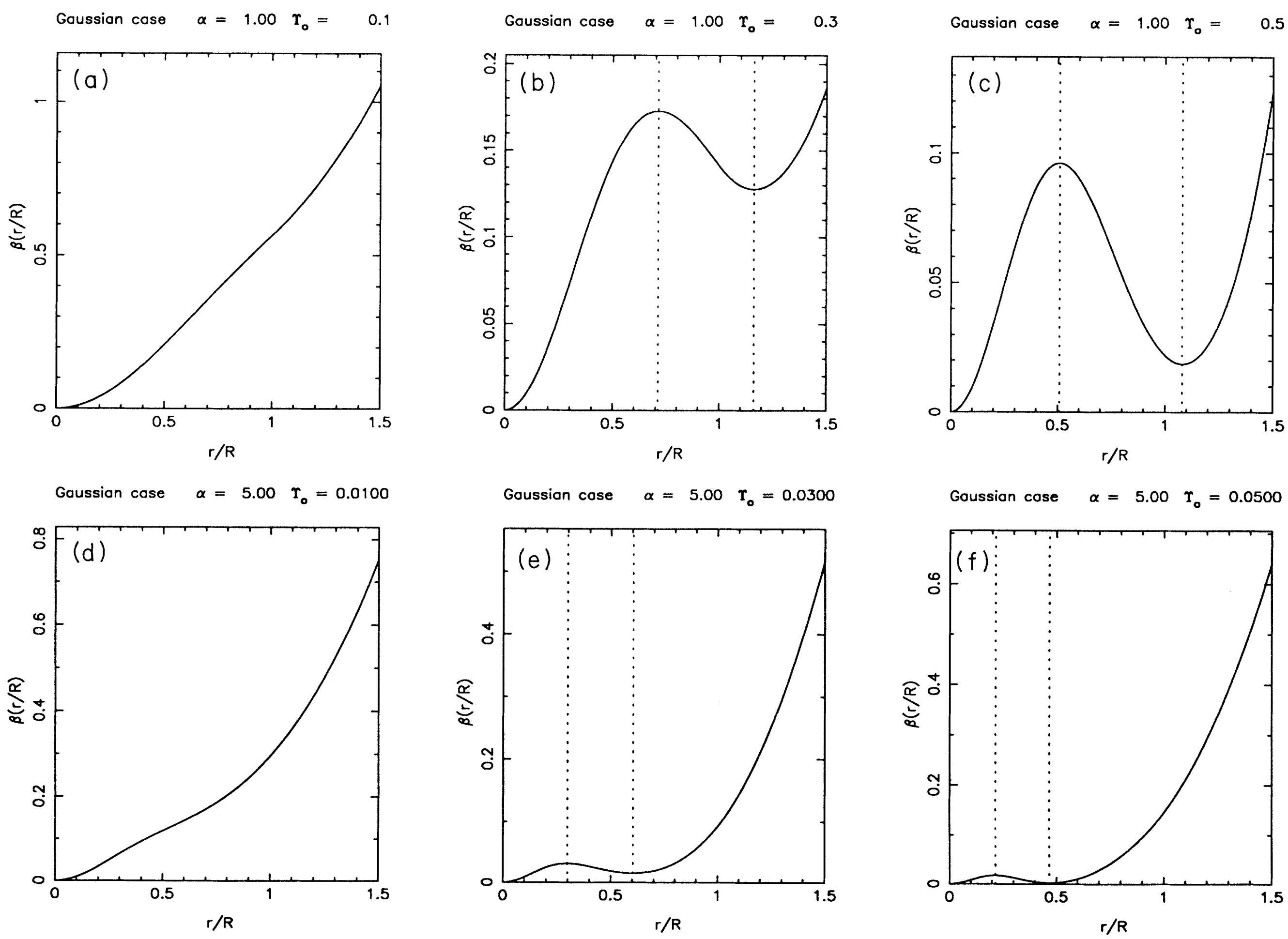

FIG. 2. Dimensionless area β as a function of the dimensionless proper distance $y=r/R$ for a Gaussian density distribution. The values of the dimensionless parameter Υ_0 and compactness parameter α are indicated. The dotted vertical lines mark local extrema. Note the presence of a maximum and minimum (indicative of a trapped surface) in the area function.

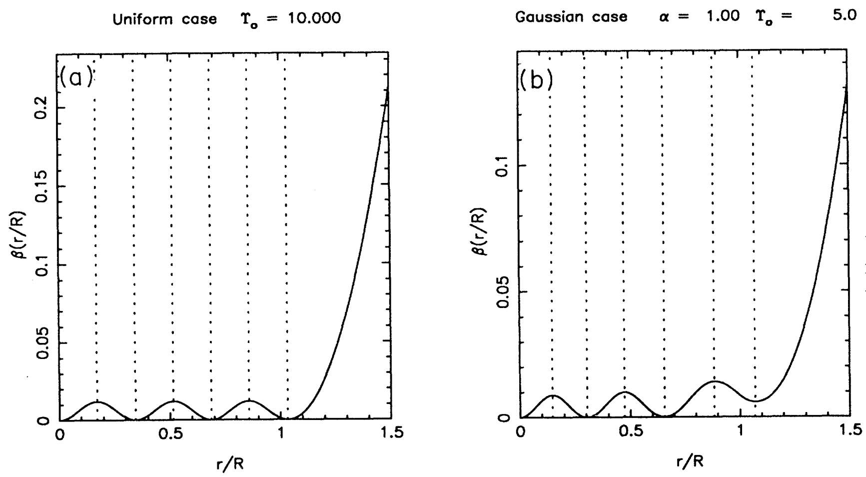

FIG. 3. Examples of configurations for which several maxima and minima are present in the dimensionless area β. (a) Uniform density configuration. (b) Gaussian density profile.

taneously keeps shrinking (i.e., its proper area is decreasing). Finally at $\Upsilon_0=3\pi/8$ the outer horizon has zero proper area and resides exactly on the surface of the star. The subsequent increase of Υ_0 causes the outer horizon to move inward while it maintains its zero area. For even larger values more critical points start to appear [see also Fig. 3(a)]. Such data, however, may be regarded as unphysical since the gravitational field generated by them is so strong that it disconnects the star from the asymptotically flat region (for further relevant comments on this point consult Ref. [8]). In summary, therefore, whenever data obey $3\pi/32<\Upsilon_0<3\pi/8$ they admit a distinct inner and outer apparent horizon. We discuss elsewhere [9] the impact of the above constraints on models of compact astrophysical objects.

Solutions of (8) and (9), with the source term the Gaussian profile of Eq. (12), exhibit the same qualitative features as those of the uniform case. In Figs. 2(a)–2(f) we present a few plots of (8) and (9) for various values of the parameters α and Υ_0. Note however, that in this case there exist data that allow both inner and outer apparent horizons of nonzero area lying within the interior of the star [see Fig. 2(e)]. This situation does not occur for the uniform density stars. From the analysis of the solution curves of Eqs. (8) and (9) there appears to emerge a common feature worth emphasizing. The systems (3) and (4) admit no solutions where two or more consecutive minima (excluding the origin) can have values different from zero. (This point may be easily verified directly for the uniform case.) In turn this property implies that a system first must disconnect itself from the asymptotically flat region before it allows a second minima to appear [see, for instance, Figs. 3(a), 3(b)]. We would like to think that the above described property of (3) and (4) is a generic property of all solutions due to the nondecreasing form of $\rho(r)$. However, we have been unable to show this analytically.

Finally, it is of interest to know how much of the above described picture is maintained if one deals with configurations lacking time symmetry or spherical symmetry. Although one may anticipate a similar overall behavior (at least for spherical systems) we ought to bear in mind that there are additional factors entering the problem, and they may alter some of the features discussed here. We hope to come back to this point elsewhere.

The research of D.V.-G. was supported by an NSERC International Fellowship while T.Z. was supported by CITA and by Principal's Development Fund of Queen's University.

[1] J. B. Hartle and D. C. Wilkins, Phys. Rev. Lett. **31**, 60 (1973); P. N. Demmie and A. I. Janis, J. Math. Phys. **14**, 973 (1973); R. Schoen and S. T. Yau, Commun. Math. Phys. **90**, 575 (1983); N. Ó Murchadha, Phys. Rev. Lett. **57**, 2466 (1986).

[2] E. Flanagan, Phys. Rev. D **44**, 2409 (1991); **46**, 1429 (1992); E. Malec, Phys. Rev. Lett. **69**, 946 (1991); Mod. Phys. Lett. A **7**, 1679 (1992).

[3] P. Bizon, E. Malec, and N. Ó Murchadha, Phys. Rev. Lett. **61**, 1147 (1988); Class. Quantum Grav. **6**, 961 (1989); **7**, 1958 (1990).

[4] T. Zannias, Phys. Rev. D **47**, 1448 (1993).

[5] P. Bizon and E. Malec, Phys. Rev. D **40**, 2559 (1989); R. Beig and N. Ó Murchadha, Phys. Rev. Lett. **66**, 2421 (1991); T. Zannias, Phys. Rev. D **45**, 2998 (1992); S. Hayward, Class. Quantum Grav. **9**, L115 (1992).

[6] This expression for the mass is the familiar Schwarzschild mass written in our coordinate system and taking into account the momentarily static nature of the configuration.

[7] N. Ó Murchadha [1]; Bizon, Malec, and N. Ó Murchadha [3]. Also the solution can be deduced from the exact four-dimensional constant density solution, or the Oppenheimer-Snyder model.

[8] B. J. Carr and S. W. Hawking, Mon. Not. R. Astron. Soc. **168**, 399 (1974).

[9] D. Valls-Gabaud and T. Zannias (unpublished).