Gauge-independent analysis of dynamical systems with Chern-Simons term

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(Received 29 October 1992; revised manuscript received 31 March 1993)

Quantization of theories involving the coupling of the Chern-Simons term to complex scalars and Dirac fermions is carried out in the Hamiltonian formalism without using gauge constraints. The covariance under the Poincaré group of transformations is established and subtleties in defining the different (canonical or symmetric) forms of the energy-momentum tensor are examined. Guage-invariant multivalued (anyon) operators obeying graded statistics are found which create the physical states with arbitrary spin. The spin-statistics connection is verified. Implications of our analysis concerning the claimed violation of translation invariance in phenomenological Lagrangians and its connection to anyon superconductivity are also discussed.

PACS number(s): 11.15.Tk, 11.10.Ef

I. INTRODUCTION

In a recent Letter [1] we formulated the basic aspects of a gauge-independent analysis of Chern-Simons (CS) theories coupled to matter fields. This work is an elaborate version of the original Letter [1]. Chern-Simons theories have timely interest because of their applications to both high energy and condensed matter physics [2]. The original version [3] included a Maxwell term. Subsequently, theories in which only the CS piece was taken as the kinetic term were considered [4]. Different approaches ranging from the path-integral to the operatorial formulations of the models have been considered [5].

In this paper we shall be primarily concerned with the canonical quantization of CS theories in the Hamiltonian formalism without invoking gauge constraints [6]. The problems and ambiguities related to gauge fixing are quite well known. Suffice it to mention that this has led to some controversy and criticisms in the literature [7–9]. Moreover, the conventional structure [7,10,11] of anyon operators which are found to be gauge dependent are obtained in the Hamiltonian formalism by choosing a specific gauge. Consequently, it is not clear whether the anyonicity is a physical effect or an artifact of the gauge. Indeed, sometimes different results with different gauge fixing have been reported [11].

We bypass all these ambiguities by doing a gauge-independent analysis. Explicit expressions for multivalued gauge-independent operators are found which create the physical states with arbitrary spin. These operators are associated with the arbitrary spin. They obey graded commutation relations which are compatible with the spin-statistics connection. It is important to mention that ambiguous manipulations with multivalued operators, which were earlier subjected to criticisms [5,8,9,12], have been completely avoided in our computations. The implications of our analysis for the recently claimed [13,14] violation of translation invariance in phenomenological Lagrangians have been discussed. We show that this claim is rather naive and not well founded.

Section II of this paper introduces the model involving the coupling of the CS term with complex scalars. The constraint structure is analyzed and the gauge-independent formulation set up. This is used in Sec. III to establish the covariance of the model under the Poincaré group of transformations. Particular care is paid to elucidate the difference in the definitions of the energy-momentum (EM) tensor which follow either from Noether's theorem or by introducing a background gravitational metric. The implications of this difference for the Dirac-Schwinger condition [15] is analyzed. The construction of the anyon operator and the physical states of the model are given in Sec. IV. It also includes a detailed algebra of the anyon operators. The entire analysis (Sec. II to Sec. IV) is compactly presented for the model describing the coupling of the CS term to Dirac fermions in Sec. V. Section VI reveals the ambiguity in defining momentum operators for phenomenological Lagrangians and their significance in anyon superconductivity. The conclusions are given in Sec. VII.

II. CHERN-SIMONS TERM COUPLED TO COMPLEX SCALARS

As an application of our ideas, let us first consider the Lagrangian density

$$\mathcal{L}=(D_\mu\phi)^*(D^\mu\phi)+(\theta/4\pi^2)\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \ , \tag{1}$$

where

$$D_\mu=\partial_\mu+iA_\mu \tag{2}$$

is the covariant derivative with the metric

$$g_{\mu\nu}=(+1,-1,-1)\ , \quad \varepsilon^{012}=1 \tag{3}$$

and θ is the CS parameter. The Lagrangian (1) is invariant (up to a total divergence) under the transformations

$$\phi(x)\to e^{i\alpha(x)}\phi(x)\ , \quad A_\mu(x)\to A_\mu(x)-\partial_\mu\alpha(x)\ . \tag{4}$$

The canonical momenta are

$$\pi^0=\frac{\partial\mathcal{L}}{\partial\dot{A}_0}=0,\quad \pi^i=\frac{\partial\mathcal{L}}{\partial\dot{A}_i}=\frac{\theta}{4\pi^2}\varepsilon^{ij}A_j\ ,$$
$$\pi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=(D_0\phi)^*,\quad \pi^*=\frac{\partial\mathcal{L}}{\partial\dot{\phi}^*}=D_0\phi\ , \tag{5}$$

so that, by Dirac's classification [16], the primary constraints are

$$P_0=\pi_0\approx 0\ ,$$
$$P_i=\pi_i-\frac{\theta}{4\pi^2}\varepsilon_{ij}A^j\approx 0\ , \tag{6}$$

where the symbol $\approx$ indicates weak equality, i.e., the constraints can be identically set equal to zero only after computing the relevant Poisson brackets (PB's). In order to check whether there are secondary constraints, the canonical Hamiltonian density is first computed from (1) by a Legendre transformation

$$\mathcal{H}_c=\sum_{\chi=\text{fields}}\pi_\alpha\dot{\chi}_\alpha-\mathcal{L}$$
$$=\pi^*\pi-A_0J_0-(D_i\phi)^*(D^i\phi)$$
$$-\frac{\theta\varepsilon^{ij}}{4\pi^2}[A_0\partial_iA_j-(\partial_iA_0)A_j]\ , \tag{7}$$

where

$$J_\mu=i[(D_\mu\phi)^*\phi-\phi^*(D_\mu\phi)] \tag{8}$$

is the conserved gauge-invariant current.

The primary Hamiltonian is given by

$$H_P=\int d^2x(\mathcal{H}_c+u_0\pi_0+u_iP_i)\ , \tag{9}$$

where u_0 and u_i are arbitrary multipliers. Conserving the primary constraints with H_P and using the basic PB's

$$\{A_\mu(x),\pi^\nu(y)\}=g_\mu^\nu\delta^{(2)}(x-y)\ ,$$
$$\{\phi(x),\pi(y)\}=\{\phi^*(x),\pi^*(y)\}=\delta^{(2)}(x-y)\ , \tag{10}$$

which are the only nonvanishing ones, yields the secondary constraints

$$S=J_0+\frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_iA_j\approx 0\ . \tag{11}$$

It may be checked that no further constraints are generated by this iterative procedure.

The next step is to classify the constraints. We find that π_0 is first class while P_i and S are second class constraints. The second class constraint S is the analogue of the Gauss constraint in usual electrodynamics which is, however, known to be first class. This apparent difference occurs because we have not yet extracted the maximal set of first class constraints in our theory. This is essential [17] because the first class constraints are associated with the gauge invariances of the model. It is easy to show that the following combination of second class constraints,

$$P=\partial^iP_i+S=\partial^i\pi_i+J_0+\frac{\theta}{4\pi^2}\varepsilon^{ij}\partial_iA_j\approx 0\ , \tag{12}$$

is first class. The complete set of first class constraints is thus given by π_0 and P while P_i are second class. The second class constraints P_i are really trivial because these may be eliminated by using Dirac brackets (DB's) instead of PB's. In that case S may be identified with the first class constraint P [see Eq. (12)]. We therefore eventually find that the Gauss constraint S may be regarded as first class.

The computation of the DB's among the fundamental variables, generically denoted by χ, now follows from the well-known formula [16]

$$\{\chi(x),\chi(y)\}_{\rm DB}=\{\chi(x),\chi(y)\}_{\rm PB}$$
$$-\int dz\,dz'\{\chi(x),P_i(z)\}P_{ij}^{-1}(z,z')$$
$$\times\{P_j(z'),\chi(y)\}_{\rm PB}\ , \tag{13}$$

where

$$P_{ij}^{-1}(z,z')=\frac{2\pi^2}{\theta}\varepsilon_{ij}\delta^{(2)}(z-z') \tag{14}$$

is the inverse of the matrix of the PB's among P_i and P_j. The DB's which differ from their PB's are

$$\{A_i(x),A_j(y)\}_{\rm DB}=\left[\frac{4\pi^2}{\theta}\right]^2\{\pi_i(x),\pi_j(y)\}_{\rm DB}$$
$$=\frac{2\pi^2}{\theta}\varepsilon_{ij}\delta(x-y)\ ,$$
$$\{A_i(x),\pi_j(y)\}_{\rm DB}=\tfrac{1}{2}g_{ij}\delta(x-y)\ , \tag{15}$$

which are compatible with setting the second class constraint P_i [Eq. (6)] strongly to zero.

The total Hamiltonian is now written as

$$\mathcal{H}_T=\mathcal{H}_c+u\pi_0+vP\ , \tag{16}$$

where u, v are arbitrary multipliers reflecting the gauge invariances of the theory associated with the two first class constraints. There are now two options for proceeding further with the quantization program. The first and the most popular is to choose two gauge-fixing conditions so that there are no first class constraints. A fresh set of DB's is computed which is consistent with setting π_0 and P strongly to zero. The freedom in the total Hamiltonian (16) is thereby eliminated. The basic problem with this approach is the choice of suitable gauge-fixing conditions. It has also led to some controversy in the literature [8,9]. Sometimes nonlocal gauge conditions [11,18] have also been suggested. As is well known, moreover, there is no standard prescription to make an optimal gauge choice. Recently, however, we [1] worked with an alternative approach [19] which bypasses these difficulties. The basic idea is to fix u and v so that the fields have the correct transformation properties,

$$\left\{\chi,\int H_T\right\}_{\rm DB}=\partial_0\chi\ ,$$
$$\chi\equiv(\phi,\phi^*,\pi,\pi^*,A_\mu,\pi_i) \tag{17}$$

calculated by using the DB's (15). Naturally it is also essential to verify that this choice of the arbitrary multipliers correctly yields the transformation properties of the fields under the other space-time symmetries (i.e., spatial translations, rotations, and boosts). Finally, it is obligatory to check the Poincaré algebra. We make a detailed discussion of these issues in the next section.

III. ENERGY-MOMENTUM TENSOR AND COVARIANCE OF THE MODEL

It is simple to verify that Heisenberg's equation (17) can be reproduced for all the canonical variables provided u and v in (16) are chosen as

$$u=\partial_0 A_0, \quad v=0 . \tag{18}$$

In order to discuss the spatial translations we proceed along similar lines. The momentum operator is first obtained from the canonical energy momentum (EM) tensor $\theta^c_{\mu\nu}$ where

$$\begin{aligned}\theta^c_{\mu\nu}&=\sum_{\chi=\phi,\phi^*,A_\mu}\frac{\partial\mathcal{L}}{\partial(\partial^\mu\chi)}\partial_\nu\chi-g_{\mu\nu}\mathcal{L}\\&=(D_\mu\phi)^*(\partial_\nu\phi)+(D_\mu\phi)(\partial_\nu\phi^*)\\&\quad+\frac{\theta}{4\pi^2}\varepsilon_{\sigma\mu\lambda}A^\sigma\partial_\nu A^\lambda-g_{\mu\nu}\mathcal{L} .\end{aligned} \tag{19}$$

Indeed, the total expressions for the generators of space-time translations may be written in a Lorentz covariant form

$$\theta^T_{0\mu}=\theta^c_{0\mu}+u_{0\mu}\pi_0+v_{0\mu}P , \tag{20a}$$

where

$$u_{0\mu}=\partial_\mu A_0, \quad v_{0\mu}=0 \tag{20b}$$

such that the proper space-time transformations for all the fields (and their conjugate momenta) are obtained:

$$\{\chi,P_\mu\}_{\rm DB}=\partial_\mu\chi , \tag{21a}$$

$$P_\mu=\int\theta^T_{0\mu} . \tag{21b}$$

It is simple to check that the constraints have vanishing DB with $\int\theta^T_{00}$ so that these are fixed in time. A straightforward extension can be done to include rotations and boosts, defined from

$$M_{\mu\nu}=\int d^2x\mathcal{M}^T_{0\mu\nu} , \tag{22a}$$

where

$$\mathcal{M}^T_{\alpha\mu\nu}=x_\mu\theta^T_{\alpha\nu}-x_\nu\theta^T_{\alpha\mu}-\frac{\partial\mathcal{L}}{\partial(\partial^\alpha A^\lambda)}\Sigma^{\lambda\sigma}_{\mu\nu}A_\sigma \tag{22b}$$

with

$$\Sigma^{\lambda\sigma}_{\mu\nu}=g^\lambda_\mu g^\sigma_\nu-g^\lambda_\nu g^\sigma_\mu . \tag{22c}$$

Using $\theta^T_{0\mu}$ from (20) and the DB (15), it is simple to show, from the above set of equations,

$$\{M_{ij},\phi(x)\}_{\rm DB}=-x_i\partial_j\phi+x_j\partial_i\phi , \tag{23a}$$

$$\{M_{0i},\phi(x)\}_{\rm DB}=-x_0\partial_i\phi+x_i\partial_0\phi \tag{23b}$$

and, likewise,

$$\{M_{ij},A_k(x)\}_{\rm DB}=-x_i\partial_j A_k+x_j\partial_i A_k+\delta_{ik}A_j-\delta_{jk}A_i , \tag{24a}$$

$$\{M_{0i},A_j(x)\}_{\rm DB}=-x_0\partial_i A_j+x_i\partial_0 A_j-\delta_{ij}A_0 . \tag{24b}$$

It is interesting to observe that the fields transform normally under rotations and boosts. In particular, there is no rotational anomaly for the matter field ϕ which is coupled to the CS term. Such an anomaly was earlier reported in Ref. [4] where a radiation gauge computation was performed. The difference in results may, therefore, be attributed to the choice of gauge-fixing conditions implying that the rotational anomaly in [4] is really an artifact of the gauge. Similarly under boosts, the potential A_j transforms with additional terms in the radiation gauge [4], which are absent in our gauge-independent approach.

Finally a straightforward calculation shows that the generators of space-time transformations satisfy the Poincaré algebra:

$$\begin{aligned}&\{P_\mu,P_\nu\}_{\rm DB}=0 ,\\&\{P_\mu,M_{\nu\lambda}\}_{\rm DB}=g_{\mu\lambda}P_\nu-g_{\mu\nu}P_\lambda ,\\&\{M_{\mu\nu},M_{\sigma\lambda}\}_{\rm DB}=g_{\nu\lambda}M_{\mu\sigma}-g_{\mu\lambda}M_{\nu\sigma}+g_{\mu\sigma}M_{\nu\lambda}-g_{\nu\sigma}M_{\mu\lambda} .\end{aligned} \tag{25}$$

Thus in the quantized version the canonical energy momentum tensor $\theta^c_{0\mu}$ is replaced by $\theta^T_{0\mu}$ [Eq. (20)] and the DB's are transformed into commutators as $i\{,\}_{\rm DB}\to[,]$. Moreover, operator symmetrization is implied whenever products of operators occur.

An alternative route of discussing the covariance of the model is to consider the Dirac-Schwinger covariance condition [15]

$$\{\theta^T_{00}(x),\theta^T_{00}(y)\}_{\rm DB}=[\theta^T_{0i}(x)+\theta^T_{0i}(y)]\partial^x_i\delta(x-y) . \tag{26}$$

Using (20), we find however,

$$\{\theta^T_{00}(x),\theta^T_{00}(y)\}_{\rm DB}|\psi\rangle=[\bar\theta_{0i}(x)+\bar\theta_{0i}(y)]\partial_i\delta(x-y)|\psi\rangle , \tag{27}$$

where the physical states $|\psi\rangle$ are ones which are annihilated by the first class constraints and

$$\bar\theta_{0i}|\psi\rangle=[\theta^T_{0i}+A_iJ_0-\tfrac{1}{2}A_0J_i+(\theta/8\pi^2)\varepsilon_{ij}(\partial^jA_0)A_0+(\theta/4\pi^2)\varepsilon_{jk}A^j\partial_iA^k]|\psi\rangle , \tag{28}$$

implying that condition (26) is violated. We have, however, established covariance via the Poincaré algebra (25). The apparent paradox that the Dirac-Schwinger condition (26) is violated may be resolved by realizing that the definition of the energy-momentum tensor is not unique. Indeed, the general (model-independent) operatorial proof of the Schwinger condition [15] demands that the full rotational transformation [e.g., (24a)] be generated by an operator such as $M_{ij}=x_i\theta^T_{0j}-x_j\theta^T_{0i}$, which does not explicitly include the spin factor [Eq. (22c)]. Our rotation generator (22b) clearly does not satisfy this prerequisite.

It is, however, possible to define a new (symmetric) EM tensor which conforms to this criterion:

$$\tilde{\theta}_{\mu\nu}=\frac{\partial S}{\partial g^{\mu\nu}}=2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}-g_{\mu\nu}\mathcal{L}$$

$$=(D_\mu\phi)^*(D_\nu\phi)+(D_\mu\phi)(D_\nu\phi)^*-g_{\mu\nu}(D_\sigma\phi)^*(D^\sigma\phi)$$

$$=\tilde{\theta}_{\nu\mu} \ . \tag{29}$$

Note that the Chern-Simons term is covariant without reference to the metric and hence does not contribute to $\tilde{\theta}_{\mu\nu}$. It did contribute, however, to the canonical expression (19). As was done earlier, the total EM tensor is obtained from (29) by adding linear combinations of the two first class constraints:

$$\tilde{\theta}^T_{\mu\nu}=\tilde{\theta}_{\mu\nu}+\tilde{u}_{\mu\nu}\pi_0+\tilde{v}_{\mu\nu}P \ . \tag{30a}$$

It is simple to show that the space-time translations are generated by $\int\tilde{\theta}^T_{0\mu}$ where

$$\tilde{u}_{0\mu}=\partial_\mu A_0, \quad \tilde{v}_{0\mu}=-A_\mu \ . \tag{30b}$$

Note that the coefficient $\tilde{v}_{0\mu}$ is different from its previous counterpart $v_{0\mu}$ given in (20b). It is significant that the difference between the densities $\tilde{\theta}^T_{0\mu}$ and $\theta^T_{0\mu}$ is a boundary term so that the integrated expressions corresponding to the translation generators are identical, i.e.,

$$\int\tilde{\theta}^T_{0\mu}=\int\theta^T_{0\mu}=P_\mu \ . \tag{31}$$

The usual definition of the generator of rotations and boosts following from (30) is given by

$$\tilde{M}_{\mu\nu}=\int d^2x\,\tilde{\mathcal{M}}^T_{0\mu\nu} \ , \tag{32a}$$

where

$$\tilde{\mathcal{M}}^T_{0\mu\nu}=x_\mu\tilde{\theta}^T_{0\nu}-x_\nu\tilde{\theta}^T_{0\mu} \ . \tag{32b}$$

Note the absence of the explicit spin term in (32) which was, however, present in (22). Once again it may be demonstrated that the integrated expressions following from (22) and (32) agree, i.e.,

$$M_{ij}=\tilde{M}_{ij} \ , \tag{33a}$$

$$M_{0i}=\tilde{M}_{0i} \tag{33b}$$

so that the fields transform normally under rotations and boosts. The EM tensor (30) is thus seen to satisfy the criteria necessary for it to obey the Dirac-Schwinger condition [15]. Indeed, an explicit computation reveals that this is true, i.e.,

$$\{\tilde{\theta}^T_{00}(x),\tilde{\theta}^T_{00}(y)\}_{\rm DB}=[\tilde{\theta}^T_{0i}(x)+\tilde{\theta}^T_{0i}(y)]\partial^x_i\delta(x-y) \ . \tag{34a}$$

Apart from this, there are two subsidiary conditions which may be verified:

$$\{\tilde{\theta}^T_{0i}(x),\tilde{\theta}^T_{0j}(y)\}_{\rm DB}=[\tilde{\theta}^T_{0i}(x)\partial^x_j+\tilde{\theta}^T_{0j}(y)\partial^x_i]\delta(x-y) \ , \tag{34b}$$

$$\{\tilde{\theta}^T_{00}(x),\tilde{\theta}^T_{0i}(y)\}_{\rm DB}=[\tilde{\theta}^T_{ij}(x)-g_{ij}\tilde{\theta}^T_{00}(y)]\partial^x_j\delta(x-y) \ . \tag{34c}$$

The three conditions (34) are sufficient to prove the validity of the complete Poincaré algebra (25).

This completes our demonstration of the covariance of the model. Both the canonical and symmetric forms of the EM tensor are modified in the presence of first class constraints. These modifications are such as to preserve the proper transformation properties of the fields and satisfy the Poincaré algebra. The Schwinger condition, however, is compatible only with the symmetric definition of the EM tensor for reasons elucidated earlier.

IV. THE PHYSICAL STATES AND THE ANYON OPERATOR

The physical states $|\psi\rangle$ of the model are defined to be those states which are annihilated by the first class constraints:

$$\pi_0|\psi\rangle=P|\psi\rangle=0 \ . \tag{35}$$

Since the first class constraints are the generators of gauge transformations, the physical states (35) must be gauge invariant. We can be assured of this if the operators which create these states from the vacuum are also gauge invariant. Moreover, following the usual convention [5] we define the one-particle states to be those states which carry one unit of the charge $Q=\int d^2x\,J_0$ [Eq. (8)], i.e., states

$$|1\rangle=\hat{\phi}|0\rangle \tag{36}$$

such that the creation operator $\hat{\phi}$ obeys

$$[J_0(x),\hat{\phi}(y)]=\delta(x-y)\hat{\phi} \ . \tag{37}$$

As a consequence of the nontrivial commutator,

$$[J_0(x),\phi(y)]=\delta(x-y)\phi \ , \tag{38}$$

following from the DB (15), a general ansatz for a gauge-invariant $\hat{\phi}$ satisfying (37) may be written [1]:

$$\hat{\phi}(x)=\exp\left[\int dy\ \Omega(x-y)J_0(y)+i\int_{x_0}^{x}dy_i\,A_i(y)\right]\phi(x) \ , \tag{39}$$

where Ω is, as yet, an undetermined function and x_0 is some reference point. The line integration in (39) is performed along the straight path:

$$y_i=(x_0)_i+(x-x_0)_i t, \quad 0\le t\le 1 \ . \tag{40}$$

It may be pointed out that the gauge invariance of $\hat{\phi}$ will be preserved for any choice of path, the straight path being chosen for algebraic simplifications. Indeed, in computing products of $\hat{\phi}$ the following equality proves to be very useful:

$$\int_{x_0}^{x} dy_i \int_{x_0}^{z} dz_j [A_i(y), A_j(z)] = i\int_0^1 dt \int_0^1 dt' (x-x_0)_i (z-x_0)_j \frac{2\pi^2}{\theta}\varepsilon_{ij}\delta((x-x_0)t-(z-x_0)t') = 0 . \quad (41)$$

following from the antisymmetry under the exchange $i \leftrightarrow j$ and where the explicit structure for the commutator among the potentials has been substituted from (15).

In order to fix the function $\Omega(x-y)$ in (39), we first compute the general n-particle state functional:

$$|\psi_n\rangle = \left[\prod_{i=1}^{n} \hat{\phi}(x_i)\right] |0\rangle . \quad (42)$$

To simplify this, note that Eq. (38) implies, by the Baker-Campbell-Hausdorff formula,

$$\exp\left[\int dy\, \Omega(x-y) J_0(y)\right] \phi(z) \exp\left[-\int dy\, \Omega(x-y) J_0(y)\right] = \exp[\Omega(x-z)]\phi(z) . \quad (43)$$

Using this formula and the equality (41), the n-particle state functional may be expressed as

$$|\psi_n\rangle = \exp\left[-\sum_{j=1}^{n}\sum_{i=1}^{j-1} \Omega(x_i-x_j)\right] \left[\exp\left[\sum_{i=1}^{n}\int dy\, \Omega(x_i-y) J_0(y)\right] \prod_{i=1}^{n} \bar{\phi}(x_i)|0\rangle\right] , \quad (44a)$$

where

$$\bar{\phi}(x) = \exp\left[i\int^{x} dy_i\, A_i(y)\right]\phi(x) . \quad (44b)$$

Now the general structure of the n-particle functional following from the representation theory of the braid group is given by [5]

$$\psi_S[\chi(x_1),\ldots,\chi(x_n);t] = \exp\left[-2iS\sum_{j=1}^{n}\sum_{i=1}^{j-1}\omega(x_i-x_j)\right]\psi_0[\chi(x_1),\ldots,\chi(x_n);t] , \quad (45)$$

where Forte and Jolicoeur [5] have shown that for CS theory with matter coupling, the generalized spin factor S is a function of θ, henceforth denoted by $S(\theta)$. The explicit form for $S(\theta)$ will be determined shortly. $\psi_0[\chi(x_1),\ldots,\chi(x_n);t]$ is an n-particle functional with Bose statistics and $\omega(x-y)$ is the multivalued polar angle of the vector $x-y$:

$$\omega(x-y) = \arctan\frac{x^2-y^2}{x^1-y^1} . \quad (46)$$

Returning to Eq. (44) we observe that the expression in the curly brackets represents a gauge-invariant functional with commuting one-particle cocycles, because

$$\left[\int dy\, \Omega(x-y) J_0(y), \int dy' \Omega(x'-y') J_0(y')\right] = 0 \quad (47)$$

and hence may be identified with ψ_0 (45). The correspondence between Eqs. (44) and (45) is complete if one substitutes

$$\Omega(x_i-x_j) = 2iS(\theta)\omega(x_i-x_j) . \quad (48)$$

Hence the final expression for the one-particle creation operator (39) is

$$\hat{\phi}(x) = \exp\left[2iS(\theta)\int dy\, \omega(x-y) J_0(y) + i\int^{x} dy_i\, A_i(y)\right]\phi(x) , \quad (49)$$

which is multivalued due to the presence of $\omega(x-y)$.

This operator $\hat{\phi}(x)$ may be regarded as the anyon field operator since it creates states (44) with arbitrary spin $S=S(\theta)$ when acting on the vacuum. Exactly as happens in the Klein-Gordon theory, we may consider $\hat{\phi}$ as comprising creation operators of particles and annihilation operators of antiparticles. This does not affect our analysis since the n-particle functional (44) is computed for distinct χ_i only so that the vacuum contributions [proportional to $\delta(x_i-x_j)$, $i \neq j$] vanish [1]. Similarly one antiparticle state will be created by $\hat{\phi}^\dagger(x)$. The anyon operator (49) which was first suggested in Ref. [1] is different from the conventional constructions [7,10,11]. The structure (49) is gauge invariant in contrast with the conventional anyon operators [7,10,11] which are gauge dependent. The latter, in fact, are obtained in the Hamiltonian formalism employing a specific gauge fixing so that it is not clear whether their anyonicity is a physical effect or an artifact of the gauge. It is not surprising, therefore, that different results with different gauge choices have been reported [11]. Moreover, in obtaining the anyon operator, we have avoided formal manipulations with multivalued operators that were earlier criticized [8,9,12].

To determine the spin factor $S(\theta)$ in (49), consider the action of the rotation operator (32) on the physical one-particle state,

$$J|\psi\rangle = \int d^2x\, \varepsilon^{ij} x_i \tilde{\theta}_{0j}^{T} |\psi\rangle = \int d^2x\, \varepsilon^{ij} x_i \tilde{\theta}_{0j} |\psi\rangle = \int d^2x\, \varepsilon^{ij} x_i (\pi\partial_j\phi + \pi^*\partial_j\phi^* + iA_j J_0)|\psi\rangle , \quad (50)$$

where, in going from the first to the second line, we have exploited the fact that the first class constraints annihilate the physical state. The first two terms in (50) are the normal canonical terms while the third can be simplified by formally solving the constraint (11), valued on the physical state:

$$\left[A_j-\frac{2\pi^2}{\theta}\varepsilon_{jk}\frac{\partial^k}{\partial^2}J_0\right]|\psi\rangle=0 \tag{51}$$

so that, concentrating only on the third factor in (50),

$$J_S|\psi\rangle=\int d^2x\,\varepsilon^{ij}x_iJ_0 i\frac{2\pi^2}{\theta}\varepsilon_{jk}\frac{\partial^k}{\partial^2}J_0|\psi\rangle\ . \tag{52}$$

This simplifies after some algebra to [20]

$$J_S|\psi\rangle=(\pi/2\theta)Q^2|\psi\rangle=(\pi/2\theta)|\psi\rangle \tag{53}$$

since the physical state (36) is an eigenstate of the charge. The angular momentum operator therefore rotates the physical state by an additional phase which may be identified with the spin factor in (49), i.e.,

$$S(\theta)=\pi/2\theta\ . \tag{54}$$

To study the statistics of $\hat{\phi}(x)$ (49) we compute the product $\hat{\phi}(x)\hat{\phi}(y)$ and exploit formula (43) to obtain

$$\begin{aligned}\hat{\phi}(x)\hat{\phi}(y)=&\exp\{2iS(\theta)[\omega(x-y)-\omega(y-x)]\}\\&\times\hat{\phi}(y)\hat{\phi}(x)\ .\end{aligned} \tag{55}$$

Substituting $S(\theta)$ from (54) and noting that the angle between two antiparallel vectors $[\omega(x-y)-\omega(y-x)]$ is $(\pi\,\mathrm{mod}2\pi)$, we find

$$\hat{\phi}(x)\hat{\phi}(y)=\exp[i(\pi/\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\phi}(y)\hat{\phi}(x)\ . \tag{56}$$

This result reveals that the fields satisfy graded commutation relations. The statistical phase in (56) has a sign ambiguity (due to the presence of mod2π) and physically reflects the arbitrariness present in the exchange of two particles which may be done either by a clockwise or an anticlockwise rotation. If $\theta=\pi/2n$ (corresponding to bosons, since $S=\pi/2\theta=n$), we find from (56) that $\hat{\phi}$ is commuting. Similarly, for $\theta=\pi/(2n+1)$ (corresponding to fermions) anticommutators are obtained. Thus the usual spin-statistics theorem is satisfied. For other values of θ, a generalized spin with abnormal statistics is revealed. Interestingly, the statistical phase in (56) coincides with our [18] earlier calculation done by completely fixing the gauge by nonlocal gauge constraints. The algebra for the antiparticle creation operator $\hat{\phi}^\dagger(x)$ can be obtained by Hermitian conjugation of (56), while the result for the particle-antiparticle case may be explicitly evaluated:

$$\phi(x)\hat{\phi}^\dagger(y)=\exp[-i(\pi/\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\phi}^\dagger(y)\hat{\phi}(x)\ . \tag{57}$$

To compute the algebra of $\hat{\phi}$ ($\hat{\phi}^\dagger$) with their canonically conjugate momenta $\hat{\pi}$ ($\hat{\pi}^\dagger$) it is first essential to define the latter variables. This can be done by recasting the original Lagrangian (1) in terms of the careted variables and then taking the necessary partial derivatives:

$$\hat{\pi}(x)=\frac{\partial\mathcal{L}}{\partial[\partial_0\hat{\phi}(x)]}=\pi(x)\exp\left[-2iS(\theta)\int dy\,\omega(x-y)J_0(y)-i\int^x dy_i\,A_i(y)\right], \tag{58a}$$

$$\hat{\pi}^\dagger(x)=\frac{\partial\mathcal{L}}{\partial[\partial_0\hat{\phi}^\dagger(x)]}=\exp\left[2iS(\theta)\int dy\,\omega(x-y)J_0(y)+i\int^x dy_i\,A_i(y)\right]\pi^\dagger(x)\ . \tag{58b}$$

Using formulas analogous to (43) it is straightforward to work out the relevant algebra:

$$\begin{aligned}&\hat{\phi}(x)\hat{\pi}(y)=\delta(x-y)+\exp[-i(\pi/\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\pi}(y)\hat{\phi}(x)\ ,\\&\hat{\phi}(x)\hat{\pi}^\dagger(y)=\exp[i(\pi/\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\pi}^\dagger(y)\hat{\phi}(x)\ ,\\&\hat{\pi}(x)\hat{\pi}^\dagger(y)=\exp[-i(\pi/\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\pi}^\dagger(y)\hat{\pi}(x)\ ,\\&\hat{\pi}(x)\hat{\pi}(y)=\exp[i(\pi/\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\pi}(y)\hat{\pi}(x)\ ,\end{aligned} \tag{59}$$

while the remaining algebra is given by Hermitian conjugation.

Before concluding this section we comment on two important aspects concerning the definition of the anyon operator $\hat{\phi}(x)$ (49). The first point is that if the solution of the constraint (11) is substituted in (49) then the exponential becomes (ignoring a total divergence) single valued so that $\hat{\phi}(x)$ can no longer represent an anyon operator. Indeed, it is simple to explicitly verify that this modified structure for $\hat{\phi}(x)$ does not yield any statistical phase in (56), simply because everything commutes. The paradox can be resolved by noting that the constraint (11) is first class and is obeyed only weakly in contrast with the second class constraints P_i (6) which are strongly valid. Hence the solution of the constraint (11) cannot be directly inserted in (49). The second point is that it is possible to recast the interaction piece of the Hamiltonian (7) in terms of the careted (anyon) variables (49). This will involve the derivative of the exponential line integral which will be defined in the manner of Mandelstam [21]:

$$\partial_i\hat{\phi}(x,P)=\lim_{dx_i\to 0}\frac{\hat{\phi}(x+dx_i,P')-\hat{\phi}(x,P)}{dx_i}\ , \tag{60}$$

where the path P' is obtained from P by extending it by dx_i in the ith direction. In terms of the anyon variables, therefore,

$$[D_i\phi(x)]^*[D^i\phi(x)]=\partial_i\left[\exp\left[-2iS(\theta)\int dy\,\omega(x-y)\hat{J}_0(y)\right]\hat{\phi}(x)\right]^*\partial^i\left[\exp\left[-2iS(\theta)\int dy\,\omega(x-y)\hat{J}_0(y)\right]\hat{\phi}(x)\right], \tag{61}$$

so that the minimal interaction has been eliminated. This is highly reminiscent of Mandelstam's [21] construction of quantum electrodynamics. Just as in QED, here we can also discuss the theory either in terms of gauge-dependent quantities (such as ϕ, π, A_i) or in terms of gauge-independent ones (such as $\hat{\phi}$, $\hat{\pi}$, F_{ij}, etc.). The latter description, as we have shown, involves the occurrence of generalized spin and statistics.

V. CHERN-SIMONS TERM COUPLED TO FERMIONS

It is possible to develop our gauge-independent formalism discussed in the preceding sections (II–IV) for fermionic matter coupled to a Chern-Simons term. Consider the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\bar{\psi}i\overleftrightarrow{\not\partial}\psi+\bar{\psi}\not{A}\psi+(\theta/4\pi^2)\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, \tag{62}$$

where we use the form

$$\bar{\psi}\overleftrightarrow{\not\partial}\psi=\bar{\psi}\overrightarrow{\not\partial}\psi-\bar{\psi}\overleftarrow{\not\partial}\psi \tag{63}$$

to preserve Hermiticity. The γ matrices in 2+1 dimensions satisfy

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}, \quad \gamma^\mu\gamma^\nu=g^{\mu\nu}-i\varepsilon^{\mu\nu\rho}\gamma_\rho \tag{64}$$

and the metric is already defined in (3).

The canonical momenta are

$$\pi^0=\frac{\partial\mathcal{L}}{\partial\dot{A}_0}=0,\quad \pi^i=\frac{\partial\mathcal{L}}{\partial\dot{A}_i}=\frac{\theta}{4\pi^2}\varepsilon^{ij}A_j, \quad \pi_\alpha=\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}=-\frac{i}{2}(\bar{\psi}\gamma_0)_\alpha,\quad \bar{\pi}_\alpha=\frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}_\alpha}=-\frac{i}{2}(\gamma_0\psi)_\alpha. \tag{65}$$

The constraint structure for the gauge field naturally remains identical to the previous example [see Eq. (6)]. However, in the matter sector, two new primary constraints are generated from (65):

$$\eta_\alpha=\pi_\alpha+(i/2)(\bar{\psi}\gamma_0)_\alpha\approx 0, \quad \bar{\eta}_\alpha=\bar{\pi}_\alpha+(i/2)(\gamma_0\psi)_\alpha\approx 0. \tag{66}$$

The canonical Hamiltonian is

$$\mathcal{H}_c=-\frac{i}{2}\bar{\psi}\gamma_k\overleftrightarrow{\partial}^k\psi-\bar{\psi}\gamma_kA^k\psi-A_0[J_0+(\theta/2\pi^2)\varepsilon^{ij}\partial_iA_j], \tag{67}$$

where J_μ is the gauge-invariant conserved current:

$$J_\mu=\bar{\psi}\gamma_\mu\psi. \tag{68}$$

The primary Hamiltonian is

$$H_P=\int d^2x(\mathcal{H}_c+u_0P_0+u_iP_i+\eta_\alpha C_\alpha+\bar{C}_\alpha\bar{\eta}_\alpha), \tag{69}$$

where u_0, u_i are ordinary multipliers [see Eq. (9)] and c_α, $\bar{c}_\alpha$ are Grassmann multipliers. Conserving the primary constraints with H_P gives the analogue of the Gauss constraint:

$$S=J_0+(\theta/2\pi^2)\varepsilon^{ij}\partial_iA_j\approx 0. \tag{70}$$

No further constraints are obtained by this iterative procedure. Analogous to the example of complex scalars, we find that the combination of second class constraints,

$$P'=\partial^iP_i+S+i(\eta_\alpha\psi_\alpha+\bar{\psi}_\alpha\bar{\eta}_\alpha)=\partial^i\pi_i+J_0+(\theta/4\pi^2)\varepsilon^{ij}\partial_iA_j+i(\eta_\alpha\psi_\alpha+\bar{\psi}_\alpha\bar{\eta}_\alpha) \tag{71}$$

is first class. Thus the first class constraints are π_0 and P' while P_i [Eq. (6)] and η_α, $\bar{\eta}_\alpha$ are second class. We next compute the DB in the usual way. The gauge sector is already evaluated in (15) while the matter sector yields

$$\begin{aligned}\{\psi_\alpha(x),\bar{\psi}_\beta(y)\}_{\rm DB}&=-4\{\pi_\alpha(x),\bar{\pi}_\beta(y)\}_{\rm DB}=-i(\gamma_0)_{\alpha\beta}\delta(x-y),\\ \{\psi_\alpha(x),\pi_\beta(y)\}_{\rm DB}&=\{\bar{\psi}_\alpha(x),\bar{\pi}_\beta(y)\}_{\rm DB}=-\tfrac{1}{2}\delta_{\alpha\beta}\delta(x-y).\end{aligned} \tag{72}$$

The other Dirac brackets are identical to the Poisson brackets. These brackets are consistent with setting the second class constraints strongly to zero. The total Hamiltonian is given by

$$\mathcal{H}_T=\mathcal{H}_c+u\pi_0+vP', \tag{73}$$

where, as before, u and v are arbitrary parameters reflecting the gauge invariances of the theory.

The gauge-independent quantization now proceeds by fixing u and v so that Heisenberg's equation

$$\{\chi,\textstyle\int\mathcal{H}_T\}_{\rm DB}=\partial_0\chi \tag{74}$$

is reproduced for all the canonical variables χ. It is easy to check that there is a unique choice for the multipliers:

$$u=\partial_0A_0,\quad v=0. \tag{75}$$

An identical exercise can be done for the momentum operator defined from the canonical EM tensor:

$$\theta^c_{\mu\nu}=\partial_\nu\psi\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}+\partial_\nu\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial^\mu\bar{\psi})}+\frac{\partial\mathcal{L}}{\partial(\partial^\mu A_\sigma)}\partial_\nu A_\sigma-\mathcal{L}g_{\mu\nu}=(i/2)\bar{\psi}\gamma_\mu\overleftrightarrow{\partial}_\nu\psi-(\theta/4\pi^2)\varepsilon_{\mu\sigma\lambda}A^\sigma\partial_\nu A^\lambda-\mathcal{L}g_{\mu\nu}. \tag{76}$$

The final expressions for the generators of space-time translations may be expressed in a Lorentz covariant form:

$$\theta^T_{0\mu}=\theta^c_{0\mu}+u_{0\mu}\pi_0+v_{0\mu}P', \tag{77a}$$

where

$$u_{0\mu}=\partial_\mu A_0, \quad v_{0\mu}=0 \tag{77b}$$

so that

$$\left\{\chi, \int \theta^T_{0\mu}\right\}_{\rm DB}=\partial_\mu \chi \ . \tag{78}$$

To study the transformation properties of the fields under rotations and boosts, it is necessary to first define the corresponding generators

$$M_{\mu\nu}=\int d^2x\, \mathcal{M}^T_{0\mu\nu} \ , \tag{79a}$$

where

$$\mathcal{M}^T_{0\mu\nu}=x_\mu \theta^T_{0\nu}-x_\nu\theta^T_{0\mu}-\frac{\partial\mathcal{L}}{\partial(\partial^0 A^\sigma)}\Sigma^{\sigma\lambda}_{\mu\nu}A_\lambda+\frac{i}{4}\psi^\dagger[\gamma_\mu,\gamma_\nu]\psi \ , \tag{79b}$$

which differs from (22) due to the presence of the last term which occurs because fermions are present. Using $\theta^T_{0\mu}$ from (77) and the DB's (72) it follows that

$$\{M_{ij},\psi_k(x)\}_{\rm DB}=-\chi_i\partial_j\psi_k+x_j\partial_i\psi_k-\tfrac{1}{4}[\gamma_i,\gamma_j]\psi_k \ , \tag{80a}$$

$$\{M_{0i},\psi_k(x)\}_{\rm DB}=-x_0\partial_i\psi_k+x_i\partial_0\psi_k-\gamma_0\gamma_i\psi_k \ , \tag{80b}$$

illustrating that the fields have normal transformation properties. The additional factor in (80) compared to (23) precisely accounts for the spin of the fermions. There is no anomalous transformation law for rotations as claimed in Ref. [4] which, as we pointed out earlier, ought to be regarded as an artifact of the gauge. Results similar to (80) can be easily obtained for the momenta conjugate to ψ while the analysis for the gauge field can be mimicked from the previous example [see Eq. (24)]. Finally, it can be shown that the generators of space-time transformations obey the Poincaré algebra (25). This proves the covariance of the model. Thus in the quantized version of the theory the canonical EM tensor $\theta^c_{0\mu}$ is replaced by $\theta^T_{0\mu}$ while the Dirac bracket is, as usual, converted to the graded relation:

$$i\{P,Q\}_{\rm DB}\to[PQ-(-1)^{\varepsilon_P\varepsilon_Q}QP] \ , \tag{81}$$

where $\varepsilon_P=0(1)$ for bosonic (fermionic) P.

A different way of analyzing the model is to check the validity of the Dirac-Schwinger condition (34a). It is straightforward to show that this condition is, however, violated by the EM tensor (77). The reason, as in the earlier example of complex scalars, is to be found in the structure of the rotation generator (79) which explicitly involves the extra spin factors to yield the proper transformation law (80). Let us therefore consider the symmetric form of the EM tensor [4,15] obtained by introducing a background gravitational metric. We find

$$\tilde\theta_{\mu\nu}=\frac{\partial S}{\partial g_{\mu\nu}} \ , \tag{82a}$$

$$\tilde\theta_{00}=-(i/2)\bar\psi\gamma_k\overleftrightarrow{\partial}^k\psi-\bar\psi\gamma_k A^k\psi \ , \tag{82b}$$

$$\tilde\theta_{0i}=(i/2)\bar\psi\gamma_0\overleftrightarrow{\partial}_i\psi+\bar\psi\gamma_0 A_i\psi+(i/8)\partial^j(\bar\psi\gamma_0[\gamma_i,\gamma_j]\psi) \ . \tag{82c}$$

The total EM tensor is obtained by adding linear combinations of the two first class constraints π_0 and P' (71). It can be shown that the space-time translations are now generated by $\int\tilde\theta^T_{0\mu}$ where

$$\tilde\theta^T_{0\mu}=\tilde\theta_{0\mu}+\tilde u_{0\mu}\pi_0+\tilde v_{0\mu}P' \tag{83a}$$

with

$$\tilde u_{0\mu}=\partial_\mu A_0, \tilde v_{0\mu}=-A_\mu \ . \tag{83b}$$

The difference between $\theta^T_{0\mu}$ (77) and $\tilde\theta^T_{0\mu}$ (83) turns out to be a boundary term so that the generators match. The generators $\tilde M_{ij}$ ($\tilde M_{0i}$) of rotations (boosts) following from (83) are

$$\tilde M_{\mu\nu}=\int d^2x(x_\mu\tilde\theta^T_{0\nu}-x_\nu\tilde\theta^T_{0\mu}) \tag{84}$$

and are identical with the expressions obtained from the modified canonical EM tensor [see Eq. (79)], i.e.,

$$\tilde M_{ij}=M_{ij}, \quad \tilde M_{0i}=M_{0i} \ . \tag{85}$$

Thus the fields have their conventional transformations intact. One can now verify that the EM tensor (83) satisfies the complete set of Schwinger conditions (34) and thus the Poincaré algebra.

To determine the Fock space we proceed, as was done in [1], by making a suitable gauge-invariant *Ansatz* for the one-particle creation operator:

$$\hat\psi(x)=\exp\left[\int dy\,\Omega(x-y)J_0(y)-i\int^x dy_i\,A_i(y)\right]\psi(x) \ . \tag{86}$$

The one-particle states

$$|1\rangle=\hat\psi(x)|0\rangle \tag{87}$$

are also an eigenstate of the charge $Q=\int J_0$ because

$$[Q,\hat\psi(x)]=\hat\psi(x) \ , \tag{88}$$

which follows from

$$[J_0(x),\psi(y)]=\delta(x-y)\psi(y) \tag{89}$$

calculated by the DB's (72).

It is possible to fix Ω in (86) by computing the n-particle functional

$$|\psi_n\rangle=\left[\prod_{i=1}^n\hat\psi(x_i)\right]|0\rangle \tag{90}$$

and comparing with the result obtained from the representation theory of Braid group [5]. Mimicking the analysis done earlier,

$$\Omega(x-y)=-2i(-S+\tfrac{1}{2})\omega(x-y) \ , \tag{91}$$

where $\omega(x-y)$ is defined in (46). As usual, the spin term is a function of the CS parameter and the factor half in (91) comes from the use of fermions. Thus the anyon operator (86) is

$$\hat{\psi}(x)=\exp\left[2i[S(\theta)-\tfrac{1}{2}]\int dy\,\omega(x-y)J_0(y)-i\int^x dy_i\,A_i\right]\psi(x) \tag{92}$$

since it creates states with arbitrary spin $S(\theta)$ when acting on the vacuum. An explicit form for $S(\theta)$ can also be found by expressing the angular momentum operator (84) in terms of a canonical piece and an additional piece. Following the steps which led to (54), we find

$$S(\theta)=\frac{\pi}{2\theta}+\frac{1}{2}\ ; \tag{93}$$

the extra factor of $\frac{1}{2}$ is the intrinsic spin of fermions. The statistics of $\hat{\psi}(x)$ is studied by computing the product,

$$\hat{\psi}(x)\hat{\psi}(y)=\exp[2iS(\theta)(\pi\,\mathrm{mod}2\pi)]\hat{\psi}(y)\hat{\psi}(x) \tag{94}$$

obtained by using (43). Integral (half-integral) values of $S(\theta)$ lead to commutators (anticommutators), thereby conforming to the spin-statistics theorem. The algebra for the other operators are calculated in an analogous fashion and results akin to the set (59) are obtained The other considerations discussed below Eq. (59) are also applicable to the construction (92).

VI. COVARIANCE PROBLEM IN PHENOMENOLOGICAL LAGRANGIANS

It has been recently claimed [13,14] that a basic concept for the superfluidity of an anyon gas was the spontaneous violation of the commutation relation involving spatial translation generators. In the phenomenological picture of the system based on quasiparticles and quasiholes, the momentum generators are not commuting. The commutativity is restored by coupling a spin-zero massless (Goldstone) boson [13] or a massive spin-one boson [14] to the system. The object of this section is to critically examine the claimed "violation" of translation invariance in phenomenological Lagrangians. Take, for instance, a typical choice involving the coupling of complex scalars to an external gauge potential:

$$\mathcal{L}=(D_\mu\phi)^*(D^\mu\phi)\ . \tag{95}$$

Then the generator of spatial translations is

$$P_i=\int\tilde{\theta}_{0i}=\int[\pi D_i\phi+\pi^*(D_i\phi)^*] \tag{96}$$

following from the definition (29). Note that in our Hamiltonian formalism, the Lagrangian (95) admits no constraints. Hence the EM tensor in (96) need not be modified by extra factors proportional to the constraints as was necessary for the examples discussed earlier [see (30) or (83)]. Moreover, the absence of constraints allows the quantization to be done by Poisson brackets.

In that case it is simple to verify that the momentum operators do not commute,

$$\{P_i,P_j\}_{\rm PB}\neq 0 \tag{97}$$

leading to a violation of spatial translation invariance. There is, however, a snag in considering (96) as the translation generator. Consider, for example, the transformation of the field ϕ which is

$$\{\phi(x),P_i\}_{\rm PB}=D_i\phi(x) \tag{98}$$

and not $\partial_i\,\phi(x)$ so that the usual interpretation of the translation generator breaks down. Apart from this there occurs an internal algebraic inconsistency. It can be shown (see the Appendix) that the Dirac-Schwinger condition [15] holds, i.e.,

$$\{\tilde{\theta}_{00}(x),\tilde{\theta}_{00}(y)\}_{\rm PB}=[\tilde{\theta}_{0i}(x)+\tilde{\theta}_{0i}(y)]\partial_i^x\delta(x-y)\ , \tag{99}$$

where

$$\tilde{\theta}_{00}(x)=\pi^*\pi(x)-(D_i\phi)^*(x)(D^i\phi)(x) \tag{100}$$

following from (29). Integrating both sides of (99) over y and using the symmetry of the EM tensor,

$$\partial_\mu\tilde{\theta}^{\mu 0}=0\ . \tag{101}$$

It is possible to compute this divergence directly by using (29) and the equations of motion:

$$\partial_\mu\tilde{\theta}^{\mu\nu}=F^{\mu\nu}J_\mu\ , \tag{102}$$

where J_μ is given in (8). The above equation is clearly incompatible with (101).

Let us now check the possibility of suitably defining the translation generator from the canonical EM tensor (19). In that case,

$$P_i=\int\theta_{0i}^c=\int(\pi\partial_i\phi+\pi^*\partial_i\phi^*)\ . \tag{103}$$

It is interesting to observe that the translation generators commute,

$$\{P_i,P_j\}_{\rm PB}=0\ , \tag{104}$$

and yield the correct transformation law

$$\{\phi(x),P_i\}_{\rm PB}=\partial_i\phi(x) \tag{105}$$

in contrast with (98). Moreover, this P_i along with the angular momentum

$$J_{kl}=\int(x_k\theta_{0l}-x_l\theta_{0k}) \tag{106}$$

satisfy the commutators appropriate to the two-dimensional translation-rotation group:

$$\{P_k,J_{lm}\}_{\rm PB}=\delta_{km}P_l-\delta_{kl}P_m\ , \tag{107a}$$

$$\{J_{kl},J_{mn}\}_{\rm PB}=\delta_{km}J_{ln}-\delta_{lm}J_{kn}-\delta_{kn}J_{lm}+\delta_{ln}J_{km}\ . \tag{107b}$$

The other commutators of the Poincaré algebra (25) involving $\{P_0,J_{0k}\}$, etc., are, however, not satisfied. Moreover, P_i (103) is not gauge invariant which is a necessary criterion for it to be a momentum operator.

The above arguments clearly reveal that it is problematic to even suitably define the translation operator. For example, the definition using the symmetric EM tensor leads to an anomalous transformation law (98) as well as an algebraic incompatibility [between (101) and (102)]. On the other hand, the canonical momentum (103) is not gauge invariant and, hence, physically irrelevant. Since a proper candidate for the translation generator is unavailable, it is meaningless to discuss issues related to the algebra of these operators; particularly the "spontaneous violation of Poincaré algebra."

VII. CONCLUSIONS

We have developed the canonical quantization program for (bosonic and fermionic) matter coupled Chern-Simons theories without the use of gauge fixing. These theories have both first and second class constraints. The second class constraints are eliminated by the use of Dirac brackets. Since we have not imposed any gauge constraints which convert the first class constraints to second class, the first class constraints are explicitly present in our theory. They appear with arbitrary multipliers in the EM tensor defined either by Noether's theorem (which yields the canonical EM tensor) or by varying the action with respect to a background gravitational metric (the symmetric EM tensor). The arbitrary multipliers are fixed so that Heisenberg's equations of motion are reproduced for all the variables and their conjugate momenta. Additionally, it is shown that the modified EM tensors correctly generate the other (i.e., rotations, boosts) space-time symmetries of the theory. At this juncture it is important to mention that the Chern-Simons term dramatically reveals the difference between the canonical and symmetric forms of the EM tensor. Since this term is covariant without reference to the metric, it does not contribute to the symmetric EM tensor. The canonical definition, on the contrary, explicitly involves a contribution from the Chern-Simons term. This difference is precisely accounted for by the difference in the arbitrary multipliers such that the actual (modified) generators which include the first class constraints become identical. These generators are also compatible with the Poincaré algebra so that there are no problems concerning covariance of the models. It is important, however, that although both forms of the EM tensor satisfy the Poincaré algebra, the Dirac-Schwinger condition is preserved by only the symmetric EM tensor. The reason of the canonical EM tensor not satisfying this condition has been elucidated.

An immediate fallout of the present analysis is the construction of gauge-invariant multivalued operators which create the physical states with arbitrary spin. These operators may, consequently, be regarded as the anyon operators of the theory. They also satisfy the spin-statistics connection. The anyon operators found here are different from the conventional constructions [7,10,11], being gauge invariant. Indeed, any viable definition of the anyon operator must be gauge invariant so that the observed effects are physically meaningful. This lack of gauge invariance of the earlier constructions [7,10,11] was a key factor in motivating the present study [1]. Formal computations with multivalued operators which were criticized in the literature [8,9,12] have never been invoked in our theory.

The detailed analysis of the structure of the EM tensors is an ideal base for critically examining the recent claims [13,14] that the Goldstone mode in anyon superconductivity may be interpreted as the outcome of the restoration of translation invariance violated in phenomenological Lagrangians. We find such a claim to be ill founded because it is difficult to even give a proper definition for the momentum operator for such Lagrangians.

We conclude by discussing some other possibilities. The first thing is that it is straightforward to extend our analysis to include the Maxwell term. The constraint structure of this theory is such that there are only two first class constraints. In our approach, thus, quantization proceeds by Poisson brackets with appropriate modifications to the energy-momentum tensor. The results of the previous sections then follow quite logically. It is equally possible to investigate theories with broken symmetries. Moreover, our approach is suited for discussing bosonic as well as fermionic matter couplings. The explicit construction of anyonic operators in either case suggests the possibility of bosonization (fermionization) in $2+1$ dimensions [22]. We hope to return to these and other problems in the future.

APPENDIX

We prove the Dirac-Schwinger condition, which appears in Eq. (99) of the main text, for complex scalars coupled to a background potential. The relevant components of the EM tensor are given in Eqs. (96) and (100), so that

$$\{\tilde{\theta}_{00}(x),\tilde{\theta}_{00}(y)\}_{\rm PB}=\{\pi^*\pi(x)-(D_i\phi)^*(x)(D^i\phi)(x),\pi^*\pi(y)-(D_j\phi)^*(y)(D^j\phi)(y)\}_{\rm PB} .$$

Using the basic PB's

$$\{\phi(x),\phi(y)\}=\{\pi(x),\pi(y)\}=0,\quad \{\phi(x),\pi(y)\}=\delta(x-y) ,$$

we find

$$\begin{aligned}\{\tilde{\theta}_{00}(z),\tilde{\theta}_{00}(y)\}_{\rm PB}&=[-\{\pi^*(x),(D_j\phi)^*(y)\}\pi(x)D^j\phi(y)-\{(D_i\phi)^*(x),\pi^*(y)\}D^i\phi(x)\pi(y)]+\text{c.c.}\\&=\{-\pi(x)D^j\phi(y)[\partial_j\delta(x-y)+iA_j\delta(x-y)]-D^i\phi(x)\pi(y)[\partial_i\delta(x-y)-iA_i\delta(x-y)]\}+\text{c.c.}\\&=[-\pi(x)D^j\phi(x)\partial_j\delta(x-y)-\pi(y)D^j\phi(y)\partial_j\delta(x-y)+(\pi\partial_iD^i\phi-\pi\partial_jD^j\phi)\delta(x-y)]+\text{c.c.}\end{aligned}$$

Using the defining Eq. (96) for $\tilde{\theta}_{0i}$, we finally obtain

$$\{\tilde{\theta}_{00}(x),\tilde{\theta}_{00}(y)\}=[\tilde{\theta}_{0i}(x)+\tilde{\theta}_{0i}(y)]\partial_i\delta(x-y) ,$$

which is the desired condition (99).

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