

# Hamiltonian quantization of effective Lagrangians with massive vector fields

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Effective Lagrangians containing arbitrary interactions of massive vector fields are quantized within the Hamiltonian path-integral formalism. It is proven that the correct Hamiltonian quantization of these models yields the same result as naive Lagrangian quantization (Matthews's theorem). This theorem holds for models without gauge freedom as well as for (linearly or nonlinearly realized) spontaneously broken gauge theories. The Stueckelberg formalism, a procedure to rewrite effective Lagrangians in a gauge-invariant way, is reformulated within the Hamiltonian formalism as a transition from a second-class constrained theory to an equivalent first-class constrained theory. The relations between linearly and nonlinearly realized spontaneously broken gauge theories are discussed. The quartically divergent Higgs self-interaction is derived from the Hamiltonian path integral.

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## I. INTRODUCTION

Effective Lagrangians containing massive vector fields with arbitrary (non-Yang-Mills) self-interactions have been investigated very intensively in the literature (see, e.g., [1-3]) in order to parametrize possible deviations of electroweak interactions from the standard model with respect to experimental tests of the  $W^\pm$ ,  $Z$ , and  $\gamma$  self-couplings. In [1-3] it is always implicitly assumed that the Feynman rules can be directly obtained from the effective Lagrangian; i.e., the quadratic terms in the Lagrangian yield the propagators and the cubic, quartic, etc., terms yield the vertices. This simple quantization rule is known as Matthews's theorem [4]. Within the framework of the Feynman path-integral (PI) formalism (where the Feynman rules follow from the generating functional) it can be expressed as follows.

*Given a Lagrangian  $\mathcal{L}$  with arbitrary interactions of massive vector fields (among each other and with other fields), the corresponding generating functional can be written as a Lagrangian PI:*

$$Z[J] = \int \mathcal{D}\varphi \exp \left\{ i \int d^4x [\mathcal{L}_{\text{quant}}(\varphi, \partial_\mu \varphi) + J\varphi] \right\} \quad (1.1)$$

*(where  $\varphi$  is a shorthand notation for all fields in  $\mathcal{L}_{\text{quant}}$ ). If  $\mathcal{L}$  has no gauge freedom, the quantized Lagrangian  $\mathcal{L}_{\text{quant}}$  occurring in the PI is identical to the primordial one:*

$$\mathcal{L}_{\text{quant}} = \mathcal{L}. \quad (1.2)$$

*If  $\mathcal{L}$  has a gauge freedom, the generating functional (1.1) is the same as the one obtained in the Faddeev-Popov*

*(FP) formalism [5] with the quantized Lagrangian*

$$\mathcal{L}_{\text{quant}} = \mathcal{L} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP ghost}}, \quad (1.3)$$

*which contains additional gauge fixing (GF) and ghost terms.*

It is well known that, in general, quantization has to be performed within the Hamiltonian PI formalism. The naive Lagrangian PI formalism, where (1.1) with (1.2) is taken as the ansatz for the generating functional, can only be directly applied to quantize physical systems without derivative couplings and without constraints. Thus, to prove Matthews's theorem, one has to derive the Lagrangian PI (1.1) with (1.2) or (1.3) within the Hamiltonian PI formalism.

Matthews's theorem has been proven by Bernard and Duncan [6] for effective interactions of scalar fields, i.e., for models given by nonsingular effective Lagrangians. Massive vector fields, however, involve constraints. Thus, one has to take into account the formalism of quantization of constrained systems, which goes back to Dirac [7] and has been formulated within the path-integral formalism by Faddeev [8] (for first-class constrained, i.e., gauge-invariant, systems) and by Senjanovic [9] (for second-class constrained, i.e., gauge-noninvariant, systems). Recent extensive treatises on this subject can be found in [10,11]. In this paper, I will prove Matthews's theorem for effective interactions of massive vector fields taking into account this formalism.

Since it is in general not possible to find closed expressions for the velocities and the Hamiltonian in terms of the fields and the generalized momenta within an effective theory (if there are higher than second powers of  $\partial_\mu \varphi$  in the Lagrangian), Bernard and Duncan assumed that the effective interaction terms are proportional to an  $\epsilon$  with  $\epsilon \ll 1$  and proved Matthews's theorem to a finite order in  $\epsilon$ . I will proceed similarly; I will assume that the vector boson self-interactions are given by Yang-Mills interactions (which can be treated straight-

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forwardly within the Hamiltonian PI formalism [8–10] plus extra non-Yang-Mills interactions, which are proportional to a small  $\epsilon$ . In the following proof I will only consider terms which are at most first order in  $\epsilon$ , neglecting higher powers of  $\epsilon$ . This treatment is justified when dealing with phenomenologically motivated effective Lagrangians [1–3], since these are considered to investigate the effects of small deviations from the standard Yang-Mills couplings.

It will turn out that the result (1.2) or (1.3) is only correct up to additional quartically divergent terms,<sup>1</sup> i.e., terms proportional to  $\delta^4(0)$ . In [6] it is argued that these terms can be neglected, since within dimensional regularization  $\delta^4(0)$  becomes zero. In fact, how to interpret divergences higher than logarithmic within an effective (nonrenormalizable) field theory is an open question [13]. In this paper I will also neglect  $\delta^4(0)$  terms when establishing the equivalence of Hamiltonian and Lagrangian quantization. To give an example of such a term, I will derive the well-known quartically divergent Higgs self-interaction term [12,14–16] from the Hamiltonian PI.

Recently, Lagrangians have been considered which contain non-Yang-Mills self-interactions of massive vector fields within a gauge-invariant framework with spontaneously broken symmetry [2,3,12,17]. Thus, to justify the treatment of these models within the (Lagrangian) Faddeev-Popov formalism [5], I will prove Matthews's theorem also for spontaneously broken gauge theories (SBGT's). To do this, I will first consider SBGT's with a nonlinear realization of the unphysical scalar fields. Each of these models can be obtained by applying a Stueckelberg transformation [18] to a Lagrangian without gauge freedom [12,17] which is obtained by removing all unphysical scalar fields from the gauge-invariant Lagrangian and which will be shown (within the Hamiltonian formalism) to be the unitary gauge ( $U$  gauge) of the original SBGT. I will reformulate the Stueckelberg formalism [18] within the Hamiltonian formalism, thereby establishing the equivalence of (nonlinear) gauge-invariant Lagrangians and the corresponding gauge-noninvariant Lagrangians. This enables a generalization of Matthews's theorem to (nonlinearly realized) SBGT's.

*A priori* it is not clear that two Lagrangians related by a Stueckelberg transformation are equivalent, since such a transformation is not a simple point transformation because it involves derivatives of the unphysical scalar fields; however, within the Hamiltonian formalism this equivalence can be properly shown. Within this formalism no more "Stueckelberg transformation" is performed; instead, when passing from the gauge-noninvariant (second-class constrained) system to the gauge-invariant (first-class constrained) system, one enlarges the phase space [19] by introducing new (unphysical) variables and additional constraints that express the new variables in terms of the old ones. Next, one uses the

extra constraints to rewrite the Hamiltonian and the primordial constraints. Then one-half of the second-class constraints can be considered as first-class constraints and the other half as gauge-fixing conditions [20].

The proof of Matthews's theorem for SBGT's goes then as follows: Using the Stueckelberg formalism described above, I will show that the generating functional corresponding to a SBGT can be written as a Lagrangian PI with the quantized Lagrangian being identical to the  $U$ -gauge Lagrangian (i.e., the Lagrangian which is obtained by removing all unphysical scalar fields from the gauge-invariant one). This generating functional has been shown to be the result of the FP procedure [5] if the ( $U$  gauge) GF conditions that all unphysical scalar fields become equal to zero are imposed<sup>2</sup> [12]. Then I will use the equivalence of all gauges, i.e., the independence of the  $S$ -matrix elements from the choice of the gauge in the FP procedure [15,21], in order to generalize the result (1.3) to any other gauge.

Finally, I will prove Matthews's theorem for Higgs models, i.e., for SBGT's with linearly realized scalar fields. Since each Higgs model is related to a nonlinear Stueckelberg model by a simple point transformation [12,15,22], which becomes a canonical transformation within the Hamiltonian formalism and leaves the Hamiltonian PI invariant, the result for nonlinearly realized SBGT's can easily be generalized to linearly realized SBGT's. As in the nonlinear case, Matthews's theorem will first be derived for the special case of the  $U$  gauge and then be generalized to any other gauge.

My proof of Matthews's theorem will be restricted to effective Lagrangians, which do not depend on higher-order derivatives of the fields and which depend on first-order derivatives of the vector fields only through the non-Abelian field strength tensor. (The latter requirement ensures that the SBGT's corresponding to such effective Lagrangians also do not involve higher-order derivatives.) This includes the phenomenologically most important interactions [1–3].

In this paper, I will only consider massive Yang-Mills fields (of course with extra non-Yang-Mills interactions) where all vector bosons have equal masses and the corresponding SBGT's. The results can easily be generalized to any other effective Lagrangian with massive vector bosons, e.g., to electroweak models. In these cases the treatment becomes formally more complicated (in electroweak models there are extra first-class constraints due to the unbroken subgroup and extra second-class constraints due to the presence of fermions, which can, however, be treated in a standard manner) but the physically important features remain the same. Thus, for clearness

<sup>1</sup>The  $\delta^4(0)$  terms can be interpreted as the contributions of the loops of static ghost fields [12]. Thus, they do not contribute in the tree approximation.

<sup>2</sup>This is due to the fact that within this special gauge there is no GF term (because the FP  $\delta$  function, which usually serves to introduce the GF term, vanishes when performing the integration over the unphysical scalar fields in order to remove these fields from the Lagrangian) and the ghost term can be expressed as a  $\delta^4(0)$  term and thus be neglected here [12].

of representation, I will restrict here to the investigation of simple massive Yang-Mills theories.

This paper is organized as follows. In Sec. II, effective Lagrangians without gauge freedom are quantized using the Hamiltonian PI formalism and Matthews's theorem is proven for such models. In Sec. III, the Stueckelberg formalism is reformulated within the Hamiltonian formalism, the equivalence of an arbitrary effective theory without gauge freedom and the corresponding nonlinear SBGT is established and Matthews's theorem is proven for nonlinearly realized SBGT's. In Section IV, Higgs models are considered and the above proof is extended to linearly realized SBGT's. In Sec. V, the quartically divergent Higgs self-interaction term is derived from the Hamiltonian PI. Section VI is devoted to a summary of the results.

## II. MATTHEWS'S THEOREM FOR MASSIVE VECTOR FIELDS

In this section, I will quantize a massive Yang-Mills theory with additional non-Yang-Mills interactions [1], which are proportional to a parameter  $\epsilon$  (with  $\epsilon \ll 1$ ), within the Hamiltonian PI formalism and derive the simple Lagrangian form (1.1) with (1.2) of the generating functional upon neglecting terms proportional to  $\epsilon^2$  or to  $\delta^4(0)$ .

The effective Lagrangian has the form

$$\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_I = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \frac{1}{2} M^2 A_a^\mu A_\mu^a + \epsilon \mathcal{L}_I(A_\mu^a, F_{\mu\nu}^a) \quad (2.1)$$

( $a = 1, \dots, N$ ) with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf_{abc} A_\mu^b A_\nu^c. \quad (2.2)$$

For the non-Yang-Mills part of the effective interactions, given by  $\mathcal{L}_I$ , I make the following assumptions: (a)  $\mathcal{L}_I$  does not depend on higher time derivatives; (b)  $\mathcal{L}_I$  depends on first time derivatives of  $A_\mu^a$  only through the non-Abelian field strength tensor<sup>3</sup>  $F_{\mu\nu}^a$  (2.2). These conditions are fulfilled by the phenomenologically most important effective interactions, especially by all nonstandard  $P$ ,  $C$ , and  $CP$  invariant trilinear interactions of electroweak vector bosons [1].

From (2.1) one finds the momenta

$$\pi_0^a = \frac{\partial \mathcal{L}}{\partial \dot{A}_0^a} = 0, \quad (2.3)$$

$$\begin{aligned} \pi_i^a &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i^a} = F_{i0}^a + \epsilon \frac{\partial \mathcal{L}_I}{\partial \dot{A}_i^a} = \dot{A}_i^a + \partial_i A_0^a \\ &\quad - gf_{abc} A_i^b A_0^c + \epsilon \frac{\partial \mathcal{L}_I}{\partial \dot{A}_i^a}. \end{aligned} \quad (2.4)$$

Equation (2.4) can be solved for  $\dot{A}_i^a$  (to first order in  $\epsilon$ ):

$$\dot{A}_i^a = \pi_i^a - \partial_i A_0^a + gf_{abc} A_i^b A_0^c - \epsilon \left. \frac{\partial \mathcal{L}_I}{\partial \dot{A}_i^a} \right|_{F_{i0}^a \rightarrow \pi_i^a} + O(\epsilon^2). \quad (2.5)$$

The Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \pi_\mu^a \dot{A}_\mu^a - \mathcal{L} \\ &= \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a + gf_{abc} \pi_i^a A_i^b A_0^c + \frac{1}{4} F_{ij}^a F_{ij}^a \\ &\quad - \frac{1}{2} M^2 (A_0^a A_0^a - A_i^a A_i^a) - \epsilon \bar{\mathcal{L}}_I + O(\epsilon^2), \end{aligned} \quad (2.6)$$

where  $\bar{\mathcal{L}}_I$  is given by

$$\bar{\mathcal{L}}_I \equiv \mathcal{L}_I|_{F_{i0}^a \rightarrow \pi_i^a}. \quad (2.7)$$

Equation (2.3) yields the primary constraints

$$\phi_1^a = \pi_0^a = 0. \quad (2.8)$$

The secondary constraints are obtained from the requirement that the primary constraints must be consistent with the equations of motion, i.e., the relations

$$\dot{\phi}_1^a = \{\phi_1^a, H\} = 0 \quad (2.9)$$

must be satisfied. This yields

$$\phi_2^a = \partial_i \pi_i^a + gf_{abc} \pi_i^b A_i^c - M^2 A_0^a - \epsilon \frac{\partial \bar{\mathcal{L}}_I}{\partial A_0^a} + O(\epsilon^2) = 0. \quad (2.10)$$

There are no further constraints. The Poisson brackets of the primary and the secondary constraints are

$$\begin{aligned} \{\phi_1^a(\mathbf{x}), \phi_2^b(\mathbf{y})\} &= \left( M^2 \delta^{ab} + \epsilon \frac{\partial \bar{\mathcal{L}}_I}{\partial A_0^a \partial A_0^b} + O(\epsilon^2) \right) \\ &\quad \times \delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (2.11)$$

Since  $\{\phi_1^a(\mathbf{x}), \phi_1^b(\mathbf{y})\} = 0$ , one finds

$$\text{Det}^{\frac{1}{2}} \{\Phi^a, \Phi^b\} = (-1)^{N+1} \text{Det} \{\phi_1^a, \phi_2^b\} \neq 0 \quad (2.12)$$

[with  $\Phi = (\phi_1, \phi_2)$ ]. Thus, the constraints are second class. This is due to the fact that  $\mathcal{L}$  is gauge noninvariant, since the mass term and (in general) the non-Yang-Mills interactions in  $\mathcal{L}_I$  break gauge invariance explicitly.

The generating functional for a second-class constrained system is generally given by [9–11]

<sup>3</sup>Actually, this requirement is equivalent to the demand that  $\mathcal{L}_I$  does not depend on  $A_0^a$ .

$$Z[J] = \int \prod_{\mu, \alpha} \mathcal{D}A_\mu^\alpha \mathcal{D}\pi_\mu^\alpha \exp \left\{ i \int d^4x [\pi_\mu^\alpha \dot{A}_\mu^\alpha - \mathcal{H} + J_\mu^\alpha A_\mu^\alpha] \right\} \prod_\alpha [\delta(\phi_1^\alpha) \delta(\phi_2^\alpha)] \text{Det}^{\frac{1}{2}} [\{\Phi^a(\mathbf{x}), \Phi^b(\mathbf{y})\} \delta(x^0 - y^0)]. \quad (2.13)$$

The determinant in (2.13) only yields  $\delta^4(0)$  terms which are neglected here. This can easily be seen from (2.11) and (2.12) by use of the identity<sup>4</sup> [23]

$$\text{Det} [M_{ab}(x) \delta^4(x - y)] = \exp \left\{ \delta^4(0) \int d^4x \ln[\det M_{ab}(x)] \right\} \quad (2.14)$$

(where “Det” expresses the functional determinant and “det” the ordinary one). Dropping the determinant, integrating out the  $\pi_0^\alpha$  due to the presence of  $\prod_\alpha \delta(\pi_0^\alpha)$  in (2.13), and using the relation

$$\prod_\alpha \delta(\phi_2^\alpha) \propto \int \prod_\alpha \mathcal{D}\lambda^\alpha \exp \left\{ -i \int d^4x \lambda^\alpha \phi_2^\alpha \right\}, \quad (2.15)$$

one finds

$$\begin{aligned} Z[J] = \int \prod_{\mu, \alpha} \mathcal{D}A_\mu^\alpha \prod_{i, \alpha} \mathcal{D}\pi_i^\alpha \prod_\alpha \mathcal{D}\lambda^\alpha \exp \left\{ i \int d^4x \left[ -\frac{1}{2} \pi_i^\alpha \pi_i^\alpha + \pi_i^\alpha [\dot{A}_i^\alpha + \partial_i(A_0^\alpha + \lambda^\alpha) - g f_{abc} A_i^b (A_0^c + \lambda^c)] - \frac{1}{4} F_{ij}^\alpha F_{ij}^\alpha \right. \right. \\ \left. \left. + \frac{1}{2} M^2 [(A_0^\alpha + \lambda^\alpha)(A_0^\alpha + \lambda^\alpha) - \lambda^\alpha \lambda^\alpha - A_i^\alpha A_i^\alpha] \right. \right. \\ \left. \left. + \epsilon \left( \bar{\mathcal{L}}_I + \lambda^\alpha \frac{\partial \bar{\mathcal{L}}_I}{\partial A_0^\alpha} \right) + O(\epsilon^2) + J_\mu^\alpha A_\mu^\alpha \right] \right\}. \quad (2.16) \end{aligned}$$

The substitution<sup>5</sup>

$$A_0^\alpha \rightarrow A_0^\alpha - \lambda^\alpha, \quad (2.17)$$

which obviously leaves the functional integration measure invariant, yields

$$\begin{aligned} Z[J] = \int \prod_{\mu, \alpha} \mathcal{D}A_\mu^\alpha \prod_{i, \alpha} \mathcal{D}\pi_i^\alpha \prod_\alpha \mathcal{D}\lambda^\alpha \exp \left\{ i \int d^4x \left[ -\frac{1}{2} \pi_i^\alpha \pi_i^\alpha + \pi_i^\alpha F_{i0}^\alpha - \frac{1}{4} F_{ij}^\alpha F_{ij}^\alpha \right. \right. \\ \left. \left. + \frac{1}{2} M^2 (A_0^\alpha A_0^\alpha - \lambda^\alpha \lambda^\alpha - A_i^\alpha A_i^\alpha) \right. \right. \\ \left. \left. - \tilde{\mathcal{H}}_I(A_\mu^\alpha, \partial_i A_\mu^\alpha, \pi_i^\alpha, \lambda^\alpha) + J_\mu^\alpha A_\mu^\alpha \right] \right\} \quad (2.18) \end{aligned}$$

with

$$\tilde{\mathcal{H}}_I(A_\mu^\alpha, \partial_i A_\mu^\alpha, \pi_i^\alpha, \lambda^\alpha) \equiv -\epsilon \left( \bar{\mathcal{L}}_I + \lambda^\alpha \frac{\partial \bar{\mathcal{L}}_I}{\partial A_0^\alpha} \right) \Big|_{A_0^\alpha \rightarrow A_0^\alpha - \lambda^\alpha} + O(\epsilon^2). \quad (2.19)$$

Now the procedure of Bernard and Duncan [6] can be generalized to the model considered here. Introducing sources  $K_i^\alpha$  and  $K_\lambda^\alpha$  coupled to  $\pi_i^\alpha$  and  $\lambda^\alpha$ , one can rewrite (2.18) as

$$\begin{aligned} Z[J] = \int \prod_{\mu, \alpha} \mathcal{D}A_\mu^\alpha \exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I \left( A_\mu^\alpha, \partial_i A_\mu^\alpha, \frac{\delta}{i\delta K_i^\alpha}, \frac{\delta}{i\delta K_\lambda^\alpha} \right) \right\} \\ \times \int \prod_{i, \alpha} \mathcal{D}\pi_i^\alpha \prod_\alpha \mathcal{D}\lambda^\alpha \exp \left\{ i \int d^4x \left[ -\frac{1}{2} \pi_i^\alpha \pi_i^\alpha + \pi_i^\alpha F_{i0}^\alpha - \frac{1}{4} F_{ij}^\alpha F_{ij}^\alpha \right. \right. \\ \left. \left. + \frac{1}{2} M^2 (A_0^\alpha A_0^\alpha - \lambda^\alpha \lambda^\alpha - A_i^\alpha A_i^\alpha) + J_\mu^\alpha A_\mu^\alpha + \pi_i^\alpha K_i^\alpha + \lambda^\alpha K_\lambda^\alpha \right] \right\} \Big|_{K_i^\alpha = K_\lambda^\alpha = 0}. \quad (2.20) \end{aligned}$$

<sup>4</sup>Another way to see this is to rewrite the determinant as a functional integral over Grassmann variables which yields a ghost term  $\mathcal{L}_{\text{ghost}} = -M^2 \eta_a^* \eta_a - \epsilon \eta_a^* \frac{\partial \bar{\mathcal{L}}_I}{\partial A_0^\alpha} \eta_b + O(\epsilon^2)$ . The ghost fields are static, i.e. there are no kinetic terms for them, only mass terms and couplings to the  $A_\mu^\alpha$  fields. This means all ghost propagators are simply inverse masses and thus all ghost loops are quartically divergent. Thus, the ghost term can be replaced by a  $\delta^4(0)$  term which yields the same contribution to matrix elements as the ghost loops [12].

<sup>5</sup>After this substitution the source  $J_0^\alpha$  becomes coupled to  $A_0^\alpha - \lambda^\alpha$  instead of  $A_0^\alpha$ . However it does not affect physical matrix elements to remove the coupling of  $\lambda^\alpha$  to  $J_0^\alpha$  [21].

Performing the Gaussian intergrations over  $\pi_i^a$  and  $\lambda^a$  one gets

$$\begin{aligned} Z[J] &= \int \prod_{\mu,a} \mathcal{D}A_\mu^a \exp \left\{ i \int d^4x [\mathcal{L}_0 + J_\mu^a A_\mu^a] \right\} \\ &\quad \times \exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I \left( A_\mu^a, \partial_i A_\mu^a, \frac{\delta}{i\delta K_i^a}, \frac{\delta}{i\delta K_\lambda^a} \right) \right\} \\ &\quad \times \exp \left\{ i \int d^4x \left[ \frac{1}{2} K_i^a K_i^a + \frac{1}{2} K_\lambda^a K_\lambda^a + K_i^a F_{i0}^a \right] \right\} \Big|_{K_i^a = K_\lambda^a = 0}, \end{aligned} \tag{2.21}$$

where  $\mathcal{L}_0$  is the massive Yang-Mills part of the effective Lagrangian (2.1). The use of the functional identity [24]

$$F \left[ \frac{\delta}{i\delta K} \right] G[K] \Big|_{K=0} = G \left[ \frac{\delta}{i\delta \rho} \right] F[\rho] \Big|_{\rho=0} \tag{2.22}$$

yields

$$\begin{aligned} Z[J] &= \int \prod_{\mu,a} \mathcal{D}A_\mu^a \exp \left\{ i \int d^4x [\mathcal{L}_0 + J_\mu^a A_\mu^a] \right\} \\ &\quad \times \exp \left\{ \int d^4x \left[ -\frac{i}{2} \sum_{i,a} \left( \frac{\delta}{\delta \rho_i^a} \right)^2 - \frac{i}{2} \sum_a \left( \frac{\delta}{\delta \rho_\lambda^a} \right)^2 + F_{i0}^a \left( \frac{\delta}{\delta \rho_i^a} \right) \right] \right\} \\ &\quad \times \exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I (A_\mu^a, \partial_i A_\mu^a, \rho_i^a, \rho_\lambda^a) \right\} \Big|_{\rho_i^a = \rho_\lambda^a = 0}. \end{aligned} \tag{2.23}$$

Since  $\tilde{\mathcal{H}}_I$  (2.19) is proportional to  $\epsilon$ , the third exponential in (2.23) can be expanded in powers of  $\epsilon$ :

$$\exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I (A_\mu^a, \partial_i A_\mu^a, \rho_i^a, \rho_\lambda^a) \right\} = 1 - i \int d^4x \tilde{\mathcal{H}}_I (A_\mu^a, \partial_i A_\mu^a, \rho_i^a, \rho_\lambda^a) + O(\epsilon^2). \tag{2.24}$$

Obviously, second-order functional derivatives with respect to the  $\rho$ 's acting on this expression yield terms which are proportional to  $\epsilon^2$  or to  $\delta^4(0)$  and which are both neglected here. Thus, the second-order derivatives in the second exponential in (2.23) can be omitted. The second and the third exponential in (2.23) together reduce to

$$\begin{aligned} &\exp \left\{ \int d^4x F_{i0}^a \left( \frac{\delta}{\delta \rho_i^a} \right) \right\} \exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I (A_\mu^a, \partial_i A_\mu^a, \rho_i^a, \rho_\lambda^a) \right\} \Big|_{\rho_i^a = \rho_\lambda^a = 0} \\ &= \exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I (A_\mu^a, \partial_i A_\mu^a, \rho_i^a, \rho_\lambda^a) \right\} \Big|_{\substack{\rho_i^a = F_{i0}^a \\ \rho_\lambda^a = 0}}. \end{aligned} \tag{2.25}$$

With the definitions of  $\tilde{\mathcal{H}}_i$  (2.19) and of  $\tilde{\mathcal{L}}_I$  (2.7) one finds

$$\tilde{\mathcal{H}}_I (A_\mu^a, \partial_i A_\mu^a, \rho_i^a, \rho_\lambda^a) \Big|_{\substack{\rho_i^a = F_{i0}^a \\ \rho_\lambda^a = 0}} = -\epsilon \mathcal{L}_I (A_\mu^a, F_a^{\mu\nu}) + O(\epsilon^2). \tag{2.26}$$

The insertion of (2.25) with (2.26) into (2.23) yields [apart from the  $\epsilon^2$  and  $\delta^4(0)$  terms]

$$Z[J] = \int \prod_{\mu,a} \mathcal{D}A_\mu^a \exp \left\{ i \int d^4x [\mathcal{L}_0 + \epsilon \mathcal{L}_I + J_\mu^a A_\mu^a] \right\}, \tag{2.27}$$

which is the expected result, namely, the naive Lagrangian path integral (1.1) with (1.2). Thus, Matthews's theorem is proven for effective self-interactions of massive vector fields (within a gauge-

noninvariant framework).

This proof can easily be generalized to Lagrangians which also contain effective interactions of the massive vector fields with other fields (scalar, fermion, or additional vector fields). To derive this result, one adds in (2.1) the kinetic and mass terms of the extra fields as well as the couplings without derivatives to  $\mathcal{L}_0$  and the derivative couplings to  $\mathcal{L}_I$  and then goes through the same procedure as above. Thus, Matthews's theorem also holds for effective vector-fermion and vector-scalar interactions.<sup>6</sup>

<sup>6</sup>An application of this result, which will become important in Sec. IV, is to consider  $\mathcal{L}_0$  as the  $U$  gauge Lagrangian of a (minimal) Higgs model, while  $\mathcal{L}_I$  contains additional effective interactions of the vector and Higgs fields.

### III. THE STUECKELBERG FORMALISM

In this section, I will generalize Matthews's theorem to SBGT's with nonlinearly realized symmetry which contain arbitrary gauge boson self-interactions within a gauge-invariant framework [2,12,17]. It has been shown in [12,17] that each theory given by an effective Lagrangian of the type (2.1) can be rewritten as a (nonlinearly realized) SBGT by applying Stueckelberg transformations [18]. On the other hand, each nonlinear SBGT (without higher derivatives) can be obtained by applying a Stueckelberg transformation to a Lagrangian of type (2.1). Thus, I will reformulate the Stueckelberg formalism within the Hamiltonian formalism in order to show the equivalence of effective Lagrangians which are related by Stueckelberg transformations.

The Stueckelberg formalism can be most easily formulated within the matrix notation. With  $t_a$  being the generators of the gauge group, which are orthonormalized due to

$$\text{tr}(t_a, t_b) = \frac{1}{2} \delta_{ab}, \quad (3.1)$$

one defines

$$A_\mu \equiv A_\mu^\alpha t_a, \quad (3.2)$$

$$\varphi \equiv i \frac{g}{M} \varphi^\alpha t_a, \quad (3.3)$$

$$U \equiv \exp(\varphi). \quad (3.4)$$

The  $\varphi^\alpha$  are the unphysical pseudo Goldstone scalars. The Stueckelberg transformation is defined as

$$\begin{aligned} A_\mu &\rightarrow -\frac{i}{g} U^\dagger D_\mu U = U^\dagger A_\mu U - \frac{i}{g} U^\dagger \partial_\mu U \\ &= U^\dagger A_\mu U + \frac{1}{M} \partial_\mu \varphi^\alpha U^\dagger Q_\alpha \end{aligned} \quad (3.5)$$

( $D_\mu U$  is the covariant derivative of  $U$ ) with

$$\begin{aligned} Q_a &\equiv \left( t_a + (\varphi t_a + t_a \varphi) \right. \\ &\quad \left. + \frac{1}{2!} (\varphi^2 t_a + \varphi t_a \varphi + t_a \varphi^2) + \dots \right). \end{aligned} \quad (3.6)$$

The Stueckelberg transformation (3.5) *formally* acts like a gauge transformation, however, with the gauge parameters being replaced by the pseudo Goldstone fields. Thus, it effects only the mass term and the effective interaction term  $\mathcal{L}_I$  in (2.1) but not the gauge-invariant Yang-Mills term  $-\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a$ . Equation (3.5) can be written in components by multiplying with  $2t_a$  and taking the trace. With (3.1) and (3.2) one finds

$$A_\mu^\alpha \rightarrow X_{ab} A_\mu^b + \frac{1}{M} Y_{ab} \partial_\mu \varphi^b, \quad (3.7)$$

where the matrices  $X$  and  $Y$  are defined as

$$X_{ab} \equiv 2\text{tr}(U^\dagger t_b U t_a), \quad (3.8)$$

$$Y_{ab} \equiv 2\text{tr}(U^\dagger Q_b t_a). \quad (3.9)$$

$X$  and  $Y$  are nonpolynomial expressions in the pseudo Goldstone fields  $\varphi^\alpha$ . They do not depend on the derivatives  $\partial_\mu \varphi^\alpha$  and, due to (3.1), they become unity matrices for vanishing  $\varphi^\alpha$ :

$$X_{ab}(\varphi^\alpha = 0) = Y_{ab}(\varphi^\alpha = 0) = \delta_{ab}. \quad (3.10)$$

The SBGT corresponding to the effective Lagrangian (2.1) is

$$\mathcal{L}^S \equiv \mathcal{L} \Big|_{A_\mu \rightarrow U^\dagger A_\mu U - \frac{i}{g} U^\dagger \partial_\mu U}. \quad (3.11)$$

$\mathcal{L}$  can be recovered from  $\mathcal{L}^S$  simply by removing all unphysical scalar fields in  $\mathcal{L}^S$ :

$$\mathcal{L} = \mathcal{L}^S \Big|_{\varphi_a = 0}. \quad (3.12)$$

The non-Yang-Mills part of the effective interactions is given by the gauge invariant term  $\mathcal{L}_I^S$ , which is obtained by applying (3.5) to  $\mathcal{L}_I$ .  $\mathcal{L}^S$  describes a generalized gauged nonlinear  $\sigma$  model with extra non-Yang-Mills vector boson self-interactions [12]. Each nonlinearly realized effective SBGT (without higher derivatives) given by a Lagrangian  $\mathcal{L}^S$  can be constructed by applying (3.5) to an effective Lagrangian  $\mathcal{L}$  (2.1), which is obtained by removing the pseudo Goldstone fields in  $\mathcal{L}^S$ . I will prove that the Lagrangians  $\mathcal{L}$  and  $\mathcal{L}^S$  are equivalent within the Hamiltonian formalism.<sup>7</sup> This is not obvious because the Stueckelberg transformation (3.5) involves derivatives of the pseudo Goldstone fields and from the Lagrangian point of view one can only argue that two Lagrangians which are related by a point transformation (i.e., a transformation which does not involve derivatives) are equivalent. I will show within the Hamiltonian formalism that  $\mathcal{L}$  is the  $U$  gauge of  $\mathcal{L}^S$ ; i.e., the  $U$  gauge of a nonlinear effective SBGT is simply obtained by dropping all unphysical scalar fields (as one naively expects).

One can easily see that, if  $\mathcal{L}$  satisfies the conditions required at the beginning of Sec. II,  $\mathcal{L}^S$  also satisfies these demands, since the field strength tensor  $F_{\mu\nu} = F_{\mu\nu}^\alpha t_a$  transforms under Stueckelberg transformations due to

$$F_{\mu\nu} \rightarrow U^\dagger F_{\mu\nu} U \quad (3.13)$$

or, written in components,

$$F_{\mu\nu}^\alpha \rightarrow X_{ab} F_{\mu\nu}^b. \quad (3.14)$$

For the subsequent treatment it is convenient to rewrite (3.11) as

$$\mathcal{L}^S = \mathcal{L} \Big|_{\substack{A_\mu \rightarrow U^\dagger A_\mu U - \frac{i}{g} U^\dagger \partial_\mu U \\ F_{\mu\nu} \rightarrow U^\dagger F_{\mu\nu} U}} = \mathcal{L} \Big|_{\substack{A_0^\alpha \rightarrow X_{ab} A_0^b + \frac{1}{M} Y_{ab} \varphi^b \\ A_i^\alpha \rightarrow X_{ab} A_i^b + \frac{1}{M} Y_{ab} \partial_i \varphi^b \\ F_{i0}^\alpha \rightarrow X_{ab} F_{i0}^b \\ F_{ij}^\alpha \rightarrow X_{ab} F_{ij}^b}}, \quad (3.15)$$

<sup>7</sup>For simple gauged nonlinear  $\sigma$  models without effective interactions this equivalence has been shown within the Hamiltonian formalism in [25] (in a very formal way).

where the following convention has been used: While in (3.11) the Stueckelberg transformation is applied to  $A_\mu$  *everywhere* in  $\mathcal{L}$  [which automatically implies the transformation of  $F_{\mu\nu}$  (3.13)] it is in (3.15) only applied to the  $A_\mu$  field where it does not occur as a part of the field strength tensor  $F_{\mu\nu}$ , and  $F_{\mu\nu}$  becomes then transformed separately. I will use this convention throughout this section.

The momenta conjugate to the fields in  $\mathcal{L}^S$  are

$$\pi_0^a = \frac{\partial \mathcal{L}^S}{\partial \dot{A}_0^a} = 0, \quad (3.16)$$

$$\dot{A}_a^i = \pi_i^a - \partial_i A_0^a + g f_{abc} A_i^b A_0^c - \epsilon \frac{\partial \mathcal{L}_I^S}{\partial \dot{A}_a^i} \Big|_{\substack{F_{i0}^a \rightarrow \pi_i^a \\ \dot{\varphi}^a \rightarrow Y_{ab}^{-1}(Y_{cb}^{-1} \pi_\varphi^c - M X_{bc} A_0^c)}} + O(\epsilon^2), \quad (3.19)$$

$$\dot{\varphi}^a = Y_{ab}^{-1} \left[ Y_{cb}^{-1} \left( \pi_\varphi^c - \epsilon \frac{\partial \mathcal{L}_I^S}{\partial \dot{\varphi}^c} \Big|_{\substack{F_{i0}^a \rightarrow \pi_i^a \\ \dot{\varphi}^a \rightarrow Y_{ab}^{-1}(Y_{cb}^{-1} \pi_\varphi^c - M X_{bc} A_0^c)}} \right) - M X_{bc} A_0^c \right] + O(\epsilon^2) \quad (3.20)$$

and the Hamiltonian

$$\begin{aligned} \mathcal{H}^S &= \pi_\mu^a \dot{A}_a^\mu + \pi_\varphi^a \dot{\varphi}^a - \mathcal{L}^S \\ &= \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a + g f_{abc} \pi_i^a A_i^b A_0^c + \frac{1}{4} F_{ij}^a F_{ij}^a \\ &\quad - \frac{1}{2} M^2 \left[ \sum_a (X_{ab} A_0^b)^2 \right. \\ &\quad \left. - \sum_a \left( X_{ab} A_i^b + \frac{1}{M} Y_{ab} \partial_i \varphi^b \right)^2 \right] \\ &\quad + \frac{1}{2} \sum_a (Y_{ba}^{-1} \pi_\varphi^b - M X_{ab} A_0^b)^2 - \epsilon \bar{\mathcal{L}}_I^S + O(\epsilon^2). \end{aligned} \quad (3.21)$$

$\bar{\mathcal{L}}_I^S$  is given by

$$\begin{aligned} \bar{\mathcal{L}}_I^S &\equiv \mathcal{L}_I^S \Big|_{\substack{F_{i0}^a \rightarrow \pi_i^a \\ \dot{\varphi}^a \rightarrow Y_{ab}^{-1}(Y_{cb}^{-1} \pi_\varphi^c - M X_{bc} A_0^c)}} \\ &= \mathcal{L}_I \Big|_{\substack{A_0^a \rightarrow \frac{1}{M} Y_{ba}^{-1} \pi_\varphi^b \\ A_i^a \rightarrow X_{ab} A_i^b + \frac{1}{M} Y_{ab} \partial_i \varphi^b \\ F_{i0}^a \rightarrow X_{ab} \pi_i^b \\ F_{ij}^a \rightarrow X_{ab} F_{ij}^b}} \end{aligned} \quad (3.22)$$

[where (3.15) has been used]. The primary constraints are

$$\phi_1^a = \pi_0^a = 0 \quad (3.23)$$

and the secondary constraints, obtained analogously to (2.9), are

$$\phi_2^a = \partial_i \pi_i^a + g f_{abc} \pi_i^b A_i^c - M X_{ba} Y_{cb}^{-1} \pi_\varphi^c = 0. \quad (3.24)$$

$$\begin{aligned} \pi_i^a &= \frac{\partial \mathcal{L}^S}{\partial \dot{A}_a^i} = F_{i0}^a + \epsilon \frac{\partial \mathcal{L}_I^S}{\partial \dot{A}_a^i} = \dot{A}_a^i + \partial_i A_0^a \\ &\quad - g f_{abc} A_i^b A_0^c + \epsilon \frac{\partial \mathcal{L}_I^S}{\partial \dot{A}_a^i}, \end{aligned} \quad (3.17)$$

$$\pi_\varphi^a = \frac{\partial \mathcal{L}^S}{\partial \dot{\varphi}^a} = M Y_{ca} \left( X_{cb} A_0^b + \frac{1}{M} Y_{cb} \dot{\varphi}^b \right) + \epsilon \frac{\partial \mathcal{L}_I^S}{\partial \dot{\varphi}^a}. \quad (3.18)$$

To first order in  $\epsilon$  one finds the velocities

There are no terms proportional to  $\epsilon$  in (3.24), since, due to (3.22),  $\bar{\mathcal{L}}_I^S$  does not depend on  $A_0^a$  (either directly or through  $F_{i0}^a$ ).<sup>8</sup> There are no further constraints. The constraints are first class due to the gauge freedom of  $\mathcal{L}^S$  (3.11).

Since in first-class constrained systems the solutions of the equations of motion contain undetermined Lagrange multipliers, one has to remove this ambiguity by imposing additional gauge fixing (GF) conditions [8,10,11], such that the number of GF conditions is equal to the number of first-class constraints. Constraints and GF conditions together have to form a system of second-class constraints being consistent with the equations of motion. A convenient way to construct these conditions [10] is to start with primary GF conditions  $\chi_1^a$  and to construct secondary GF conditions by demanding

$$\{\chi_1^a, H^S\} = 0, \quad (3.25)$$

which ensures consistency with the equations of motion. To prove the equivalence of  $\mathcal{L}$  (2.1) and  $\mathcal{L}^S$  (3.11) it is most convenient to construct the  $U$  gauge by imposing the primary GF conditions

$$\chi_1^a = \varphi^a = 0. \quad (3.26)$$

Equation (3.25) yields then the secondary GF conditions<sup>9</sup>

$$\chi_2^a = Y_{ab}^{-1} (Y_{cb}^{-1} \pi_\varphi^c - M X_{bc} A_0^c) - \epsilon \frac{\partial \bar{\mathcal{L}}_I^S}{\partial \pi_\varphi^a} + O(\epsilon^2) = 0. \quad (3.27)$$

<sup>8</sup>In fact, to all orders in  $\epsilon$ ,  $A_0^a$  becomes replaced by  $\frac{1}{M} Y_{ba}^{-1} \pi_\varphi^b$  and  $F_{i0}^a$  by  $X_{ab} \pi_i^b$ . Thus,  $\mathcal{H}_I^S$  does not depend on  $A_0^a$  and (3.24) holds exactly.

<sup>9</sup>The GF conditions do not fulfil Faddeev's requirement  $\{\chi^a, \chi^b\} = 0$  [8]. In fact, this restriction is unnecessary [10,11,26].

Using the primary GF conditions<sup>10</sup> (3.26), the relation (3.10) and the definitions of  $\tilde{\mathcal{L}}_I^S$  (3.22) and  $\tilde{\mathcal{L}}_I$  (2.7) one can express the Hamiltonian (3.21), the secondary constraints (3.24), and GF conditions (3.27) as

$$\begin{aligned} \mathcal{H}^S = & \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a + g f_{abc} \pi_i^a A_i^b A_0^c + \frac{1}{4} F_{ij}^a F_{ij}^a \\ & - \frac{1}{2} M^2 (A_0^a A_0^a - A_i^a A_i^a) + \frac{1}{2} \sum_a (\pi_\varphi^a - M A_0^a)^2 \\ & - \epsilon \tilde{\mathcal{L}}_I + O(\epsilon^2), \end{aligned} \quad (3.28)$$

$$\phi_2^a = \partial_i \pi_i^a + g f_{abc} \pi_i^b A_i^c - M \pi_\varphi^a = 0, \quad (3.29)$$

$$\chi_2^a = \pi_\varphi^a - M A_0^a - \epsilon \frac{\partial \tilde{\mathcal{L}}_I}{\partial \pi_\varphi^a} + O(\epsilon^2) = 0 \quad (3.30)$$

with

$$\tilde{\mathcal{L}}_I \equiv \tilde{\mathcal{L}}_I^S \Big|_{\varphi_a=0} = \tilde{\mathcal{L}}_I \Big|_{A_0^a \rightarrow \frac{1}{M} \pi_\varphi^a}. \quad (3.31)$$

Equation (3.30) can be inserted into itself and becomes

$$\chi_2^a = \pi_\varphi^a - M A_0^a - \epsilon \frac{\partial \tilde{\mathcal{L}}_I}{\partial A_0^a} + O(\epsilon^2) = 0. \quad (3.32)$$

Applying the secondary GF condition (3.32), one can rewrite the Hamiltonian (3.28) as (2.6) and the constraints (3.23), (3.29) as (2.8), (2.10) (to first order in  $\epsilon$ ), i.e., as the Hamiltonian and the constraints corresponding to the gauge-noninvariant Lagrangian  $\mathcal{L}$  (2.1). Finally, the GF conditions (3.26) and (3.32) can be omitted, since they involve the fields  $\varphi^a$  and  $\pi_\varphi^a$  and neither the Hamiltonian nor the constraints depend on these fields anymore. Thus, the Lagrangians  $\mathcal{L}$  and  $\mathcal{L}^S$  in (3.11) describe equivalent physical systems,  $\mathcal{L}$  being the  $U$  gauge of  $\mathcal{L}^S$ .

Because of the equality of the Hamiltonians and the constraints corresponding to  $\mathcal{L}^S$  and  $\mathcal{L}$ ,  $\mathcal{L}^S$  can be quantized as described in the previous section; the generating functional turns out to be (2.27). This, however, is identical [apart from  $\delta^4(0)$  terms, which are neglected here] to the generating functional obtained in the (Lagrangian) Faddeev-Popov formalism [5] if one imposes the ( $U$  gauge) GF conditions (3.26) [12]. Because of the equivalence of all gauges [15,21], (2.27) yields the same  $S$ -matrix elements as the Faddeev-Popov PI corresponding to  $\mathcal{L}^S$  in any other gauge (e.g.,  $R_\xi$  gauge, Lorentz gauge, Coulomb gauge). Thus, Matthews's theorem is also valid for SGBT's with nonlinearly realized symmetry.

It should be noted that this result cannot be *directly* obtained within the  $R_\xi$  gauge (which is most adequate for loop calculations) because the  $R_\xi$  gauge GF condition

$$\chi^a = \partial^\mu A_\mu^a - \xi M \varphi^a = 0 \quad (3.33)$$

is *not* a GF condition within the Hamiltonian formalism. This is because (3.33) cannot be written as a relation among the fields and the conjugate fields alone, due to the presence of the velocity  $A_0^a$  in  $\chi^a$  that is not expressible through the momenta [remember the constraint (3.23)]. Therefore, when quantizing  $\mathcal{L}^S$  within the  $R_\xi$  gauge, one has to proceed *indirectly* as outlined above by first constructing the  $U$  gauge and then using the equivalence of all gauges (i.e., the invariance of the  $S$ -matrix element under a change of the gauge in the FP procedure<sup>11</sup>) [15,21] in order to rewrite this result in the  $R_\xi$  gauge.

The procedure outlined in this section shows how to interpret the Stueckelberg formalism on the Hamiltonian level. While the gauge-noninvariant Lagrangian  $\mathcal{L}$  is related to the gauge-invariant Lagrangian  $\mathcal{L}^S$  by a Stueckelberg transformation (3.5), one can pass from the second-class constrained Hamiltonian  $\mathcal{H}$  to the first-class constrained Hamiltonian  $\mathcal{H}^S$  by the following procedure: One enlarges the phase space by introducing the unphysical variables  $\varphi^a$  and  $\pi_\varphi^a$  and the extra constraints (3.26) and (3.27), which make the new variables dependent on the others and leaves the number of physical degrees of freedom unchanged. Next, one rewrites, using the additional constraints (3.26) and (3.27), the Hamiltonian as (3.21) and the primordial constraints as (3.23) and (3.24). Finally, half of the constraints, namely the new ones, are considered as GF conditions.<sup>12</sup>

#### IV. HIGGS MODELS

Finally, Matthews's theorem has to be proven for SGBT's with linearly realized symmetry, i.e., Higgs models, which contain effective (non-Yang-Mills) gauge boson self-interactions [3,12]. This result will simply be ob-

<sup>11</sup>Actually, loop calculations within the  $U$  gauge suffer from ambiguities in the determination of the finite part of an  $S$ -matrix element [27,28]. Therefore, for practical calculations it is useful to rewrite the PI obtained within in the  $U$  gauge in the  $R_\xi$  gauged form in order to remove these ambiguities. In fact, loop calculations within the  $U$  gauge yield the same  $S$ -matrix elements as loop calculations within the  $R_\xi$  gauge, but only if the correct renormalization prescription is used [28]; other renormalization prescriptions yield distinct results. In the  $R_\xi$  gauge these ambiguities are absent.

<sup>12</sup>A similar transition from a second-class constrained system to a first-class constrained system has recently been investigated in several works [20]. However, there no phase space enlargement is performed with the outcome that the resulting model contains only half as many first-class constraints as the original model has second-class constraints. In my treatment the number of constraints remains unchanged, since, due to the phase space enlargement, new constraints are introduced. The method of connecting first- and second-class constrained systems by performing a phase space enlargement goes back to [19].

<sup>10</sup>The insertion of the GF conditions into the Hamiltonian and the other constraints corresponds de facto to a redefinition of the Lagrange multipliers in the total Hamiltonian, i.e., the Hamiltonian from which follow the equations of motion  $\mathcal{H}_T^S = \mathcal{H}^S + \lambda_a \Phi_a + \tilde{\lambda}_a \chi_a$  (where  $\Phi_a$  stands for all constraints,  $\chi_a$  for all GF conditions and the  $\lambda_a$  and  $\tilde{\lambda}_a$  are the Lagrange multipliers). In the Hamiltonian PI (2.13) this insertion is justified due to the presence of the  $\delta$  functions.



tained by showing the equivalence of a linear Higgs model to a nonlinear Stueckelberg model [with (an) additional physical scalar(s)].

Since the Higgs model corresponding to a massive Yang-Mills theory cannot be written in a general form for an arbitrary gauge group, I restrict to the case of SU(2) symmetry (i.e.,  $t_a = \frac{1}{2}\tau_a$ ,  $a = 1, 2, 3$ ). The extension to other gauge groups is straightforward.

Any effective Lagrangian (2.1) can be extended to a Higgs model by constructing the Stueckelberg Lagrangian (3.11) and then introducing a physical scalar field  $h$  and linearizing the scalar sector of the theory [12] via

$$\frac{v}{\sqrt{2}}U \equiv \frac{v}{\sqrt{2}} \exp\left(\frac{i\varphi_a\tau_a}{v}\right) \rightarrow \Phi \equiv \frac{1}{\sqrt{2}}[(v+h)\mathbf{1} + i\varphi_a\tau_a], \quad (4.1)$$

where  $v$  is the vacuum expectation value of the Higgs field,  $v = \frac{2M}{g}$ . The Lagrangian of the Higgs model corresponding to (2.1) becomes

$$\mathcal{L}^H = \mathcal{L}^S \Big|_{\frac{v}{\sqrt{2}}U \rightarrow \Phi} - V(\Phi) \quad (4.2)$$

with the Higgs self-interaction potential  $V(\Phi)$  which yields the nonvanishing vacuum expectation value. In distinction from  $\mathcal{L}^S$  (3.11),  $\mathcal{L}^H$  is *not* equivalent to the effective Lagrangian  $\mathcal{L}$ , since there is an additional physical degree of freedom. However,  $\mathcal{L}^H$  contains the same effective vector boson self-interactions as  $\mathcal{L}$  and  $\mathcal{L}^S$ . In fact,  $\mathcal{L}^S$  is the limit of  $\mathcal{L}^H$  for infinite Higgs boson mass [2,12]. Each effective Higgs model (without higher derivatives) can be constructed this way from a Lagrangian of type (2.1).

To extend the results of the previous two sections to the Lagrangian (4.2), one uses the fact that even within a linearly realized SBGT the scalar fields can be parametrized nonlinearly [12,15,22] by the point transformation

$$\Phi \rightarrow \frac{v+h}{\sqrt{2}}U \quad (4.3)$$

[with  $U$  and  $\Phi$  given by (4.1)]. The Lagrangian of the Higgs model in which the scalar sector is nonlinearly realized,

$$\mathcal{L}^{H,S} \equiv \mathcal{L}^H \Big|_{\Phi \rightarrow \frac{v+h}{\sqrt{2}}U}, \quad (4.4)$$

describes a Stueckelberg model with one additional physical scalar  $h$ . Thus (remembering the last paragraph of Sec. II), the results of the previous two sections can be used to quantize  $\mathcal{L}^{H,S}$ ; the generating functional takes the Lagrangian form (1.1) with the quantized Lagrangian

$$\mathcal{L}_{\text{quant}} = \mathcal{L}_U^H \equiv \mathcal{L}^{H,S} \Big|_{\varphi_a=0} = \mathcal{L}^H \Big|_{\varphi_a=0}. \quad (4.5)$$

It is now easy to establish the equivalence between  $\mathcal{L}^H$  and  $\mathcal{L}^{H,S}$  since a point transformation (i.e., a transformation which does not involve derivatives) such as (4.3) be-

comes a canonical transformation within the Hamiltonian formalism; i.e., the Hamiltonians and also the constraints corresponding to  $\mathcal{L}^H$  and  $\mathcal{L}^{H,S}$  are related by canonical transformations.<sup>13</sup> Thus, the physical systems described by both Lagrangians are equivalent on the Hamiltonian level.  $\mathcal{L}_U^H$  becomes the  $U$  gauge of  $\mathcal{L}^H$ ; i.e., also for a linearly realized Higgs model the  $U$  gauge is obtained naively by removing all unphysical scalar fields.

Because of the invariance of the Hamiltonian PI under canonical transformations [8], the generating functional obtained when quantizing the linear Lagrangian  $\mathcal{L}^H$  also has the form (1.1) with (4.5), which is again identical [apart from  $\delta^4(0)$  terms] to the result of the Faddeev-Popov procedure if the ( $U$  gauge) GF conditions (3.26) are applied [12]. As in the previous section, this result can be generalized to any other gauge. This completes the proof of Matthews's theorem for any effective Lagrangian, which satisfies the conditions required at the beginning of Sec. II.

The treatment of this section shows that the Stueckelberg formalism, which was originally introduced in order to construct Higgs-boson-less SBGT's [17,18], also represents a powerful tool when dealing with Higgs models [12,22].

## V. THE QUARTICALLY DIVERGENT HIGGS SELF-INTERACTION

In all previous sections, I have neglected the quartically divergent  $\delta^4(0)$  terms. In this section I will quantize the SU(2) Higgs model (without effective non-Yang-Mills vector boson self-interactions) *thereby taking into account the  $\delta^4(0)$  terms* to derive the well-known quartically divergent Higgs self interaction [12,14–16], which serves as a simple example for such a term.

From the discussion of the previous two sections it is clear that it makes no difference to quantize the gauge-invariant Lagrangian of a SBGT or the corresponding  $U$ -gauge Lagrangian<sup>14</sup> which is obtained by setting  $\varphi_a = 0$ . Thus, for simplicity, I start from the  $U$ -gauge Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{8}g^2(v+h)^2 A_a^\mu A_\mu^a - V(h, \varphi_a = 0). \quad (5.1)$$

The momenta are given by

$$\pi_0^a = \frac{\partial \mathcal{L}}{\partial \dot{A}_a^0} = 0, \quad (5.2)$$

<sup>13</sup>For the Hamiltonian and the primary constraints this statement is obvious and the secondary constraints are obtained from the Poisson brackets (2.9) which are invariant under canonical transformations.

<sup>14</sup>When establishing this equivalence, no  $\delta^4(0)$  terms have been neglected, thus, even concerning the quartically divergent extra terms, quantization of both Lagrangians yields the same result.

$$\pi_i^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{A}_i^\alpha} = F_{i0}^\alpha = A_a^i + \partial_i A_0^\alpha - g f_{abc} A_i^b A_0^c, \quad (5.3)$$

$$\pi_h = \frac{\partial \mathcal{L}}{\partial \dot{h}} = \dot{h}, \quad (5.4)$$

and the Hamiltonian is

$$\begin{aligned} \mathcal{H} &= \pi_\mu^\alpha \dot{A}_\mu^\alpha + \pi_h \dot{h} - \mathcal{L} \\ &= \frac{1}{2} \pi_i^\alpha \pi_i^\alpha + \frac{1}{2} \pi_h^2 - \pi_i^\alpha \partial_i A_0^\alpha + g f_{abc} \pi_i^\alpha A_i^b A_0^c \\ &\quad + \frac{1}{4} F_{ij}^\alpha F_{ij}^\alpha + \frac{1}{2} (\partial_i h) (\partial_i h) \\ &\quad - \frac{1}{8} g^2 (v+h)^2 (A_0^\alpha A_0^\alpha - A_i^\alpha A_i^\alpha) + V(h, \varphi_\alpha = 0). \end{aligned} \quad (5.5)$$

The constraints turn out to be

$$\phi_1^\alpha = \pi_0^\alpha = 0, \quad (5.6)$$

$$\phi_2^\alpha = \partial_i \pi_i^\alpha + g f_{abc} \pi_i^b A_i^c - \frac{1}{4} g^2 (v+h)^2 A_0^\alpha = 0. \quad (5.7)$$

The Poisson brackets of the primary and the secondary constraints are given by

$$\{\phi_1^\alpha(\mathbf{x}), \phi_2^b(\mathbf{y})\} = \frac{1}{4} g^2 (v+h)^2 \delta^{ab} \delta^3(\mathbf{x}-\mathbf{y}). \quad (5.8)$$

The constraints are second class.

To quantize this, one starts from the Hamiltonian PI (2.13), integrates out the  $\pi_0^\alpha$  to eliminate  $\delta(\phi_1^\alpha)$ , uses (2.15) to rewrite  $\delta(\phi_2^\alpha)$ , performs the substitution (2.17), and rewrites the determinant using (2.12) and (5.8). The generating functional becomes

$$\begin{aligned} Z[J] &= \int \prod_{\mu,a} \mathcal{D}A_\mu^\alpha \mathcal{D}h \prod_{i,a} \mathcal{D}\pi_i^\alpha \mathcal{D}\pi_h \prod_a \mathcal{D}\lambda^a \exp \left\{ i \int d^4x \left[ -\frac{1}{2} \pi_i^\alpha \pi_i^\alpha - \frac{1}{2} \pi_h^2 + \pi_i^\alpha F_{i0}^\alpha + \pi_h \dot{h} \right. \right. \\ &\quad \left. \left. - \frac{1}{4} F_{ij}^\alpha F_{ij}^\alpha - \frac{1}{2} (\partial_i h) (\partial_i h) + \frac{1}{8} g^2 (v+h)^2 (A_0^\alpha A_0^\alpha - \lambda^\alpha \lambda^\alpha - A_i^\alpha A_i^\alpha) \right. \right. \\ &\quad \left. \left. - V(h, \varphi_\alpha = 0) + J_\mu^\alpha A_\mu^\alpha + J_h h \right] \right\} \text{Det}^3 \left( \frac{1}{4} g^2 (v+h)^2 \delta^4(x-y) \right). \end{aligned} \quad (5.9)$$

Now one can perform the Gaussian intergrations over  $\pi_i^\alpha$ ,  $\pi_h$ , and  $\lambda^\alpha$ . Intergrating out  $\lambda^\alpha$  yields an extra factor  $\text{Det}^{-\frac{3}{2}} \left( \frac{1}{4} g^2 (v+h)^2 \delta^4(x-y) \right)$ . One finds

$$\begin{aligned} Z[J] &= \int \prod_{\mu,a} \mathcal{D}A_\mu^\alpha \mathcal{D}h \exp \left\{ i \int d^4x \left[ \mathcal{L} + J_\mu^\alpha A_\mu^\alpha + J_h h \right] \right\} \\ &\quad \times \text{Det} \left( \frac{1}{8} g^3 (v+h)^3 \delta^4(x-y) \right). \end{aligned} \quad (5.10)$$

Using (2.14) to exponentiate the determinant,  $Z[J]$  becomes a Lagrangian PI (1.1) with the quantized Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{quant}} &= \mathcal{L} - 3i\delta^4(0) \ln \left( 1 + \frac{h}{v} \right) \\ &= \mathcal{L} - 3i\delta^4(0) \ln \left( 1 + \frac{g}{2M} h \right) \end{aligned} \quad (5.11)$$

(after dropping a constant). Thus, the quantized Lagrangian contains, in addition to the primordial Lagrangian, an extra quartically divergent Higgs self-interaction term.

Alternatively, the determinant in (5.10) can be exponentiated by introducing Grassmann variables, which yields the ghost term

$$\mathcal{L}_{\text{ghost}} = -M \eta_a^* \eta_a - \frac{g}{2} \eta_a^* \eta_a h. \quad (5.12)$$

The ghost fields are static due to the absence of a kinetic term. Thus, all ghost loops are quartically divergent. In [12] it has been shown that the ghost loops following from (5.12) yield the same contribution to the  $S$ -matrix

elements as the  $\delta^4(0)$  term in (5.11) and thus  $\mathcal{L}_{\text{ghost}}$  can be replaced by this term.

For the renormalizable Lagrangian (5.1), however, the quartic divergences from the extra term in (5.11) cancel against other quartically divergent Higgs self-interactions arising from vector boson loops [29]. Thus, in this case it is completely justified to neglect the quartic divergences altogether as in [6].

## VI. SUMMARY

The quantization of Lagrangians containing arbitrary interactions of massive vector fields (that do not depend on higher-order derivatives) within the Hamiltonian PI formalism yields the following results.

The generating functional corresponding to an effective Lagrangian without gauge freedom is a simple Lagrangian PI with the quantized Lagrangian being identical to the primordial one [apart from the  $\epsilon^2$  and  $\delta^4(0)$  terms]. Thus, the Feynman rules follow directly from the various terms in the effective Lagrangian.

(Linearly or nonlinearly realized) SBGT's containing effective vector boson interactions, which are embedded in a gauge-invariant framework, can be quantized within the (Lagrangian) Faddeev-Popov PI formalism.

The  $U$  gauge of such an effective SBGT is obtained by removing all unphysical pseudo Goldstone fields.

Lagrangians that are related by a Stueckelberg transformation are physically equivalent.

Using the Stueckelberg formalism, one can rewrite each effective Lagrangian as a (nonlinearly realized) SBGT and extend it, by introduction of (a) physical scalar(s), to a (linearly realized) Higgs model.

These statements seem to be obvious from the naive Lagrangian point of view. However, one has to go through the more elaborate Hamiltonian treatment of this article to derive them correctly. The results obtained in this paper will be generalized to effective Lagrangians with higher-order derivatives in a forthcoming work [30].

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