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Equations of motion for spinning particles in external electromagnetic and gravitational fields

Karl Yee and Myron Bander*
Department of Physics, University of California, Irvine, California 92717
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The equations of motion for the position and spin of a classical particle coupled to an external electromagnetic and gravitational potential are derived from an action principle. The constraints ensuring a correct number of independent spin components are automatically satisfied. In general the spin is not Fermi-Walker transported nor does the position follow a geodesic, although the deviations are small for most situations.

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The derivation of the equations of motion of *classical* spinning particles in external fixed fields has occupied physicists for over 50 years. In special relativity it was first attacked by Frenkel [1]. Using his work as a basis Bargmann, Michel, and Telegdi [2] discussed the precession of spinning particles in external electromagnetic fields. It is amusing to note that, even today, there is a controversy as to the torque and force on such particles in space- and time-dependent fields [3]. The discussion of a spinning particle in an external gravitational potential goes back to Papapetrou [4] who endowed a particle with spin by considering a rotating mass-energy distribution in the limit of vanishing volume but with the angular momentum remaining finite. Results were obtained using Grassmann variables and supersymmetry [5, 6]. In Ref. [7] a crucial constraint, Eq. (5) below, is satisfied by expressing the spin tensor as a product of two Grassman variables; the equations of motion are then derived using a Dirac-Poisson brackets formalism. There is an earlier work relying on finite representations of the Lorentz group [8]. An attempt at a general procedure was made by Khriplovich [9]. Much of the emphasis in the above works is on the equations for the spin components; in Refs. [1,3,4,6–8] the equations for the motion of the position of the particle are discussed while in the other works these are either ignored or are incomplete. In Refs. [7, 8] either the mass or the relation between proper and ordinary time has a spin and field dependence. Recently, the phenomenological effects of an *ad hoc*, parity violating term in the equations for the position were studied [10]. We shall present a canonical procedure, not relying on any analogy with relativistic quantum equations or on the use of Grassmann variables, for obtaining the equations of motion, both for the spin and position. The results are identical to those that would have been obtained using the method of Ref. [1].

We shall obtain the equations of motion in terms of the proper time τ of the particle. The position of the particle will be denoted by x^μ and the spin will be described by the antisymmetric spin matrix S_{ab}. As usual, Greek indices will denote covariant vectors, tensors, etc., and Latin ones those in local Lorentz frames; these are connected by the vierbein field $e^a_\mu(x)$. The spin matrix satisfies the Poisson relation

$$\{S_{ab}, S_{cd}\} = \eta_{ac}S_{bd} + \eta_{bd}S_{ac} - \eta_{ad}S_{bc} - \eta_{bc}S_{ad}\,, \tag{1}$$

with η_{ab} the flat space metric. It will prove to be convenient to obtain the equations of motion for the position from a Lagrangian and those for the spin from a Hamiltonian; such a combined procedure calls for the introduction of a Routhian [11] $\mathcal{R}(x^\mu, S_{ab})$. At the end of this work we shall provide an expression for the action. The equations of motion are

$$\begin{aligned} \frac{\delta\int d\tau\mathcal{R}}{\delta x^\mu} &= 0\,,\\ \frac{dS_{ab}}{d\tau} &= \{\mathcal{R}, S_{ab}\}\,, \end{aligned} \tag{2}$$

τ is the proper time. All the variables we have considered are not independent, but satisfy various constraints. With u^μ, the four-velocity, these are

$$u^\mu u_\mu = 1\,, \tag{3}$$

$$S_{ab}S^{ab} = 2s^2\,, \tag{4}$$

$$S_{ab}u^a = 0\,. \tag{5}$$

Equation (3) guarantees that τ is the proper time and will be satisfied as long as $\mathcal{R}$ is written in a reparametrization invariant form and Eq. (5) is satisfied. Equation (4) ensures that the spin of the particle is constant and is equal to s; it is automatically satisfied for all situations considered. Equation (5) results from the fact that in the particle's rest frame the spin tensor has only three independent components; it is this constraint that causes all the complications.

For a particle in an external field derived from a vector potential A_μ and a gravitational field specified by the spin connection ω^{ab}_μ a tempting Routhian is

$$\begin{aligned} \mathcal{R}_0 = {} & m\sqrt{u^2} + eu^\mu A_\mu - \frac{u^\mu}{2}\omega^{ab}_\mu S_{ab} + \frac{eg}{4m}F^{ab}S_{ab}\sqrt{u^2}\\ & -\frac{\kappa}{8m}R^{abcd}S_{ab}S_{cd}\sqrt{u^2}\,. \end{aligned} \tag{6}$$

*Electronic address: mbander@funth.ps.uci.edu

m is the mass of the particle and e is its charge; g is the gyromagnetic ratio and κ specifies the strength of gravitational magnetic moment coupling introduced in Ref. [9]. This is the most general Routhian not involving derivatives of the field strength tensor or of the Riemann tensor and not involving terms of the form $S_{ab}u^a$. Unfortunately, the equations of motion derived using $\mathcal{R}_0$ do not satisfy the constraints of Eq. (5). There are two procedures that will guarantee this constraint; both give the same equations of motion. One can follow the method of Ref. [1] and add to $\mathcal{R}_0$ the constraint multiplied by a Lagrange multiplier. We shall follow a different procedure. First define

$$\tilde{S}_{ab} = S_{ab} - S_{ac}\frac{u^c u_b}{u^2} - S_{cb}\frac{u^c u_a}{u^2} \,. \tag{7}$$

The Poisson brackets of the $\tilde{S}_{ab}$'s is the one given in Eq. (1) with the metric tensor η_{ab} replaced by $\eta_{ab} - u_a u_b/u^2$. The $\tilde{S}_{ab}$ are related to the spin vector s^a by

$$\begin{aligned} s^a &= -\frac{1}{2}\epsilon^{abcd}\tilde{S}_{bc}u_d \,, \\ \tilde{S}_{ab} &= \epsilon_{abcd}s^c u^d \,. \end{aligned} \tag{8}$$

The spin vector satisfies $s^a s_a = s^2$ and the constraint $s^a u_a = 0$.

A desired Routhian is obtained by replacing all S_{ab}'s in Eq. (6) by $\tilde{S}_{ab}$'s and by adding $(du^a/d\tau)\,S_{ab}u^b/u^2$ to it:

$$\begin{aligned} \mathcal{R} = {} & m\sqrt{u^2} + eu^\mu A_\mu - \frac{u^\mu}{2}\omega_\mu^{ab}S_{ab} + \frac{Du^a}{D\tau}\frac{S_{ab}u^b}{u^2} \\ & + \frac{eg}{4m}F^{ab}\tilde{S}_{ab}\sqrt{u^2} - \frac{\kappa}{8m}R^{abcd}\tilde{S}_{ab}\tilde{S}_{cd}\sqrt{u^2} \,. \end{aligned} \tag{9}$$

Here, D denotes a covariant derivative and we have used the identity

$$\frac{du^a}{d\tau}\frac{S_{ab}u^b}{u^2} - \frac{u^\mu}{2}\omega_\mu^{ab}\tilde{S}_{ab} = \frac{Du^a}{D\tau}\frac{S_{ab}u^b}{u^2} - \frac{u^\mu}{2}\omega_\mu^{ab}S_{ab} \,. \tag{10}$$

The fourth term in Eq. (9) involves the acceleration explicitly; it adds the Thomas precession term to the equations of motion for the spin and ensures that $S_{ab}u^a$ may be set equal to zero.

The equations of motion, with Eqs. (3)–(5) satisfied, for the spin tensor are

$$\frac{DS_{ab}}{D\tau} + (u_b S_{ac} - u_a S_{bc})\frac{Du^c}{D\tau} = -\left(\frac{eg}{2m}F^{cd} + \frac{\kappa}{2m}R^{cdef}S_{ef}\right)[S_{ac}(\eta_{bd} - u_b u_d) + S_{bd}(\eta_{ac} - u_a u_c)] \,. \tag{11}$$

This expression is consistent with Eq. (4) and Eq. (5). The equations of motion for the coordinates of the particle are

$$\begin{aligned} m\frac{Du_\mu}{D\tau} - eF_{\mu\nu}u^\nu - \frac{1}{2}R_{\mu\nu}^{cd}S_{cd}u^\nu = {} & \frac{D}{D\tau}\left[\left(-\frac{eg}{4m}F^{cd}S_{cd} + \frac{\kappa}{8m}R^{cdef}S_{cd}S_{ef}\right)u_\mu\right] \\ & \times\frac{D}{D\tau}\left(e_\mu^c S_{cd}\frac{Du^d}{D\tau}\right) + \frac{eg}{4m}F^{cd}{}_{;\mu}S_{cd} - \frac{\kappa}{8m}R^{cdef}{}_{;\mu}S_{cd}S_{ef} \\ & + \frac{D}{D\tau}\left[e_\mu^h\left(\frac{eg}{2m}F^{cd}u_c S_{hd} - \frac{\kappa}{2m}R^{cdef}S_{cd}u_e S_{hf}\right)\right] \,. \end{aligned} \tag{12}$$

These equations are exact. Except in the case of large gravitational field gradients the modifications due to the spin will be small [12]. It is interesting to study various limits of Eq. (12). If we ignore the right-hand side of that equation and plug the results into Eq. (11) we obtain

$$\frac{DS_{ab}}{D\tau} = -\left(\frac{eg}{2m}F^{cd} + \frac{\kappa}{2m}R^{cdef}S_{ef}\right)(S_{ac}\eta_{bd} + S_{bd}\eta_{ac}) + \left[\frac{e(g-2)}{2m}F^{cd} + \frac{(\kappa-1)}{2m}R^{cdef}S_{ef}\right](S_{ac}u_b u_d + S_{bd}u_a u_c) \,. \tag{13}$$

We know that the electromagnetic part of the equations of motion for the spin simplify in the case $g = 2$; we also see that there is a simplification for the gravitational part in the case $\kappa = 1$. That the Dirac equation

$$\gamma^a e_a^\mu\left(i\partial_\mu - eA_\mu + \frac{1}{2}\omega_\mu^{cd}S_{cd}\right)\psi - m\psi = 0 \,, \tag{14}$$

with S_{cd} expressed in terms of the Dirac γ matrices, yields $g = 2$ is well known; it also yields $\kappa = 1$. We note that even in the presence of only gravitational couplings, but with $\kappa \neq 0$, the spin is *not* Fermi-Walker transported [12, 13] and Eqs. (11) and (12) differ from those in Refs. [7, 6]. The corrections to Fermi-Walker transport are very small, except as mentioned earlier, in the presence of large gravitational fields and gradients.

Another interesting limit is the situation of no electromagnetic field and $\kappa = 0$. The equation of motion for the position of the particle is

$$m\frac{Du_\mu}{D\tau} - \frac{1}{2}R_{\mu\nu}^{cd}S_{cd}u^\nu - \left(e_\mu^c S_{cd}\frac{D^2u^d}{D\tau^2}\right) = 0 \,. \tag{15}$$

This agrees with the equations in Ref. [4] but differ from those of Refs. [7, 6]; we note that spinning particles do not follow geodesics. The term involving the time derivative of the acceleration has been interpreted, in Ref. [4], as being responsible for classical *Zitterbewegung*; in the following sense we agree with this interpretation: in the absence of gravitational interactions a spinning particle will oscillate in the plane perpendicular to the spin direction with a frequency $\omega = E/s$; $s^2 = S^{ab}S_{ab}/2$ is the magnitude of the spin vector. If we set $|s| = \hbar/2$ we recover the quantum mechanical *Zitterbewegung* frequency.

In the nonrelativistic limit Eq. (12), for a purely magnetic dipole interaction, is

$$m\frac{d\mathbf{v}}{dt} = \frac{eg}{2m}\boldsymbol{\nabla}\left(\mathbf{H}\cdot\mathbf{s}\right) + \frac{eg}{2m}\frac{d}{dt}\left(\mathbf{E}\times\mathbf{s}\right) , \qquad (16)$$

$\mathbf{s}$ is defined in Eq. (8). This expression agrees with the force equation advocated in Ref. [3].

For completeness we present an action which corresponds to the Routhian of Eq. (9). A convenient approach is to add a Wess-Zumino [14] term. For closed paths in proper time we introduce a two-dimensional manifold $\mathcal{M}$ parametrized by y_1, y_2 whose boundary is the path τ. The action is

$$\mathcal{A} = \frac{1}{s^2}\int_{\mathcal{M}} dy_1\; dy_2 \epsilon^{\alpha\beta}\mathrm{Tr}S\frac{\partial S}{\partial y_\alpha}\frac{\partial S}{\partial y_\beta} + \int d\tau\mathcal{R}\,. \qquad (17)$$

The canonical equations obtained from the double integral yield the Poisson brackets of Eq. (1). The above action may be amenable to quantization by path integral methods.

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