Information-theoretic measure of uncertainty due to quantum and thermal fluctuations

Arlen Anderson* and Jonathan J. Halliwell†

Theory Group, Blackett Laboratory, Imperial College, South Kensington, London SW7 2BZ, United Kingdom

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We study an information-theoretic measure of uncertainty for quantum systems. It is the Shannon information I of the phase-space probability distribution $\langle z|\rho|z\rangle$, where $|z\rangle$ are coherent states and ρ is the density matrix. As shown by Lieb $I \geq 1$, and this bound represents a strengthened version of the uncertainty principle. For a harmonic oscillator in a thermal state, I coincides with von Neumann entropy, $-\mathrm{Tr}(\rho\ln\rho)$, in the high-temperature regime, but unlike entropy, it is nonzero (and equal to the Lieb bound) at zero temperature. It therefore supplies a nontrivial measure of uncertainty due to both quantum and thermal fluctuations. We study I as a function of time for a class of nonequilibrium quantum systems consisting of a distinguished system coupled to a heat bath. We derive an evolution equation for I. For the harmonic oscillator, in the Fokker-Planck regime, we show that I increases monotonically, if the width of the coherent states is chosen to be the same as the width of the harmonic oscillator ground state. For other choices of the width, and for more general Hamiltonians, I settles down to a monotonic increase in the long run, but may suffer an initial decrease for certain initial states that undergo "reassembly" (the opposite of quantum spreading). Our main result is to prove, for linear systems, that I at each moment of time has a lower bound $I_t^{\min}$, over all possible initial states. This bound is a generalization of the uncertainty principle to include thermal fluctuations in nonequilibrium systems, and represents the least amount of uncertainty the system must suffer after evolution in the presence of an environment for time t. $I_t^{\min}$ is an envelope, equal for each time t, to the time evolution of I for a certain initial state, which we calculate to be a nonminimal Gaussian. $I_t^{\min}$ coincides with the Lieb bound in the absence of an environment, and is related to von Neumann entropy in the long-time limit. The form of $I_t^{\min}$ indicates that the thermal fluctuations become comparable with the quantum fluctuations on a time scale equal to the decoherence time scale, in agreement with earlier work of Hu and Zhang. Our results are also related to those of Zurek, Habib, and Paz, who looked for the set of initial states generating the least amount of von Neumann entropy after a fixed period of nonunitary evolution.

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I. INTRODUCTION

One of the most important features of quantum mechanics is the uncertainty principle:

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}\ . \tag{1.1}$$

Although frequently interpreted as a statement about the precision of measurements, it may also be taken to mean that there is intrinsic uncertainty in any phase-space description of quantum systems. This uncertainty may be especially significant for systems in certain states, such as the ground state. However, in many quantum systems of interest there is additional uncertainty due to thermal fluctuations, and moreover, there may be regimes in which the thermal fluctuations dominate. A number of questions then naturally arise: Is there a useful measure of uncertainty due to both quantum and thermal fluctuations? And, if so, what is the lower bound on this uncertainty, analogous to (1.1)? What are the regimes in which each type of fluctuations dominate? This paper addresses these questions.

Apart from being of interest in their own right, there are a number of specific motivations for studying these issues. The principal one concerns the general question of the emergence of classical behavior in quantum systems. Understanding this issue is one of the main aims of the decoherent histories approach to quantum mechanics [1–4]. There (and in other approaches [5–9]), the process of decoherence is held to play an essential role. This process typically occurs as a result of interaction of the system under scrutiny with a wider environment. But this same interaction also leads to essentially random disturbances of the system, driving it off its classical path. The probabilities for histories are typically found to be peaked about classical histories, with some width determined by quantum effects and broadened by thermal fluctuations induced by interaction with the environment [1]. It therefore becomes important to gain a quantitative understanding of both types of fluctuations, and to find the regimes in which each are important.

In this paper we will explore an information-theoretic measure of uncertainty due to both quantum and thermal effects, suitable for the nonequilibrium quantum systems

*Electronic address: arley@ic.ac.uk

†Electronic address: j__halliwell@vax1.physics.imperial.ac.uk

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used in decoherence models.

We begin in Sec. II by describing the necessary background. We first review some aspects of information theory. We then introduce a quantum-mechanical phase-space distribution. It is the distribution

$$\mu(p,q)=\langle z|\rho|z\rangle \ , \tag{1.2}$$

where ρ is the density matrix of the system and $|z\rangle$ are the coherent states. Our chosen measure of uncertainty is the Shannon information I of this distribution:

$$I=-\int \frac{dp\,dq}{2\pi\hbar}\mu(p,q)\ln\mu(p,q) \ . \tag{1.3}$$

This is sometimes called Wehrl entropy [10]. As we shall explain, the uncertainty principle manifests itself through the inequality

$$I\geq 1 \tag{1.4}$$

with equality if and only if ρ is a coherent state [10,11]. Our main aim is to generalize (1.4) to include the effects of thermal fluctuations in nonequilibrium systems.

In Sec. III we study the properties of I for a simple equilibrium system—the harmonic oscillator in the thermal state. This simple example clearly illustrates how I supplies a useful measure of both thermal and quantum fluctuations. We then go on, in Sec. IV, to consider nonequilibrium systems, the main topic of this paper. We describe an important class of nonequilibrium systems consisting of a distinguished system coupled to a heat bath (often referred to as open quantum systems).

In Sec. V, we discuss the time evolution of I for nonequilibrium systems. We show that I_t generally settles down to monotonic increase. There is, however, the possibility of an initial period of decrease for specially chosen initial states which reassemble (the opposite of wave-packet spreading).

In Sec. VI, we describe our main result. This is the demonstration that I_t has a nontrivial lower bound, the generalization of the Lieb result (1.4) to include thermal fluctuations in nonequilibrium systems. The function $I_t^{\min}$ bounding I_t from below is generally not the time evolution of I for some particular initial state, but is an envelope. The initial state which achieves $I_t^{\min}$ at time t (but generally not at any other time) is a nonminimal Gaussian t. $I_t^{\min}$ is a measure of the least amount of quantum and thermal noise the system must suffer after nonunitary evolution for time t. The bound reduces to the Lieb bound in the absence of an environment.

As we shall explain, there are three contributions to the uncertainty.

(1) There is the uncertainty intrinsic to quantum mechanics, expressed through the uncertainty principle (1.1). This is not dependent on the dynamics. It is this uncertainty that is referred to by the expression "quantum fluctuations."

(2) There is uncertainty that arises due to the spreading or reassembly (the reverse of spreading) of the wave packet. This effect depends on the dynamics, and because quantum mechanics is time symmetric, it may increase or decrease the uncertainty.

(3) There is the uncertainty due to the coupling to a thermal environment. This has two components: dissipation and diffusion (the latter being responsible for the process of decoherence). This generally tends to increase the uncertainty as time evolves.

The point is that the lower bound $I_t^{\min}$ includes the effects (1) and (3), but avoids (2).

Finally, in Sec. VII, we summarize and discuss our results. We compare our results with calculations of Hu and Zhang [12], who calculated the time evolution of the usual uncertainty function for a particular initial state, and determined the time scale on which the thermal fluctuations catch up with the quantum fluctuations. We also compare with the results of Zurek, Habib, and Paz [7,8], who looked for the set of initial states which generate the smallest amount of von Neumann entropy after a fixed period of nonunitary evolution.

II. BACKGROUND

We now review the necessary background.

A. Information theory

Suppose one has a set of probabilities p_i for a data set S consisting of discrete set of alternatives labeled by i, $i=1,2,\ldots,N$. One has $0<p_i<1$ and $\sum_i p_i=1$. Then the Shannon information of the data set is defined to be

$$I(S)=-\sum_{i=1}^{N} p_i\ln p_i \ . \tag{2.1}$$

Here, ln is the logarithm to base e. $I(S)$ satisfies the inequalities

$$0\leq I(S)\leq \ln N \ . \tag{2.2}$$

It reaches its minimum if and only if $p_i=1$, for one particular value of i, and so $p_i=0$ for all the other values. It reaches its maximum when $p_i=1/N$ for all i. The information of a probability distribution is therefore a measure of how strongly peaked it is about a given alternative. For this reason, $I(S)$ is sometimes referred to as *uncertainty*, being large for spread out distributions and small for concentrated ones. This nomenclature is appropriate for purposes of this paper. The expression (2.1) is also often referred to as the *entropy* of the distribution, but we will not do so here, reserving the word entropy for the von Neumann entropy of quantum statistical mechanics (discussed in later sections).

In a similar manner for continuous distributions, let X be a random variable with probability density $p(x)$. Then $\int dx\, p(x)=1$. The information of X is defined to be

$$I(X)=-\int dx\, p(x)\ln p(x) \ . \tag{2.3}$$

Unlike the discrete case, $I(X)$ is no longer positive, since $p(x)$ is not a probability, but a probability density, so may be greater than 1. However, it retains its utility as a measure of uncertainty. This is exemplified by a Gaussian distribution of variance Δx:

$$p(x)=\frac{1}{[2\pi(\Delta x)^2]^{1/2}}\exp\left[-\frac{(x-x_0)^2}{2(\Delta x)^2}\right] . \quad (2.4)$$

It has information

$$I(X)=\ln[2\pi e(\Delta x)^2]^{1/2} . \quad (2.5)$$

From this we see that $I(X)$ is unbounded from below, and indeed, approaches $-\infty$ as $\Delta x \rightarrow 0$ and $p(x)$ approaches a δ function. $I(X)$ is also unbounded from above, as may be seen by taking the width Δx to be very large. However, if the variance is fixed, then a straightforward variational calculation shows that $I(X)$ is maximized by the Gaussian distribution (2.4). We therefore have the important inequality

$$I(X)\leq \ln[2\pi e(\Delta x)^2]^{1/2} . \quad (2.6)$$

The generalization to probability distributions of more than one variable is straightforward. For example, one has

$$I(X,Y)=-\int dx\, dy\, p(x,y)\ln p(x,y) \quad (2.7)$$

and it is easy to show that

$$I(X,Y)\leq I(X)+I(Y) , \quad (2.8)$$

where $I(X)$ is the information of the distribution $\int dy\, p(x,y)$, and similarly for $I(Y)$. We also record another useful result. Let $f(x),g(x)\geq 0$ and let $\int dx\, g(x)=1$. Then

$$-\left[\int dx\, f(x)g(x)\right]\ln\left[\int dy\, f(y)g(y)\right] \geq -\int dx\, f(x)g(x)\ln f(x) . \quad (2.9)$$

This is essentially due to the convexity of the function $x\ln x$, and also holds in the discrete case. Further details on information theory may be found in the literature [13].

B. Phase-space distributions in quantum mechanics

As stated above, our work is partly aimed at discussing the emergence of classical behavior. In this connection, it is often useful to introduce quantum-mechanical phase-space distributions. There are a variety of phase-space distributions that may be employed in quantum mechanics [14]. In this paper we shall focus on the function

$$\mu(p,q)=\langle z|\rho|z\rangle , \quad (2.10)$$

where

$$\langle x|z\rangle=\langle x|p,q\rangle = \left[\frac{1}{2\pi\sigma_q^2}\right]^{1/4}\exp\left[-\frac{(x-q)^2}{4\sigma_q^2}+ipx\right]$$

are the coherent states, with $\sigma_p\sigma_q=\frac{1}{2}\hbar$. We find it useful to work with units with dimension, and for this reason it is necessary to introduce the parameter σ_q into the coherent state wave functions. The function $\mu(p,q)$ is normalized according to

$$\int\frac{dp\, dq}{2\pi\hbar}\mu(q,p)=1 .$$

It is readily shown that $\mu(p,q)$ is also equal to

$$\mu(p,q)=2\int dp'dq'\exp\left[-\frac{(p-p')^2}{2\sigma_p^2}-\frac{(q-q')^2}{2\sigma_q^2}\right] \times W_\rho(p',q') , \quad (2.11)$$

where $W_\rho(p,q)$ is the Wigner function of ρ, defined by [14]

$$W_\rho(p,q)=\frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\rho(q+\tfrac{1}{2}\xi,q-\tfrac{1}{2}\xi) . \quad (2.12)$$

The distribution $\mu(q,p)$ is therefore a Wigner function, smeared over an $\hbar$ sized region of phase space. This smearing renders the distribution function positive, even though the Wigner function is not in general [15]. The distribution (2.11) is sometimes known as the Husimi distribution [16], and has appeared frequently in discussions of the Wigner function (e.g., Refs. [1,15,17]).

The utility of the distribution function $\mu(p,q)$ will become apparent as we expose some of its properties. We remark, however, that μ is of the form

$$\mu(p,q)=\mathrm{Tr}[P_z\rho] , \quad (2.13)$$

where $P_z=|z\rangle\langle z|$ is a coherent state projector (actually only an approximate projector due to the overcompleteness of the coherent states). $\mu(p,q)$ therefore has the interpretation as the probability of a simultaneous but approximate sampling of position and momentum. Moreover, it may be shown that by taking suitably weighted sums over p and q of (2.13), an object of the form

$$p(\bar{x}_2,t_2,\bar{x}_1,t_1)=\mathrm{Tr}[P_{\bar{x}_2}(t_2)P_{\bar{x}_1}(t_1)\rho P_{\bar{x}_1}(t_1)] \quad (2.14)$$

may be obtained, where $P_{\bar{x}}(t)$ denotes an imprecise position sampling at time t. Equation (2.14) is the probability for the history characterized by the initial state ρ, and samplings of position at times t_1 and t_2. The distribution $\mu(p,q)$ is therefore closely connected with the decoherent histories approach to quantum mechanics, which focuses on objects of the form (2.14). In particular, it may be shown that the degree to which expressions of the form (2.14) are peaked about classical paths is limited by the degree of peaking of $\mu(p,q)$ in phase space. This is discussed in another paper [18].

C. An information-theoretic measure of uncertainty

We are interested in the extent to which $\mu(p,q)$ is peaked about some region of phase space. As we have discussed, the Shannon information is a natural measure of the extent to which a probability distribution is peaked. We shall therefore take as our measure of uncertainty, the information

$$I(P,Q)=-\int\frac{dp\, dq}{2\pi\hbar}\mu(p,q)\ln\mu(p,q) . \quad (2.15)$$

The uncertainty principle strongly suggests that a genuine phase-space probability distribution in quantum mechanics cannot be arbitrarily peaked about a point in phase space. We therefore expect the information (2.15) to possess a lower bound. Furthermore, since coherent states are normally regarded as the states most concentrated in phase space, we expect the lower bound to be the value of I on a coherent state. It turns out that both of these expectations are true. It was conjectured by Wehrl [10], and proved by Lieb [11], that

$$I(P,Q)\geq 1 \tag{2.16}$$

with equality if and only if ρ is the density matrix of a coherent state, $|z'\rangle\langle z'|$.

The inequality (2.16) may be related to the usual uncertainty principle (1.1). One has the inequalities

$$\ln\left[\frac{e}{\hbar}\Delta_\mu q\,\Delta_\mu p\right]\geq I(Q)+I(P) \geq I(P,Q)\ . \tag{2.17}$$

The second inequality is an elementary property of information (2.8); the first is the inequality (2.6) applied to each of the marginal distributions for p and q, where $\Delta_\mu q$ and $\Delta_\mu p$ are the variances of the distribution $\mu(p,q)$ (the difference by a factor of $2\pi\hbar$ is due to our choice of phase space measure). These variances are, however, not the quantum-mechanical variances, since they include the variances of the coherent states. Indeed, one has

$$(\Delta_\mu q)^2=(\Delta_\rho q)^2+\sigma_q^2\ , \tag{2.18}$$

$$(\Delta_\mu p)^2=(\Delta_\rho p)^2+\sigma_p^2\ , \tag{2.19}$$

where Δ_ρ denotes the quantum-mechanical variance. Now (2.16)–(2.19) together imply that

$$[(\Delta_\rho q)^2+\sigma_q^2][(\Delta_\rho p)^2+\sigma_p^2]\geq\hbar^2\ . \tag{2.20}$$

Now note that the width σ_q in the coherent state is so far arbitrary. Minimizing (2.20) over σ_q (and recalling that $\sigma_q\sigma_p=\frac{1}{2}\hbar$), we thus obtain the standard uncertainty relations, (1.1). An alternative method of connecting the standard uncertainty relations with (2.16) has been given by Grabowski [19].

Suppose now we have a state which is genuinely mixed. It may therefore be written

$$\rho=\sum_n p_n|n\rangle\langle n| \tag{2.21}$$

for some basis of states $|n\rangle$, and where $p_n<1$. One has

$$\mu(q,p)=\sum_n p_n|\langle z|n\rangle|^2\ . \tag{2.22}$$

The information of (2.22) will always satisfy (2.16), but this will be a very low lower bound for a mixed state. However, from the inequality (2.9), one has

$$\begin{aligned} I &\geq -\int\frac{dp\,dq}{2\pi\hbar}\sum_n|\langle z|n\rangle|^2 p_n\ln p_n \\ &= -\sum_n p_n\ln p_n \\ &= -\mathrm{Tr}(\rho\ln\rho)\equiv S[\rho]\ . \end{aligned}\tag{2.23}$$

That is, I is bounded from below by the von Neumann entropy $S[\rho]$. As we shall see in the following section, this inequality can be close to equality in the regime where thermal fluctuations are large. This close connection with von Neumann entropy is one of the virtues of our chosen measure of uncertainty, over other measures one might contemplate [e.g., the usual uncertainty function $U=(\Delta_\rho q)^2(\Delta_\rho p)^2$].

From the above we therefore see that I is a useful measure of both quantum and thermal fluctuations. It possesses a lower bound expressing the effect of quantum fluctuations, and is closely connected to entropy, which in turn is a measure of thermal fluctuations. In the following sections we will explore the further properties of I, especially for nonequilibrium systems.

III. FLUCTUATIONS AT THERMAL EQUILIBRIUM

To see some of the features of I more clearly, consider the equilibrium case. Let the density matrix be thermal, $\rho=Z^{-1}e^{-\beta H}$, where $Z=\mathrm{Tr}(e^{-\beta H})$ is the partition function, and $\beta=1/kT$. One has

$$\langle z|\rho|z\rangle=\frac{1}{Z}\sum_n e^{-\beta E_n}|\langle z|n\rangle|^2\ , \tag{3.1}$$

where $|n\rangle$ are a set of energy eigenstates with eigenvalues E_n. For simplicity we restrict attention to the simple harmonic oscillator, for which

$$H=\frac{1}{2}\left[\frac{p^2}{M}+M\omega^2q^2\right] \tag{3.2}$$

and so $E_n=\hbar\omega(n+\frac{1}{2})$, and

$$|\langle z|n\rangle|^2=\frac{|z|^{2n}}{n!}e^{-|z|^2}\ . \tag{3.3}$$

Here, $z=\frac{1}{2}(q/\sigma_q+ip/\sigma_p)$, where $\sigma_q\sigma_p=\frac{1}{2}\hbar$, and we have made the choice $\sigma_q=(\hbar/2M\omega)^{1/2}$. See Ref. [20] for details about the coherent states. One thus has

$$\begin{aligned} \mu(q,p)&=\langle z|\rho|z\rangle \\ &=(1-e^{-\beta\hbar\omega})\exp[-(1-e^{-\beta\hbar\omega})|z|^2]\ . \end{aligned}\tag{3.4}$$

The information (2.15) may then be computed explicitly. It is

$$I=1-\ln(1-e^{-\beta\hbar\omega})\ . \tag{3.5}$$

Equation (3.5) is exactly the sort of result one would expect. As the temperature goes to zero, $\beta\to\infty$, and the uncertainty reduces to the Lieb-Wehrl result (2.16) expressing purely quantum fluctuations. But for nonzero temperature, the uncertainty is larger, tending to the value $-\ln(\beta\hbar\omega)$, as the temperature goes to infinity. This limit expresses purely thermal fluctuations. For more general Hamiltonians we expect the information I of the equilibrium thermal state to behave similarly (although we have not been able to derive its explicit form).

It is of interest to compare (3.5) with the entropy,

$$S=-\mathrm{Tr}(\rho\ln\rho)\ . \tag{3.6}$$

The partition function is readily shown to be

$$Z=\frac{1}{2\sinh(\frac{1}{2}\beta\hbar\omega)} \tag{3.7}$$

and the entropy is

$$S=-\beta\frac{\partial}{\partial\beta}(\ln Z)+\ln Z$$
$$=-\ln[2\sinh(\tfrac{1}{2}\beta\hbar\omega)]+\tfrac{1}{2}\beta\hbar\omega\coth(\tfrac{1}{2}\beta\hbar\omega)\ . \tag{3.8}$$

For large temperatures (small β)

$$S\approx-\ln(\beta\hbar\omega)\ , \tag{3.9}$$

S therefore coincides with I in the high-temperature limit. On the other hand, $S\to0$ as the temperature goes to zero, while I goes to a nontrivial lower bound.

We therefore see that I is a useful measure of uncertainty, in both the quantum and thermal regimes. Entropy, by contrast, supplies a measure of uncertainty due only to thermal fluctuations. It is therefore good in the thermal regime, but in the quantum regime, it underestimates the intrinsic quantum uncertainty since it goes to zero for pure states.

It is also useful to compare this measure of uncertainty with the more standard measure:

$$U=(\Delta_\rho q)^2(\Delta_\rho p)^2\ . \tag{3.10}$$

Here, $(\Delta_\rho q)^2$ is computed using $\langle q^2\rangle=\mathrm{Tr}(q^2\rho)$, etc. One readily finds that

$$(\Delta_\rho p)^2=\omega^2(\Delta_\rho q)^2\ . \tag{3.11}$$

Now Eq. (3.4) is a product of Gaussians in p and q, with variances $\Delta_\mu q$, $\Delta_\mu p$, say. The information of such a distribution may be written

$$I=\ln\left[\frac{e}{\hbar}\Delta_\mu q\,\Delta_\mu p\right]\ . \tag{3.12}$$

As in (2.18) and (2.19), the variances of q and p in (3.12) are not the same as the quantum-mechanical variances, because they also include the variances of the coherent state:

$$(\Delta_\mu q)^2=(\Delta_\rho q)^2+\frac{\hbar}{2M\omega}\ , \tag{3.13}$$

$$(\Delta_\mu p)^2=(\Delta_\rho p)^2+\frac{M\omega\hbar}{2}\ . \tag{3.14}$$

Inserting these in (3.12) and using (3.11) one obtains

$$I=\ln\left[\frac{e}{\hbar}\left[U^{1/2}+\frac{1}{2}\hbar\right]\right]\ . \tag{3.15}$$

This shows that, in this simple case, there is a complete equivalence between U and I as measures of uncertainty. We do not expect this equivalence to hold more generally, however.

Finally, we note that an information-theoretic uncertainty relation including the effects of thermal fluctuations at thermal equilibrium has been derived by Abe and Suzuki [21], using thermofield dynamics. Their information-theoretic measure is different from the one used here.

IV. NONEQUILIBRIUM SYSTEMS

Consider now the case of nonequilibrium systems, the main topic of this paper. An important class of such systems in the present context are those in which the total system naturally decomposes into a distinguished system, $\mathcal{S}$ say, and the rest, summarily referred to as the environment. $\mathcal{S}$ is then often referred to as an open quantum system. One is interested only in the behavior of $\mathcal{S}$ and not in the detailed behavior of the environment. The distinguished system is most completely described by the reduced density matrix ρ obtained by tracing out over the environment. The environment leaves its mark, however, in that the effective evolution of the reduced density matrix alone is nonunitary.

A useful model of the type described above consists of a particle moving in one dimension in a potential $V(x)$, linearly coupled to a bath of harmonic oscillators in a thermal state. The environment is characterized by a temperature T and a dissipation coefficient γ. This model has been the subject of many papers, so we will give only the briefest of accounts here (for further details, see Refs. [22–27]).

After tracing out the environment, the reduced density matrix ρ of the distinguished system evolves nonunitarily, according to the relation

$$\rho_t(x,y)=\int dx_0dy_0J(x,y,t|x_0,y_0,0)\rho_0(x_0,y_0)\ . \tag{4.1}$$

Here, J is the reduced density matrix propagator. It is given by the path integral expression

$$J(x_f,y_f,t|x_0,y_0,0)=\int\mathcal{D}x\,\mathcal{D}y\,\exp\left[\frac{i}{\hbar}S[x]-\frac{i}{\hbar}S[y]+\frac{i}{\hbar}W[x,y]\right]\ , \tag{4.2}$$

where

$$S[x]=\int dt[\tfrac{1}{2}M\dot{x}^2-V(x)] \tag{4.3}$$

and $W[x(t),y(t)]$ is the Feynman-Vernon influence functional phase,

$$W[x(t),y(t)]=-\int_0^t ds\int_0^s ds'[x(s)-y(s)]\eta(s-s')[x(s')+y(s')]+i\int_0^t ds\int_0^s ds'[x(s)-y(s)]\nu(s-s')[x(s')-y(s')]\ . \tag{4.4}$$

The explicit forms of the nonlocal kernels η and ν may be found in Refs. [26,23]. We have assumed, as is typical in these models, that the initial density matrix of the total system is simply a product of the initial system and environment density matrices.

Considerable simplifications occur in a purely Ohmic environment at high temperature. Take a regularized Ohmic environment with cutoff frequency Λ having the spectral density

$$C(\omega)=\frac{2M\gamma\omega}{\pi}e^{-\omega^2/\Lambda^2} . \tag{4.5}$$

In the Fokker-Planck limit (see Ref. [26]), one first takes the high-temperature limit $\hbar/kT \ll \Lambda^{-1}$ and then lets the cutoff go to infinity, $\Lambda \to \infty$. One finds

$$\eta(s-s')=M\gamma\delta'(s-s') , \tag{4.6}$$

$$\nu(s-s')=\frac{2M\gamma kT}{\hbar}\delta(s-s') . \tag{4.7}$$

This limit is a simple and useful one, but our main results do not depend on it.

The propagator J may be evaluated exactly for the case of the simple harmonic oscillator $V(x)=\frac{1}{2}M\omega^2x^2$. Introducing $X=x+y$, $\xi=x-y$, one has

$$J(X_f,\xi_f,t|X_0,\xi_0,0)=F^2(t)\exp\left[\frac{i}{\hbar}\tilde{S}-\frac{\phi}{\hbar}\right] , \tag{4.8}$$

where

$$\tilde{S}=\tilde{K}(t)X_f\xi_f+\hat{K}(t)X_0\xi_0-L(t)X_0\xi_f-N(t)X_f\xi_0 \tag{4.9}$$

and

$$\phi=A(t)\xi_f^2+B(t)\xi_f\xi_0+C(t)\xi_0^2 . \tag{4.10}$$

Explicit expressions for the coefficients $\tilde{K}$, $\hat{K}$, L, N, A, B, and C are given in Refs. [23,26]. $F^2(t)=N/\pi$ is a normalization factor, fixed by imposing the condition

$$\int dx\, dy\,\delta(x-y)J(x,y,t|x_0,y_0,0)=\delta(x_0-y_0) . \tag{4.11}$$

This ensures that $\mathrm{Tr}\rho_t=1$ at all times. On the other hand, tracing over the initial arguments of J leads to

$$\int dx_0\, dy_0\,\delta(x_0-y_0)J(x,y,t|x_0,y_0,0)=\frac{N}{L}\delta(x-y) . \tag{4.12}$$

We remark that $\tilde{S}$ is in fact the action of the solution to the boundary value problem for the harmonic oscillator with (nonlocal) dissipation, for which the equation of motion is

$$\ddot{X}+\omega^2X+2\int_0^s ds'\eta(s-s')X(s')=0 . \tag{4.13}$$

In the classical limit we expect that the quantum system reduces to motion described by this equation.

One may also derive an evolution equation for ρ, for general potentials. Its most general form is [26]

$$i\hbar\frac{\partial\rho}{\partial t}=-\frac{\hbar^2}{2M}\left[\frac{\partial^2\rho}{\partial x^2}-\frac{\partial^2\rho}{\partial y^2}\right]+[V_R(x)-V_R(y)]\rho-i\hbar\Gamma(t)(x-y)\left[\frac{\partial\rho}{\partial x}-\frac{\partial\rho}{\partial y}\right]-i\Gamma(t)h(t)(x-y)^2\rho +\hbar\Gamma(t)f(t)(x-y)\left[\frac{\partial\rho}{\partial x}+\frac{\partial\rho}{\partial y}\right] . \tag{4.14}$$

Here $V_R(x)$ is the renormalized potential $V_R(x)=V(x)+\frac{1}{2}M\delta\Omega^2(t)x^2$. The explicit forms for the time-dependent coefficients $\delta\Omega(t)$, $\Gamma(t)$, $f(t)$, $h(t)$ are in general rather complicated. Explicit expressions for them may be found in Ref. [26]. In the Fokker-Planck limit one has

$$\Gamma(t)=\gamma,\quad h(t)=\frac{2MkT}{\hbar},\quad f(t)=0 . \tag{4.15}$$

The first two terms on the right-hand side of (4.14) generate purely unitary evolution (but with a renormalized potential). The third term is the dissipative term, and the fourth and fifth terms are diffusive terms. In particular, the fourth term is responsible for the process of decoherence discussed elsewhere [5–9].

V. TIME EVOLUTION OF I_t

We now study the evolution of I as the density matrix ρ evolves under the nonunitary evolution discussed in the previous section. For simplicity, consider first the unitary evolution of ρ, without an environment. One has

$$\mu_t(p,q)=\langle z|e^{-iHt}\rho_0e^{iHt}|z\rangle , \tag{5.1}$$

where ρ_0 is the density matrix at $t=0$, and may be pure or mixed. The operators $e^{\mp iHt}$, evolving ρ_0 forward in time, may be equally regarded as evolving the coherent states backward in time. For a harmonic oscillator, the width σ_q of the coherent states $|z\rangle$ may be chosen to be the width of the ground state (although this choice is by no means obligatory). With this choice, the coherent states are preserved under unitary evolution, with their centers following the classical evolution:

$$e^{-iHt}|p,q\rangle=|p_{\rm cl}(t),q_{\rm cl}(t)\rangle . \tag{5.2}$$

The same is true for evolution backward in time, with $t\to -t$. It is a standard result that the transformation from (p,q) to $(p_{\rm cl}(t),q_{\rm cl}(t))$ is a classical canonical transformation. The effect of unitary evolution in (5.1) is therefore to perform a canonical transformation on the arguments of $\mu_t(p,q)$ at $t=0$. It is straightforward to see that our measure of uncertainty (2.15) is invariant under canonical transformations of the variables of integration. We therefore find that I is constant under unitary evolution for the harmonic oscillator, with the above special choice of σ_q.

If the width σ_q is not set to the above special value, then the coherent states are not preserved under evolution by the harmonic oscillator Hamiltonian. Likewise for more general Hamiltonians. For example, if the initial state is a coherent state, it will spread as time evolves, and thus I will increase from its initial value, $I=1$. Whether I increases or decreases, however, depends very much on the initial state. For example, the pure state $e^{+iHt}|z\rangle$, which could have a very large value of I, will evolve under e^{-iHt} into the coherent state $|z\rangle$, possessing the minimum value of I. This "reassembly" of a state sharply peaked in phase space from a very spread out state will therefore cause I to decrease with time.

One would in fact expect initial states undergoing an initial decrease of I to be just as likely as ones undergoing an initial increase, since quantum mechanics is a completely time-symmetric theory. However, I does in a certain sense capture the intuitive notion that "entropy increases," even for pure states, in that it will increase for initial states which might reasonably be described as highly organized or special (namely, states that are sharply peaked in phase space).

Now consider the coupling to an environment, as described in the previous section. We shall derive an evolution equation for I_t. We will first use the evolution equation for ρ, (4.14), to derive an evolution for the Wigner function of ρ (2.12). Performing the Wigner transform of (4.14), one obtains

$$\frac{\partial W}{\partial t}=-\frac{p}{M}\frac{\partial W}{\partial q}+V_R'(q)\frac{\partial W}{\partial p}+2\Gamma(t)\frac{\partial}{\partial p}(pW)+\hbar\Gamma(t)h(t)\frac{\partial^2 W}{\partial p^2}+\hbar\Gamma(t)f(t)\frac{\partial^2 W}{\partial q\partial p}+\sum_{k=1}^{\infty}\left[\frac{i\hbar}{2}\right]^{2k}\frac{1}{(2k+1)!}V^{(2k+1)}(q)\frac{\partial^{2k+1}W}{\partial p^{2k+1}}\ . \tag{5.3}$$

The infinite power series incurred for general potentials makes progress rather difficult. We shall therefore restrict attention to the harmonic oscillator, $V(q)=\frac{1}{2}M\omega^2q^2$, returning at the end to a heuristic discussion of the possible effects of more general potentials. Now using the expression for $\mu(\bar{p},\bar{q})$, (2.11), one obtains

$$\frac{\partial\mu}{\partial t}=-\frac{\bar{p}}{M}\frac{\partial\mu}{\partial\bar{q}}+M\omega_R^2(t)\bar{q}\frac{\partial\mu}{\partial\bar{p}}-\left[\frac{\sigma_p^2}{M}-M\omega_R^2(t)\sigma_q^2-\hbar\Gamma(t)f(t)\right]\frac{\partial^2\mu}{\partial\bar{p}\partial\bar{q}}+2\Gamma(t)\mu+2\Gamma(t)\left[\bar{p}+\sigma_p^2\frac{\partial}{\partial\bar{p}}\right]\frac{\partial\mu}{\partial\bar{p}}+\hbar\Gamma(t)h(t)\frac{\partial^2\mu}{\partial\bar{p}^2}\ . \tag{5.4}$$

Here, $\omega_R^2(t)=\omega^2+\delta\Omega^2(t)$ is the renormalized frequency. Differentiating the expression for I, (2.15), one obtains, at some length,

$$\dot{I}=-2\Gamma(t)-\left[\frac{\sigma_p^2}{M}-M\omega_R^2(t)\sigma_q^2-\hbar\Gamma(t)f(t)\right]\int\frac{d\bar{p}\,d\bar{q}}{2\pi\hbar}\frac{1}{\mu}\frac{\partial\mu}{\partial\bar{p}}\frac{\partial\mu}{\partial\bar{q}}+[\hbar\Gamma(t)h(t)+2\Gamma(t)\sigma_p^2]\int\frac{d\bar{p}\,d\bar{q}}{2\pi\hbar}\frac{1}{\mu}\left[\frac{\partial\mu}{\partial\bar{p}}\right]^2\ . \tag{5.5}$$

This is the exact result for the time evolution of I for linear systems.

Now the interesting question is whether we can say anything definite about the monotonicity properties of I, given Eq. (5.5). First, note that in the case of no environment, and for the harmonic oscillator (i.e., $\omega\neq0$), it is possible to make the choice

$$\sigma_q^2=\frac{\hbar}{2M\omega},\quad \sigma_p^2=\tfrac{1}{2}M\omega\hbar\ , \tag{5.6}$$

and thus $\dot{I}=0$, as expected.

The next interesting case to consider is the Fokker-Planck limit (4.15) in which it is again useful to make the choice (5.6), and the second term in (5.5) vanishes. Consider the remaining terms in (5.5). The first term is -2γ and the coefficient of the last term is approximately $2M\gamma kT$ (the σ_p^2 term is negligible in the Fokker-Planck limit). Now the question is, what are the relative sizes of the first and last terms in (5.5)? Introduce the time scales

$$t_{\rm dec}=\frac{\hbar^2}{2M\gamma kT\sigma_q^2},\quad t_{\rm rel}=\frac{1}{\gamma}\ . \tag{5.7}$$

The time scale $t_{\rm dec}$ frequently emerges in studies of decoherence and is therefore called the decoherence time scale. We are not of course discussing decoherence *per se* here, but we will use the nomenclature. $t_{\rm rel}$ is the relaxation time scale. On dimensional grounds it is clear that the first term will cause I to decrease on a time scale $t_{\rm rel}$ and the last term will cause it to increase on a time scale $t_{\rm dec}$. The important point is that the relaxation time is typically very much longer than the decoherence time [6], so the decoherence term will dominate in (5.5). Thus for the harmonic oscillator, with the choice (5.6), and in the Fokker-Planck limit, I will increase monotonically for any initial state.

Now consider the case in which the choice (5.6) is not made. Closely related is the case of the free particle, in which $\omega=0$ in (5.5), and σ_p is arbitrary. The question is whether $\dot{I}$ may be rendered negative by the indefinite term in the integrand (which is associated with spreading or reassembly). Physically, it is reasonably clear how this may come about. As discussed above, it is possible to choose special initial states that reassemble, at least under unitary evolution, and will cause I to decrease. One would expect to be able to identify a spreading or reassembly time scale, t_s. If the decoherence time scale is much shorter than the spreading time scale, one would expect I to increase monotonically, since the environment acts before the system has time to undergo reassembly. On the other hand, if the spreading time is shorter than the decoherence time, an initial decrease may occur for carefully chosen initial states, but this will eventually go over to increase after a time of order t_d. A similar situation could be expected to hold for more general Hamiltonians. The Hamiltonian terms [in (5.3), say] may make I increase or decrease, but eventually the diffusive terms will take over and cause I to increase.

These statements all apply to the high-temperature regime, in which thermal effects will eventually dominate. Equation (5.5) is valid for all regimes, and it would be of interest to explore these, although we do not do so here.

We now have a general picture of the behavior of I under time evolution. This sets the stage for the next section, in which we derive a lower bound on the behavior of I.

Finally, we note that the analogue of Eq. (5.5) for von Neumann entropy is very hard, if not impossible, to derive, even for linear systems. Generally it can be obtained only if explicit diagonalization of ρ is possible, e.g., for Gaussian density matrices. For this reason, I may be more practically useful than S as a measure of uncertainty, quite simply because it is easier to calculate.

VI. A LOWER BOUND FOR I_t

We now come to the main point of this paper, which is to establish a lower bound over all possible initial states for I_t, thus generalizing (2.16) to include thermal fluctuations in time-evolving nonequilibrium systems. We therefore seek a time-dependent function $I_t^{\min}$ such that, for every time t,

$$I_t \geq I_t^{\min} . \tag{6.1}$$

$I_t^{\min}$ represents the least amount of uncertainty the system must suffer, after evolution for time t in the presence of an environment. Clearly for consistency we must have $I_t^{\min}=1$ in the absence of an environment.

To fix ideas, consider first the case of no environment, for which the evolution is unitary. The Lieb-Wehrl result is that the information (2.15) at a fixed time is minimized by a system in a coherent state $\rho_0=|z'\rangle\langle z'|$. A harmonic oscillator initially in a coherent state with a width given by Eq. (5.6) evolves so that it remains in a coherent state, and therefore $I_t=1=I_t^{\min}$. It is easy to see that this behavior is very special and cannot be realized for other Hamiltonians. This is because Hamiltonian evolution generally does not preserve the coherent states. As described in the previous section, for every time τ, there is an initial state $e^{+iH\tau}|z'\rangle$, with nonminimal I_t at $t=0$, which evolves to a coherent state at time τ, there minimizing I_t. After this, it disperses, and I_t is no longer minimal. I_t is only minimized at $t=\tau$.

The implication of this is that $I_t^{\min}$ is actually an envelope. No particular ρ_0 realizes the minimum for all time—instead there are a succession of states which achieve the minimum. The minimum $I_t^{\min}=1$ is realized, at each time t, by the value of I_t for the initial state $e^{+iHt}|z'\rangle$; that is, for the initial state obtained by evolving the coherent state $|z'\rangle$ at time t backward to $t=0$.

Now consider the situation with an environment, as discussed in the previous section. Instead of unitary evolution under e^{-iHt}, we now have nonunitary evolution under the propagator J. As we have seen, interaction with the environment will cause I_t to increase in the long run, but there is the possibility of an initial decrease of I_t, due to the reassembly effect. We therefore expect $I_t^{\min}$ to again be an envelope: there will be many initial states which achieve $I_t^{\min}$ for some value of t, but there will be no initial state for which $I_t=I_t^{\min}$ for all t.

To find $I_t^{\min}$ we will exploit the Lieb-Wehrl inequality (2.16). It cannot, however, be applied immediately to the case at hand. To see why, consider again the case of no environment. One is interested in the information (2.15). Application of the inequality (2.9) shows that the minimum is achieved for a pure rather than mixed state. One is thus minimizing the integral

$$I=-\int \frac{dp\,dq}{2\pi\hbar}|\langle z|\psi\rangle|^2\ln|\langle z|\psi\rangle|^2 \tag{6.2}$$

over all square-integrable wave functions ψ. The minimum is found to be achieved for $|\psi\rangle=|z'\rangle$, a coherent state. If one expresses the state at a later time in terms of unitary evolution from its initial value, $|\psi\rangle=e^{-iHt}|\psi_0\rangle$, one has the expression for the information at time t:

$$I_t=-\int \frac{dp\,dq}{2\pi\hbar}|\langle z|e^{-iHt}|\psi_0\rangle|^2\ln|\langle z|e^{-iHt}|\psi_0\rangle|^2 . \tag{6.3}$$

Minimizing this over all square-integrable wave functions $|\psi_0\rangle$ is easy because $e^{-iHt}|\psi_0\rangle$ is itself a square-integrable wave function, so the previous result applies, giving $|\psi_0\rangle=e^{iHt}|z'\rangle$, as discussed above.

Now we are interested in the more general case in which the propagator is not unitary. We would like to know what the new lower bound on the uncertainty is for systems that have undergone interaction with the environment for time t. Denoting the coherent state density matrix by $\rho_z=|z\rangle\langle z|$, and the initial density matrix by ρ_0, the information at time t is given by

$$I_t = -\int \frac{dp\,dq}{2\pi\hbar} \mathrm{Tr}[\rho_z J_t(\rho_0)] \ln \mathrm{Tr}[\rho_z J_t(\rho_0)] \ . \tag{6.4}$$

Here, $J_t(\rho_0)$ denotes the nonunitary evolution of ρ_0, Eq. (4.1). For each time t we seek the ρ_0 that minimizes (6.4). Differently put, we need to minimize (6.4) over all density matrices of the form $\rho_t = J_t(\rho_0)$, where ρ_0 is an arbitrary density matrix. The feature that distinguishes this case from the Lieb-Wehrl case discussed above is that this class of density matrices is smaller than the class of *all* density matrices, since evolution under J is not invertible. It is therefore difficult to characterize the class over which to do the minimization. Since $J_t(\rho_0)$ is linear in ρ_0, and using the convexity property (2.9), we again deduce that the minimizing ρ_0 must be pure. This simplifies the problem somewhat, but the inconvenience stated still remains.

To get around this difficulty we adopt the following strategy. We are interested in the quantity

$$\begin{aligned}\mu_t(\bar{p},\bar{q}) &= \langle z|\rho_t|z\rangle \\ &= \int dx\, dy\, dx_0 dy_0 \langle z|x\rangle\langle y|z\rangle \\ &\quad \times J(x,y,t|x_0,y_0,0)\rho_0(x_0,y_0)\ ,\end{aligned} \tag{6.5}$$

where J is the reduced density matrix propagator. μ_t is then conveniently written in the form

$$\begin{aligned}\mu_t(\bar{p},\bar{q}) &= \int dx_0 dy_0\, A_t^z(y_0,x_0)\rho_0(x_0,y_0) \\ &= \mathrm{Tr}(A_t^z\rho_0)\ ,\end{aligned} \tag{6.6}$$

where

$$A_t^z(y_0,x_0) = \int dx\, dy \langle z|x\rangle\langle y|z\rangle J(x,y,t|x_0,y_0,0)\ . \tag{6.7}$$

The quantity A_t^z is therefore the final density operator $|z\rangle\langle z|$ brought back from time t to time zero using J. Note, however, that A_t^z is not a physical density matrix, since from (4.12), $\mathrm{Tr} A_t^z = N/L$ [although one does have $\int (d\bar{p}\, d\bar{q}/2\pi\hbar) A_t^z = 1$]. Using the Wigner representation (2.12) one may write

$$\mu_t(\bar{p},\bar{q}) = 2\pi\hbar \int dp\, dq\, W_{A_t^z}(p,q) W_{\rho_0}(p,q)\ . \tag{6.8}$$

Since J is the Gaussian for the linear case considered here, $A_t^z(x_0,y_0)$ and $W_{A_t^z}(p,q)$ are also Gaussian.

Compare this to the Lieb-Wehrl result (2.16). The latter may be regarded as stating that the information of the distribution,

$$\mu(p,q) = 2\int dp' dq' \exp\left[-\frac{(p'-p)^2}{2\sigma_p^2} - \frac{(q'-q)^2}{2\sigma_q^2}\right] W_{\tilde{\rho}}(p',q')\ , \tag{6.9}$$

is bounded from below by $I=1$, with equality if and only if $\tilde{\rho}$ is a coherent state. Now the point is that (6.8) and (6.9) have a very similar form: they are both Wigner functions of an arbitrary density matrix with a Gaussian smearing, but the Gaussian factors are not the same. Our aim, therefore, is to perform a series of transformations to bring (6.8) into the form (6.9), and then apply the Lieb-Wehrl result (2.16). As we shall see, the information of μ is not preserved under these transformations, and thus we obtain a nontrivial lower bound, different from (2.16), and depending on the quantity A_z^t. The difficulty outlined above is avoided because the evolution under J is contained entirely in A_t^z, and the minimization is now over all pure ρ_0, a well-defined class to which the Lieb-Wehrl result may be applied.

Turn now to the details. Consider first Eq. (6.7). The final density matrix is

$$\langle x|z\rangle\langle z|y\rangle = \frac{1}{(2\pi\sigma_q^2)^{1/2}} \exp\left[-\frac{\xi^2}{8\sigma_q^2} - \frac{(X-2\bar{q})^2}{8\sigma_q^2} + \frac{i}{\hbar}\bar{p}\xi\right]\ , \tag{6.10}$$

where as in Sec. IV, $X = x+y$, $\xi = x-y$. Under evolution backward in time by the nonunitary propagator J it yields

$$A_t^z(y_0,x_0) = \frac{N}{L}\left[\frac{2\alpha_0}{\pi}\right]^{1/2} \exp\left[-\alpha_0(X_0-2\bar{q}_0)^2 - \beta_0\xi_0^2 + \frac{i}{\hbar}\xi_0[\Gamma_0(X_0-2\bar{q}_0)+\bar{p}_0]\right]\ , \tag{6.11}$$

where

$$\alpha_0 = \frac{L^2}{32\sigma_q^2\Delta}\ , \tag{6.12}$$

$$\beta_0 = C + 2\sigma_q^2 N^2 - \frac{(B-4\sigma_q^2 N\tilde{K})^2}{32\sigma_q^2\Delta}\ , \tag{6.13}$$

$$\Gamma_0 = \hat{K} + \frac{L}{4\Delta}\left[\frac{B}{4\sigma_q^2} - N\tilde{K}\right]\ . \tag{6.14}$$

Here,

$$\Delta = \frac{1}{8\sigma_q^2}\left[A + \frac{1}{8\sigma_q^2}\right] + \frac{1}{4}\tilde{K}^2\ . \tag{6.15}$$

(These coefficients may be obtained by a straightforward modification of the calculations described in Ref. [4].) Also, $\bar{p}_0, \bar{q}_0$ are the classical evolution of $\bar{p}, \bar{q}$, evolved backward in time under the dissipative equation of motion (4.13). They are given explicitly by

$$\bar{p}_0 = -\frac{2}{L}(NL - \hat{K}\tilde{K})\bar{q} + \frac{\hat{K}}{L}\bar{p} \ , \tag{6.16}$$

$$\bar{q}_0 = \frac{\tilde{K}}{L}\bar{q} + \frac{1}{2L}\bar{p} \ , \tag{6.17}$$

where the various quantities appearing are defined in Sec. IV. This transformation from $\bar{p},\bar{q}$ to $\bar{p}_0,\bar{q}_0$ is noncanonical, because the evolution is dissipative:

$$\frac{\partial(\bar{q}_0,\bar{p}_0)}{\partial(\bar{q},\bar{p})} = \frac{N}{L} = e^{2\gamma t} \ . \tag{6.18}$$

Performing the Wigner transformation, one thus obtains the explicit form of (6.8):

$$\mu_t(\bar{p},\bar{q}) = 2\frac{N}{L}\left[\frac{\alpha_0}{\beta_0}\right]^{1/2} \int dp\, dq \exp\left[-\frac{1}{4\hbar^2\beta_0}[p - \bar{p}_0 - 2\Gamma_0(q - \bar{q}_0)]^2 - 4\alpha_0(q - \bar{q}_0)^2\right] W_{\rho_0}(p,q) \ . \tag{6.19}$$

We would like to bring this expression into the form (6.9). Introduce

$$\lambda = \left[\frac{\beta_0}{\alpha_0}\right]^{1/2}, \quad \mu = \sqrt{8}\,\sigma_q(\beta_0\alpha_0)^{1/4} \ . \tag{6.20}$$

Now perform the following canonical transformation on the integration variables, together with the same change of variables on $\bar{p}_0,\bar{q}_0$:

$$q' = \mu q, \quad p' = \frac{1}{\mu}(p - 2\Gamma_0 q) \ , \tag{6.21}$$

$$\tilde{q} = \mu\bar{q}_0, \quad \tilde{p} = \frac{1}{\mu}(\bar{p}_0 - 2\Gamma_0\bar{q}_0) \ , \tag{6.22}$$

Eq. (6.19) thus becomes

$$\mu_t(\bar{p},\bar{q}) = 2\frac{N}{L}\left[\frac{1}{\lambda}\right]^{1/2} \int dp' dq' \exp\left[-\frac{(p'-\tilde{p})^2}{2\lambda\sigma_p^2} - \frac{(q'-\tilde{q})^2}{2\lambda\sigma_q^2}\right] \times W_{\tilde{\rho}}(p',q') \ , \tag{6.23}$$

where we have introduced

$$W_{\tilde{\rho}}(p',q') = W_{\rho_0}\left[\mu p' + 2\frac{\Gamma_0}{\mu}q', \frac{q'}{\mu}\right] \ . \tag{6.24}$$

There arises the question of whether $W_{\tilde{\rho}}(p',q')$ defined by (6.24) is still a Wigner function, i.e., of whether there exists a density matrix $\tilde{\rho}$ whose Wigner transform is (6.24). The answer is in the affirmative: linear canonical transformations on the arguments of the Wigner function are readily shown to correspond to unitary transformations of ρ.

The dependence on $\bar{p},\bar{q}$ on the right-hand side of (6.23) resides entirely in $\tilde{p},\tilde{q}$, via the transformations (6.16), (6.17), and (6.22). It is convenient to write (6.23) as

$$\mu_t(\bar{p},\bar{q}) = \frac{N}{L}\tilde{\mu}_t(\tilde{p},\tilde{q}) \ . \tag{6.25}$$

The factor N/L is nothing more than the Jacobian of the transformation from $\tilde{p},\tilde{q}$ to $\bar{p},\bar{q}$. The transformation (6.21) and (6.22) is canonical so the only contribution to the Jacobian comes from (6.16) and (6.17), whose Jacobian is (6.18). The information of μ_t, I_t, is then simply related to that of $\tilde{\mu}_t,\tilde{I}_t$. It is

$$I_t = \tilde{I}_t - \ln\left[\frac{N}{L}\right] \ . \tag{6.26}$$

The distribution $\tilde{\mu}_t$ is almost of the desired form (6.9), but fails to be because of the presence of the factor of λ. The positivity of the density matrix (6.11) implies that $\beta_0 \geq \alpha_0$, and in fact equality holds only at $t=0$, and thus one has $\lambda > 1$. One might have thought that the next step is to simply scale p' and q' by $\lambda^{1/2}$, thus taking λ into the Wigner function. However, this scaling would lead to a phase-space distribution function which is *not* a Wigner function; i.e., it is not the Wigner transform of a density matrix. This is easy to see: under such a scaling, the degree to which the Wigner function may be peaked about a region of phase space becomes enhanced by a factor of $\lambda > 1$, and thus it is possible to violate the uncertainty principle. Wigner functions scaled in this way cannot therefore correspond to density matrices.

Instead, the next step is carried out using the following simple fact about convolution integrals: when two Gaussians with variances σ_1 and σ_2 are convoluted, the variance of their convolution, σ_3, satisfies $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$. Let us therefore express the Gaussian smearing function in $\tilde{\mu}_t$ as the convolution of two Gaussians:

$$\frac{1}{\pi\hbar\lambda}\exp\left[-\frac{(p'-\tilde{p})^2}{2\lambda\sigma_p^2} - \frac{(q'-\tilde{q})^2}{2\lambda\sigma_q^2}\right] = \int dp\, dq \frac{1}{\pi\hbar}\exp\left[-\frac{(p-p')^2}{2\sigma_p^2} - \frac{(q-q')^2}{2\sigma_q^2}\right]\frac{1}{\pi\hbar(\lambda-1)}\exp\left[-\frac{(p-\tilde{p})^2}{2(\lambda-1)\sigma_p^2} - \frac{(q-\tilde{q})^2}{2(\lambda-1)\sigma_q^2}\right] \ . \tag{6.27}$$

We may therefore write $\tilde{\mu}_t$ as

$$\tilde{\mu}_t(\tilde{p},\tilde{q}) = \frac{1}{\pi\hbar(\lambda-1)}\int dp\, dq \exp\left[-\frac{(p-\tilde{p})^2}{2(\lambda-1)\sigma_p^2} - \frac{(q-\tilde{q})^2}{2(\lambda-1)\sigma_q^2}\right]\hat{\mu}_t(p,q) \ , \tag{6.28}$$

where $\hat{\mu}_t$ is precisely of the form (6.9), with Wigner function $W_{\bar{\rho}}(p',q')$, given above by (6.24).

The result (6.28) is as close as we can get to casting $\mu(\bar{p},\bar{q})$ in the form (6.9). However, the form (6.28) may be exploited: it is the convolution of a Gaussian with the function $\hat{\mu}_t$. We may therefore appeal to a theorem of Lieb on the information of convolutions [11]. Let f and g be functions defined in $L^s(R^n)$, where $s>1$, and let $f*g$ denote their convolution. Then the information of $f*g, I(f*g)$, satisfies the inequality

$$\exp\left[\frac{2}{n}I(f*g)\right]\ge\exp\left[\frac{2}{n}I(f)\right]+\exp\left[\frac{2}{n}I(g)\right]. \tag{6.29}$$

Equality holds if f and g are both Gaussians differing only in the location of their centers and in an overall scale of their covariance matrices.

In our case the Gaussian function in (6.28) has information

$$I=\ln\left[\frac{e}{2}(\lambda-1)\right]. \tag{6.30}$$

Since $\hat{\mu}_t$ is of the form (6.9), it satisfies the Lieb-Wehrl inequality (2.16), with equality if and only if the Wigner function (6.24) is the Wigner function of a coherent state:

$$W_{\rho_0}\left[\mu p'+2\frac{\Gamma_0}{\mu}q',\frac{q'}{\mu}\right]=\frac{1}{\pi\hbar}\exp\left[-\frac{(p'-p_1')^2}{2\sigma_p^2}-\frac{(q'-q_1')^2}{2\sigma_q^2}\right]. \tag{6.31}$$

Applying (6.29) we therefore have the lower bound on the information of $\tilde{\mu}_t$:

$$\tilde{I}_t\ge 1+\ln[\tfrac{1}{2}(\lambda+1)]. \tag{6.32}$$

Finally, inserting this in (6.26) we obtain the desired lower bound on I_t:

$$I_t\ge 1+\ln\left[\frac{L}{2N}\left[\left[\frac{\beta_0}{\alpha_0}\right]^{1/2}+1\right]\right]. \tag{6.33}$$

This is our main result. The right-hand side is the value of $I_t^{\min}$ at time t.

Now consider the conditions for equality in (6.33), to determine the initial state which meets the envelope at time t. The information of $\hat{\mu}_t$ achieves its lower bound when (6.31) is satisfied. $\hat{\mu}_t$ is then a Gaussian, differing only from the smearing Gaussian in (6.28) by an overall scaling of their covariance matrices. The conditions for equality in (6.29) are therefore also satisfied. This means that the inequality (6.33) achieves equality when the initial state is given by (6.31). Inverting the Wigner transform we find that the initial state is the pure state:

$$\Psi_t(x)=\left[\frac{4(\alpha_0\beta_0)^{1/2}}{\pi}\right]^{1/4}\exp\left[-\left[2(\alpha_0\beta_0)^{1/2}+\frac{i}{\hbar}\Gamma_0\right]\times(x-\bar{q}_1)^2+\frac{i}{\hbar}\bar{p}_1x\right]. \tag{6.34}$$

VII. DISCUSSION AND SUMMARY

We first discuss the properties of the lower bound (6.33).

Consider Eq. (6.19). We have been seeking the Wigner function W_{ρ_0} that minimizes the information of (6.19). Loosely speaking, this means finding the Wigner function which has the best overlap with the exponential in (6.19), and hence gives the most peaked probability distribution $\mu_t(\bar{p},\bar{q})$. We found that the initial state doing the job is (6.34), whose Wigner transform is, from (6.31),

$$W_\Psi(p,q)=\frac{1}{\pi\hbar}\exp\left[-\frac{(p-2\Gamma_0q-\mu p_1')^2}{2\mu^2\sigma_q^2}-\frac{\mu^2}{2\sigma_q^2}\left[q-\frac{q_1'}{\mu}\right]^2\right]. \tag{7.1}$$

Now consider the exponential function in (6.19). It is the Wigner function of the final coherent state evolved backward by J. The contours of the Wigner function start out as circles. Each contour suffers three effects under this nonunitary evolution: it is distorted into an ellipse, its axes are rotated, and its area increases. The distortion factor is given by μ in (6.20), the amount of rotation is given by Γ_0, and the area increase is given by λ. (There is in addition a translation of the contours, but this preserves the information.)

Now the point is that the Wigner function (7.1) giving the least overlap in (6.19) is the Gaussian pure state which matches two out of three of these effects: it has the same distortion and rotation factors. It does not have the same expansion factor λ—it cannot because we know that the minimizing state must be pure, and pure Gaussian states must have $\lambda=1$. The minimizing state is therefore the state whose Wigner function is close as possible to the exponential in (6.19) subject to the constraint that it be pure.

Turn now to the explicit form of the lower bound. Using the result of Refs. [4,12,26,23] it may be shown that in the Fokker-Planck limit, and for short times, one has

$$\left[\frac{\beta_0}{\alpha_0}\right]^{1/2}\approx 1+2\gamma t+\frac{8\sigma_q^2M\gamma kT}{\hbar^2}t+O(t^2). \tag{7.2}$$

Setting σ_q to the value (5.6) one thus has

$$I_t^{\min}=1+\ln\left[1+\left[\frac{2kT}{\hbar\omega_R}-1\right]\gamma t+O(t^2)\right]. \tag{7.3}$$

The Fokker-Planck limit involves $kT \gg \hbar\omega_R$, so $I_t^{\min}$ increases with time. Equation (7.3) indicates that the thermal contributions to the uncertainty principle start to become appreciable after a time

$$t \sim \frac{\hbar^2}{\sigma_q^2 M\gamma kT} \sim \frac{\hbar\omega_R}{\gamma kT} . \tag{7.4}$$

The important thing to note is that this is the decoherence time scale defined in Eq. (5.7)—the time scale on which interference is destroyed by the interaction with the environment.

Our results should be compared with the work of Hu and Zhang [12]. They calculated the usual uncertainty function (3.10) for the density matrix obtained by evolving an initial coherent state for time t in the presence of an environment. They found that for short times and high temperatures

$$U = \frac{\hbar^2}{4}\left[1+\left[\frac{2kT}{\hbar\omega_R}-1\right]\gamma t + O(t^2)\right] . \tag{7.5}$$

It was these authors who first noted, on the basis of this calculation, the significance of the decoherence time scale for the comparative sizes of thermal and quantum fluctuations. We thus find close agreement with their work.

This result has a consequence for the decoherence program. A reasonable question to ask in decoherence models is whether there is a regime in which the interaction with the environment is sufficient to induce decoherence, yet induces a noise level less than that due to intrinsic quantum fluctuations. Our results, and those of Hu and Zhang [12], show that this is not the case: in the Fokker-Planck regime, decoherence and thermal fluctuations become important on the same time scale. This means, loosely speaking, that the uncertainty principle plays little role in these models.

It is of interest to explore the form of the lower bound in other regimes. Consider for example, the low-temperature regime. In the Fokker-Planck (high-temperature) regime discussed above, the diffusion is controlled by the diffusion constant $D = 2M\gamma kT$. However, as argued by Caldeira and Leggett [23], in the low-temperature regime the appropriate diffusion constant is $D = M\gamma\hbar\omega_R$. An order of magnitude estimate on the size of $I_t^{\min}$ is therefore obtained by substitution of diffusion constants. One thus discovers that in the low-temperature regime, the environmentally induced fluctuations (we can no longer call them thermal) grow on a time scale γ^{-1}, the relaxation time scale. This shows that $I_t^{\min}$ is not just a measure of quantum fluctuations of the distinguished system plus thermal fluctuations of the environment: it also includes the quantum fluctuations of the environment (although these are of course negligible in the high-temperature regime).

Another question to ask is whether it is possible to express our new uncertainty principle (6.33) in terms of the usual uncertainty function U. Recall that the Lieb-Wehrl inequality (2.16) may be shown to imply the standard uncertainty principle (1.1) via the steps (2.17)–(2.20). Can a similar derivation be carried out in the case of (6.33)? Steps analogous to (2.17)–(2.20) can be carried out, and one obtains

$$\Delta_\mu p \, \Delta_\mu q \geq \frac{L}{2N}\left[\left[\frac{\beta_0}{\alpha_0}\right]^{1/2}+1\right]\hbar . \tag{7.6}$$

However, as before this is not the proper form of the uncertainty principle, because the variances on the left-hand side also include the variances of the coherent state, Eqs. (2.18) and (2.19). The final step of minimizing over σ_q is rather tricky to carry out because the right-hand side of (7.6) depends on σ_q in a nontrivial way, and one ends up with a fifth-order polynomial in σ_q^2. Also, the alternative method suggested by Grabowski [19] cannot obviously be generalized so as to apply to this case. Therefore, we do not give an explicit form of our uncertainty relation in terms of the variances of ρ. The possibility of deriving such a relation directly [rather than from (7.6)] will be considered elsewhere [28].

We should also compare with the work of Paz, Habib, and Zurek [7,8], who looked for the set of initial states which generated the least amount of physical entropy, $S[\rho]$, after evolution in the presence of an environment for time t. The motivation for doing this is that these states are in a sense the ones most stable under evolution in the presence of an environment. This is clearly closely related to our work, since we essentially looked for the set of initial states with the smallest value of I after time t. Indeed, Paz *et al.* claimed that the minimizing states are coherent states, whereas for us the minimizing states are more general Gaussian states. It turns out that the quantity Γ_0 in (6.34) can go to zero quite quickly (on the time scale ω^{-1}). In this case we thus see that the results are in agreement.

In summary, we have discussed the properties of an information-theoretic measure of uncertainty (1.3) for a class of nonequilibrium quantum systems. Our measure is closely related to von Neumann entropy in the thermal regime, but unlike entropy, it supplies a nontrivial measure of uncertainty in the quantum regime. It is also easier to work with calculationally than entropy. Our main result is the demonstration that, for linear systems, our measure has a nontrivial lower bound, the generalization of the uncertainty principle to include thermal (or more generally, environmentally induced) fluctuations for a class of nonequilibrium systems. We have examined the form of the lower bound in some regimes of interest. A more detailed examination is best carried out numerically, but this is beyond the scope of the present work.

Note added in proof. Some mathematical results closely related to the result (2.16) may be found in Refs. [29,30]. We are grateful to E. Lieb for bringing these papers to our attention. References [31,32] discuss other applications of the measure of uncertainty (1.3). An alternative measure of the comparative sizes of quantum and thermal fluctuations may be found in Ref. [33].

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