Basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantum-gravity model is the loop representation basis

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In three dimensions (3D) the Ponzano-Regge-Turaev-Viro-Ooguri model provides a combinatorial definition of quantum gravity. The model is written in terms of a specific basis in the Hilbert spaces associated with the 2D boundaries of spacetime. We show that this basis is the same as the one that defines the loop representation of quantum gravity. We extend this construction to the physical 4D case, by defining a modification of Regge calculus in which areas, rather than lengths, are taken as independent variables. We provide an expression for the scalar product in the loop representation in 4D. We discuss the general form that a nonperturbative quantum theory of gravity should have, and argue that this should be given by a generalization of Atiyah's topological quantum-field-theory axioms.

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The problem of describing physics at the Planck scale and the quantum properties of gravity is the problem of understanding what is a nontrivial generally covariant quantum-field theory. The last years have seen several developments in our understanding of these theories: Witten's introduction of topological field theories, in their two versions, Chern-Simons [1] and Donaldson [2]; Atiyah's axiomatization of these [3]; dynamical triangulation techniques [4]; Ashtekar's reformulation of general relativity [5], which opened the way to the loop representation [6], which led to the discovery of solutions of the Wheeler-DeWitt equation, of the relevance of knot theory in quantum gravity, and of a discrete structure of space at the Planck scale [7]; the Turaev-Viro [8] reformulation of the Ponzano-Regge model [9] in terms of quantum groups, which provides a combinatorial definition of three-dimensional (3D) topological field theories; the Crane-Vetter [10] extension of this construction to 4D; the pioneering work of Ooguri, which in 3D has tied the Euclidean, combinatorial, canonical, and topological definitions of quantum gravity [11]. These results share a remarkable common flavor, in addition, of course, to the common long-term aim of quantizing gravity. In this paper, we find the bridge between the 3D Ponzano-Regge-Turaev-Viro-Ooguri (PRTVO) model and the loop representation, and we sketch a general theory of physical 4D gravity in which all these lines may converge.

Ponzano and Regge [9] considered Regge calculus [13] in 3D, but made the ansatz that the lengths l_i of the Regge-calculus links be constrained to be half-integers: $l_i=j_i=\frac{1}{2}n_i$ (integer n_i). Half-integers j_i can be interpreted as labels of SU(2) representations. The Regge-calculus action can then be written as a very simple expression, which is essentially a sum over the triangulation's tetrahedra of the 6-j symbols of the six (half-integer) lengths $l_i=j_i$ of the links of each tetrahedron. The partition function of quantum gravity can then be constructed by fixing a sufficiently thin triangulation Δ, and summing over its colorings c (assignments of half-integers to every link). The reason for taking half-integer lengths, as well as the relation between lengths of links and SU(2) representations, appeared to be quite mysterious at the time. In this paper we throw some light on this relation. Turaev and Viro [8] were able to show that the Ponzano-Regge partition function is independent from the triangulation chosen [and transformed it in a finite sum by replacing SU(2) with a corresponding quantum group with a finite number of representations]. Ooguri [11] related the quantization of 3D quantum gravity based on this model to the Witten quantization of the same theory. Ooguri construction can be summarized in short as follows. The quantum states of the Ponzano-Regge theory have to be taken, following Atiyah's general formulations of topological quantum field theories, as quantum combinations $\Phi_\Delta(c)$ of the colored triangulations (Δ,c) induced on the 2D boundary ∂M of the 3D manifold M. In Witten's theory, quantum states are wave functions $\Psi(\omega)$ over the moduli space of the flat SU(2) connections A_a^I on the 2D boundary (ω being the equivalent classes of A's). Ooguri relates the two representations of the theory by

$$\Psi(\omega)=\sum_c \Phi_\Delta(c)\Psi_{\Delta,c}(\omega)\ , \tag{1}$$

where we have absorbed in $\Phi_\Delta(c)$ a normalization factor appearing in Ooguri's Eq. (16). The "matrix elements of the change of basis" $\langle\omega|\Delta,c\rangle=\Psi_{\Delta,c}(\omega)$ will be described in a moment. The relations between the Ooguri construction and the loop representation was suggested by Crane and Smolin [14]; in this paper, we show, indeed, that (1) is nothing but the loop transform, which was introduced in Ref. [6], and relates the connection representation of quantum gravity to the loop representation. The Ooguri representation $\Phi_\Delta(c)$ is thus essentially equivalent to the loop representation. In doing so, we provide a physical interpretation to the Ponzano-Regge ansatz that the links have half-integer length, and are related to SU(2) representations, and we find the physical justification of Ooguri's construction.

Ooguri's functions $\Psi_{\Delta,c}(\omega)$, are constructed as follows. Given the triangulation Δ, we construct the corresponding dual trivalent graph. The arc C_i of this graph crosses the ith link of the triangulation; we associate with C_i the SU(2) representation j_i, j_i being the half-integer that colors the ith link of Δ. We associate with C_i a function over flat SU(2) connections A_a^I given by the Wilson line $U_{j_i}[A,C_i]=P\exp(\int c A_a^I t_{j_i}^I dx^a)$, where the SU(2) generators t^I are taken in the j_i representation. To the full graph, we associate the product of all these Wilson lines, where 6-j symbols are used to contract the indices at the trivalent intersections. The resulting object is a function of the triangulation, the coloring, and the connection. It is gauge invariant, and thus it defines a function over the moduli space of the flat SU(2) connections for every (Δ,c); this function is $\Psi_{\Delta,c}(\omega)$.

In order to relate this construction with the loop representation, the first observation is that, since any representation of SU(2) is obtained by tensor multiplication of the $j=\frac{1}{2}$ representation with itself, a Wilson line $U_j[A,C]$ in the j representation is equivalent to $2j$ Wilson lines $U_{1/2}[A,C]$ in the $\frac{1}{2}$ representation. Thus, we replace each arc C_i of the trivalent graph with precisely $2j_i$ lines. Accordingly, the sum at the trivalent intersections obtained with the 6-j symbols can simply be replaced by the sum over all the possible rootings of these lines at the intersection; and $\Psi_{\Delta,c}(\omega)$ can be expressed as a combination of products of traces of holonomies of A along the resulting closed loops, all taken in the $\frac{1}{2}$ representation. This follows from elementary properties of SU(2) representation theory. In other words, the colored triangulation (Δ,c) uniquely determines an ensemble $E_{\Delta,c}=\{\alpha_1,\alpha_2,\ldots\}$ of multiple loops (sets of closed loops) $\alpha_i=(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{iN})$, where α_{ij} are (single) loops: Each multiple loop α_i has the property that precisely $2j$ single loops cross a link of the triangulation with color j. The ensemble $\{\alpha_1,\alpha_2,\ldots\}$ is formed by all the homotopically inequivalent multiple loops with this property. By construction we have the main relation

$$\Psi_{\Delta,c}(\omega)=\sum_{\alpha_i\in E(\Delta,c)}\prod_j \mathrm{Tr}U_{1/2}[A,\alpha_{ij}]\ .$$

Now, given a multiple loop α_i, the product $\prod_j \mathrm{Tr}U_{1/2}[A,\alpha_{ij}]$ is nothing but the loop state $|\alpha_i\rangle$, written in the connection representation, namely,

$$\langle A|\alpha_i\rangle=\Psi_{\alpha_i}(A)=\prod_j \mathrm{Tr}U_{1/2}[A,\alpha_{ij}]\ .$$

This relation is at the root of the loop representation. Using this relation, and its gauge invariance, we have

$$\begin{aligned}\langle\omega|\Delta,c\rangle&=\Psi_{\Delta,c}(\omega)\\&=\sum_{\alpha_i\in E(\Delta,c)}\langle A|\alpha_i\rangle=\sum_{\alpha_i\in E(\Delta,c)}\langle\omega|\alpha_i\rangle\ ,\end{aligned}$$

or (2)

$$|\Delta,c\rangle=\sum_{\alpha_i\in E(\Delta,c)}|\alpha_i\rangle\ .$$

This equation provides the identification between the loop representation states $|\alpha\rangle$ and the Ooguri states $|\Delta,c\rangle$. (Recall that the loop representation states are not independent.) This is our first result.

In Ooguri's work, relation (1) is postulated, and the equivalence of the combinatorial theory with Witten's quantization is derived *a posteriori* by showing the isomorphism of the two structures. Still, the half-integer lengths remain as mysterious as they were in the original Ponzano-Regge paper. To provide an interpretation of this fact, let us *calculate* the lengths of the links of a triangulation in a fixed quantum state $\Psi_{\Delta,c}(\omega)$ of the gravitational field. [Gravity *is* geometry, and in a quantum state of gravity lines *have* (expectation values of) length.] A recent calculation in (3+1)-dimensional quantum gravity indicates that the area of any surface is quantized in the loop representation in multiples of $\frac{1}{2}$ (in Planck units); the area being precisely given by the number of intersections of the surface with the loops of the quantum state. It is natural to suspect that a similar relation may work in one dimension less. In fact, let us show it does. The length l of a curve C in ∂M is given in 3D gravity by

$$\begin{aligned}l[C]&=\int_C dt\left[\frac{dC^a}{dt}\frac{dC^b}{dt}g_{ab}\right]^{1/2}\\&=\int_C dt\left[\frac{dC^a}{dt}\frac{dC^b}{dt}E^{Ic}E^{Id}\varepsilon_{ac}\varepsilon_{bd}\right]^{1/2}\ ,\end{aligned}$$

where E is the variable conjugate to the connection and ε_{ac} is the antisymmetric two-dimensional pseudotensor. We refer to [6] for the notation. In order to promote $l[C]$ to an operator, we have to deal with the product of the two E's. Following Ref. [7], we point split the product $E(x)E(x)$, by means of a gauge-invariant two-point object:

$$\begin{aligned}E^{ia}(x)E^{ib}(x)&:=\lim_{\varepsilon\to0}\mathrm{Tr}\{U[A,\gamma_x'^{\varepsilon}]E^a(\gamma_x^\varepsilon(0))U[A,\gamma_x''^{\varepsilon}]E^b(\gamma_x^\varepsilon(\pi))\}\\&=\lim_{\varepsilon\to0}T^{ab}[\gamma_x^\varepsilon](0,\pi)\ .\end{aligned}$$

Here γ_x^ε is a loop with radius ε, and center in x, $\gamma_x'^{\varepsilon}$ and $\gamma_x''^{\varepsilon}$ are its two components from the value 0 to π and from π to 0 of the loop parameter, and in the second line we have introduced the standard loop representation notation [6] for this point-split observable. The operator corresponding to the observable $T^{ab}[\gamma_x^\varepsilon](0,\pi)$ is well defined: using the loop representation, it is given [6] by

$$\langle \alpha | T^{ab}[\gamma_x^\varepsilon](0,\pi) = \int_\alpha ds \frac{d\alpha^a}{ds} \delta^2(\alpha(s),\gamma_x^\varepsilon(0)) \int_\alpha du \frac{d\alpha^b}{du} \delta^2(\alpha(u),\gamma_x^\varepsilon(\pi)) \sum_i \langle \alpha \#_{st,i} \gamma_x^\varepsilon | \ .$$

Following Ref. [7], we may regularize the δ functions by a further replacement of γ_x^ε with a one-parameter family of loops. We can then compute the action of the operator $l[C]$ on a loop state $\langle \alpha|$. The square root of the square of the (regularized) δ function gives an absolute value; in the limit, the intersection rearrangement $\sum_i$ gives just a multiplicative factor, and in the limit we obtain, in Planck units,

$$\langle \alpha | l[C] = \tfrac{1}{2} \int_\alpha ds \int_C dt \left| \frac{d\alpha^a}{ds} \frac{dC^b}{dt} \varepsilon_{ab} \right| \langle \alpha | \ .$$

The double integral is precisely the positive intersection number between C and α. Thus, we arrive at the following results: (i) the length of every[1] curve C is quantized in units of $\frac{1}{2}$; (ii) if the gravitational field is in the state $\langle \alpha|$, the length of a curve l is given by

$$l[C] = n[C,\alpha] \frac{L_{\text{Planck}}}{2} \ ,$$

where $n[C,\alpha]$ is the number of times α crosses C.

Now we can return to the PRTVO model. Result (i) above implies that summing over independent states in quantum gravity means to sum just over quantized half-integer lengths, precisely as in the Ponzano-Regge ansatz. Second, the relation between the two-dimensional colored triangulation on the boundaries of M and the Witten theory now becomes physically transparent: Recall that the ith link C_i of the triangulation, which has color j_i, is crossed precisely by $2j_i$ loops; thus, $n[C_i,\alpha]=2j_i$; therefore, using result (ii), the physical length in Planck units in the quantum state defined by these loops is

$$l_i = \tfrac{1}{2} n[C,\alpha] = \tfrac{1}{2} 2j_i = j_i \ .$$

Thus, the physical length of the ith link in the Ooguri state is precisely equal to its coloring. Therefore, the state $\Psi_{\Delta,c}(\omega)$, which Ooguri associates with the colored triangulation (Δ,c), is a quantum state in which the *physical length* of the links of the triangulation is precisely equal to their coloring, in Planck units. This is our main result.

Before getting to the second part of this paper, let us note that the relation between the loop representation theory and the PRTVO model allows us to write the scalar product of two-loop states $|\gamma\rangle$ and $|\gamma'\rangle$ by means of a sum over colorings: we put γ and γ' on the boundaries of a $(\partial M \times [0,1])$ three-manifold M, we fix a triangulation Δ, and we have

$$\langle \gamma, | \gamma' \rangle = \sum_{c(\gamma,\gamma')} \prod P_{\text{PR}}(c) \ ,$$

where the sum is over all the triangulations $c(\gamma,\gamma')$ such that the coloring of the links of the boundaries is determined by the number of times the loops γ and γ' cross the links, and $\prod P_{\text{PR}}(c)$ is the Ponzano-Regge product for the coloring c.

The above result indicates a direction for constructing the physical 4D theory. We will now briefly sketch this theory; a similar set of ideas is discussed by Carfora, Martellini, and Marzuoli in Ref. [15]. Let us consider the 4D spacetime manifold M, with, say, two boundaries ∂M_1 and ∂M_2, and begin by fixing a 4D triangulation Δ of spacetime, which induces 3D triangulations of the two boundaries. In 4D, they are the areas of surfaces, not the lengths, that are naturally quantized in $\frac{1}{2}$ the Planck unit. Thus, it is natural to use the areas of the faces, rather than the lengths of the links, as independent variables for Regge calculus. A key observation is that a 4D simplex has the same number (10) of faces (2D simplices) and links (1D simplices). Therefore, we can generically invert the relation between lengths and areas, and express the lengths of the 10 links of each four-simplex as functions of the areas of the 10 faces. Let $a_1 \cdots a_{10}$ be the areas of the 10 faces of a fixed four-simplex s. The Regge action of the simplex can be expressed as a function of these areas: $S_{\text{Regge}}(s) = S_{\text{Regge}}(a_1 \cdots a_{10})$. Note that S_{Regge} must be a function of 10 variables with the full symmetry of the four-simplex. We expect that the corresponding quantity $S_{\text{Regge}}(a_1 \cdots a_{10})$, seen as a function of half-integer variables, has an interpretation in terms of group representation theory, at least in the large a_i limit, analogous to the 6-j symbol's interpretation of its 3D analogue.[2] Concrete results along these lines are obtained in Ref. [15]. The above construction defines a combinatorial quantum theory in 4D (for a fixed triangulation).[3] In the absence of boundaries, we have

$$Z(\Delta) = \sum_c Z_{\text{Regge}}(c) = \sum_c \prod_s \exp[-S_{\text{Regge}}(s)] \ ,$$

where the sum is over the colorings, the product over the four-simplices. The states of this theory are given by the induced colorings of the induced triangulation on the 3D boundary of the 4D triangulated spacetime. These states

[1] Here we disregard states for which the loops have self-intersections on the line [7].

[2] The reason the group SU(2) is still the relevant group, in spite of the fact we are one dimension above, is in the very fact that it is the roots of Ashtekar's magic construction: the (complexified) Lorentz group splits naturally in two (complexified) SO(3) groups, its self-dual and anti-self-dual parts, and, as shown by Ashtekar, the full theory can be constructed using only one of these two SO(3) components.

[3] We disregard the important problem of the fact that the areas of a triangulation are not independent.

are physically determined by the fact that there is a three-dimensional triangulation of space such that the (2D) surfaces of the triangulation have assigned areas, these areas being half-integers in Planck units. These states are therefore in correspondence with the states of the loop representation. The correspondence being given by the result of Ref. [7]: the physical area of any surface is $\frac{1}{2}$ the number of loops that cross the surface. Since the states can be written in terms of loop states, and vice versa, this theory defines a scalar product in loop space by

$$\langle \gamma, | \gamma' \rangle_\Delta = \sum_{c(\gamma,\gamma')} Z_{\text{Regge}}(c) \ ,$$

where γ is on ∂M_1, γ' is on ∂M_2, and the sum is over all the colorings of the interior areas, the colorings of the areas on the boundaries being determined as $\frac{1}{2}$ the number of times the loops cross the surface. The idea that states of the loop representation can be better understood in terms of the area they induce on a 3D triangulation was proposed by Smolin in Ref. [16].

We still do not have a well-defined diffeomorphism invariant theory, since the above construction depends on the triangulation Δ.[4] We eliminate this dependence by summing over all the triangulations of M. The key point is that this is now possible because we have a way of determining the quantum states on the boundaries, which is independent from the triangulation of the boundary itself, namely, the loop basis. Thus, we define the theory as

$$Z = \sum_\Delta \sum_c Z_{\text{Regge}}(c) \ , \tag{3}$$

and the scalar product between two loop states by

$$\langle \gamma, | \gamma' \rangle = \sum_\Delta \sum_{c(\gamma,\gamma')} Z_{\text{Regge}}(c) \ . \tag{4}$$

[If in Eq. (3) we perform the sum over the colorings first, the theory takes a form strictly related to the dynamical triangulations approaches to quantum gravity [4]; this relation, we believe, deserves to be explored.] The important point is that the scalar product defined in this way is invariant under independent diffeomorphisms on each of the two loops, because the sum is also. Therefore, it defines a scalar product $\langle K | K' \rangle$ between knot states by

$$\langle K(\gamma) | K(\gamma') \rangle = \langle \gamma, | \gamma' \rangle \ .$$

These equations provide an expression for the Hilbert structure of quantum gravity. We expect that Eq. (4) defines a projector on the knot states, which projects on the solutions of the Wheeler-DeWitt equation, as happens in 3D [11], so that both canonical constraints are implemented in the combinatorial theory. This is our main proposal for a 4D theory.

To clarify the meaning of Eqs. (3) and (4), let us rewrite them in terms of the original continuous Ashtekar's variables A and E. We have[5]

$$\langle \gamma, | \gamma' \rangle = \int [d^4 A][d^4 E] \exp\{-S_E[{}^4A, {}^4E]\} \times \Psi_\gamma(A)^* \Psi_{\gamma'}(A) \ ,$$

where

$$\Psi_\gamma(A) = \Psi_\gamma({}^3A) = \langle {}^3A | \gamma \rangle = \text{Tr} P \exp\left[\oint_\gamma A\right]$$

and each loop is in one of the two boundaries of spacetime. It is easy to convince oneself that this is the correct formal expression for the scalar product in the loop representation, up to the problem of definition and finiteness of the functional integration. To our knowledge, this expression was first proposed by Martellini [17]. This expression is the connection representation equivalent to Hawking's expression [18]

$$(\Psi, \Phi) = \int [d^4 g] \exp\{-S_E[{}^4g]\} \Psi({}^3g)^* \Phi({}^3g') \ ,$$

where the integration is over all the 4D metrics ${}^4g$ on M, and ${}^3g$ and ${}^3g'$ are the restrictions of ${}^4g$ to the two components of the boundary of M. Of course, these functional integrals do not mean anything until one has a definition for them. The combinatorial expression given above could be used to find this definition.[6] In this continuum case, Hawking does indeed give a formal derivation of the fact that the functional integral defines a projection on the solutions of the Wheeler-DeWitt equation. This indirectly supports our claims about the sum (4).

We conclude with a consideration on the formal structure of 4D quantum gravity, which is important to understand the above construction. Standard quantum-field theories, as QED and QCD, as well as their generalizations such as quantum-field theories on curved spaces and perturbative string theory, are defined on metric spaces. Witten's introduction of the topological quantum-field theories has shown that one can construct quantum-field theories defined on a manifold which has only its differential structure, and no fixed metric structure. The theories introduced by Witten and axiomatized by Atiyah have the following peculiar feature: they have a finite

[4]There is an optimistic scenario, which is that the scalar product above does not depend on the triangulation. While we do not hope for so much (general relativity is not topological in the naive sense), still we do not think that this is totally impossible, as some considerations below may suggest.

[5]Ooguri suggests [12] that one may have two conjugate variables in the continuum version of the theory, which correspond, respectively, to the choice of the triangulation and the choice of the coloring in the combinatorial version of the theory. Note the similarity between Ashtekar's $\int F \wedge E \wedge E$ action, with the (topological) BF theory action $\int F \wedge B$ which seems to underlie the Ooguri-Crane-Vetter invariants [10,12]. The relation between the construction proposed here and the (triangulation-independent) Ooguri-Crane-Vetter construction deserves to be studied in detail.

[6]Of course, each of the above equations can be immediately generalized to nontrivial topologies, Hartle-Hawking states, disconnected universes, and so on, if one is interested in those directions.

number of degrees of freedom, or, equivalently, their quantum-mechanical Hilbert spaces are finite dimensional; classically this follows from the fact that the number of fields is equal to the number of gauge transformations. But not every diffeomorphism-invariant field theory on a manifold has a finite number of degrees of freedom. Witten's gravity in (2+1)-D is given by the action $S[A,E]=\int \mathrm{Tr} F\wedge E$, which has a finite number of degrees of freedom; consider the action $S[A,E]=\int \mathrm{Tr} F\wedge E\wedge E$ in (3+1)-D.[7] This theory has a strong resemblance with its (2+1)-D analogue: It is still defined on a differential manifold without any fixed metric structure, and is diffeomorphism invariant. We expect its quantization to be more similar to the quantization of the $\int \mathrm{Tr} F\wedge E$ theory than to the quantization of theories on flat space based on the Wigthman axioms, namely on n-point functions and related objects. However, the theory has genuine field degrees of freedom: its physical phase space is infinite dimensional, and we expect that its Hilbert state space will also be infinite dimensional. There is a popular impression that a theory defined on a differential manifold without metric and diffeomorphism invariant has necessarily a finite number of degrees of freedom ("because thanks to general covariance we can gauge away any local excitation"). This belief is, of course, wrong. A theory as the one defined by the action $\int \mathrm{Tr} F\wedge E\wedge E$ is a theory that shares many features with the topological theories, in particular, no quantity defined "in a specific point" is gauge invariant; but it has genuinely infinite degrees of freedom. Indeed, this theory is, of course, nothing but (Ashtekar's form of) standard general relativity.

The fact that "local" quantities such as the n-point functions are not appropriate to describe quantum gravity nonperturbatively has been repeatedly noted in the literature. As a consequence, the issue of what are the quantities in terms of which a quantum theory of gravity can be constructed is a returning issue. The above discussion indicates a way to face the problem: The topological quantum-field theories studied by Witten and Atiyah provide a framework in terms of which quantum gravity itself may be framed, in spite of the infinite degrees of freedom. In particular, Atiyah's axiomatization of the topological field theories provides us with a clean way to formulate the problem. Of course, we have to relax the requirement that the theory has a finite number of degrees of freedom. These considerations lead us to propose that the correct general axiomatic scheme for a physical quantum theory of gravity is simply Atiyah's set of axioms [3] up to the finite dimensionality of the Hilbert state space. We denote a structure that satisfies all the Atiyah's axioms, except the finite dimensionality of the state space, as a *generalized topological theory.*

The theory we sketched in this paper is an example of such a generalized topological theory. We associate with the components ∂M_i of the boundary of M the infinite dimensional state space of the loop representation of quantum gravity. Equation (4) then provides a map, in Atiyah's sense, between two of these boundary components. Equivalently, it provides the definition of the Hilbert product in the state space.

Finally, we must recall that the computation of the evolution of expectation values in physical time (as opposed to coordinate time, which has no diffeomorphism invariant meaning) requires the use of a physical clock coupled to the theory (in principle, this could also be a component of the gravitational field itself) [19]. In this sense the integration (or the sum) over M is physically very analogous to the derivation of the propagator of a relativistic particle by means of an integral over the paths $x^\mu(\tau)$, where $\mu=0,\ldots,3$; in the particle case too, indeed, the scalar product between two wave functions on Minkowski space can be obtained by integrating over a five-dimensional manifold that interpolates two Minkowski spaces (see, for instance, [20]). Physical evolution, of course, is in x^0, not in τ. In addition, the quantization of the physical area is a non-gauge-invariant result, unless reinterpreted in a suitable gauge-invariant context [21].

Summarizing, we have shown the following. (i) The "colored triangulation" basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantization of 3D gravity is precisely the loop representation basis. (ii) We can interpret the quantization of the length in half-integer units in physical terms: the spectrum of the length operator has discrete half-integer eigenvalues. (iii) These lengths are related to the SU(2) representations because the quantum states that diagonalize the lengths are given (in the connection representation) by Wilson lines that cross the curve $2j$ times, or, equivalently, by one Wilson line in the j representation that crosses the curve. Motivated by these results, we have sketched a 4D combinatorial theory, based on a modification of Regge calculus. Many questions remain open as far as this theory is concerned, the most relevant ones being the relation between the Euclidean and the Lorentzian theory, and the convergence properties of the sum (3). Finally, we have proposed that Atiyah's axiomatization of topological field theories can be extended to theories with infinite degrees of freedom, and that this extension can be taken as the general formal structure of a quantum theory of gravity.

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