Noncircular axisymmetric stationary spacetimes

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A formalism is presented to treat axisymmetric stationary spacetimes in the most general case, when the stress-energy tensor is not assumed to be circular, so that one cannot make the usual foliation of spacetime into two orthogonal families of two-surfaces. Such a study is motivated by the consideration of rotating relativistic stars with strong toroidal magnetic field or meridional circulation of matter (convection). The formulation is based on a "(2+1)+1" slicing of spacetime and the corresponding projections of the Einstein equation. It offers a suitable frame to discuss the choice of coordinates appropriate for the description of asymptotically flat and noncircular axisymmetric spacetimes. We propose a certain class of coordinates which is interpretable in terms of extremal three- and two-surfaces. This choice leads to well-behaved elliptic operators in the equations for the metric coefficients. Consequently, in the case of a starlike object, the proposed coordinates are global ones, i.e., they can be extended to spatial infinity. These coordinates are also appropriate for obtaining initial conditions for (instability triggered) evolution, since they match naturally with coordinates proposed for dynamical evolution, especially with the "maximal time slicing" condition. The formulation is written in an entirely two-dimensional covariant form, but, in order to obtain numerical solutions, we also give the complete system of partial differential equations obtained by specialization of the equations to a certain subclass of the proposed coordinates.

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I. INTRODUCTION

In a recent work on "baby" rotating neutron stars (i.e., very hot neutron stars just formed at the end of the collapse of a massive star's core), we found that a very strong *toroidal* magnetic field ($\sim 10^{16}$ to 10^{17} G) is likely to be created by the differential rotation in the core of the neutron star [2]. Such a strong magnetic field is able to deform appreciably the neutron star. In order to describe these objects within the framework of general relativity, we have to be able to find axisymmetric steady-state solutions of Einstein equations when the stress-energy tensor is not circular, since a toroidal magnetic field results in a noncircular electromagnetic stress-energy tensor. Let us recall that, in an axisymmetric stationary spacetime, the stress-energy tensor $\mathbf{T}$ is said to be *circular* or *nonconvective* [3] if

$$\varepsilon_\alpha T^{\alpha[\beta}\varepsilon^\gamma \xi^{\delta]}=0 \ , \tag{1.1a}$$

$$\xi_\alpha T^{\alpha[\beta}\varepsilon^\gamma \xi^{\delta]}=0 \ , \tag{1.1b}$$

where $\varepsilon=\partial/\partial t$ and $\xi=\partial/\partial\varphi$ are two Killing vector fields associated, respectively, with stationarity and axisymmetry and square brackets denote antisymmetrization. The conditions (1.1) are equivalent to the absence of momentum currents in the meridional planes orthogonal to both ε and ξ. In the case of a fluid, this means that there is no convective motion but only circular motion around the axis of symmetry. Papapetrou [4] and Carter [5,3] have shown that when the stress-energy tensor is circular, the two-parameter group of spacetime isometries, $R(1)\times SO(2)$, is *orthogonally transitive,* i.e., there exists a family of two-surfaces everywhere orthogonal to the plane defined by the two Killing vectors ε and ξ. In this case, one may choose coordinates (t,x^1,x^2,φ) such that $\varepsilon=\partial/\partial t$, $\xi=\partial/\partial\varphi$ and (x^1,x^2) span the two-surfaces orthogonal to ε and ξ. The orthogonality property means that the components g_{01}, g_{02}, g_{31}, g_{32} of the metric tensor $\mathbf{g}$ in these coordinates are identically zero, so that

$$g_{\alpha\beta}dx^\alpha dx^\beta=-N^2dt^2+g_{\phi\phi}(d\phi-N^\phi dt)^2+g_{11}(dx^1)^2 +2g_{12}dx^1dx^2+g_{22}(dx^2)^2 \ , \tag{1.2}$$

where the functions N, N^ϕ, $g_{\phi\phi}$, g_{11}, g_{12}, and g_{22} depend on x^1 and x^2 only. As a consequence, the Einstein field equations written in these coordinates simplify dramatically.

The circular case, with the above adapted coordinates, has been studied widely in the past two decades, either in the slow rotation approximation [6–8] or in the exact case [9–20]. Especially, it has been shown [10,11,20] that the Einstein equation can be reduced to a set of four Poisson-like quasilinear partial differential equations (PDE's).

As far as we know, no study has been made in the more general case of noncircular axisymmetric stationary spacetime. Now, it can be seen easily that the necessary and sufficient condition for circularity, as stated by Carter [3] in terms of the electromagnetic field tensor $F_{\alpha\beta}$, is

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violated by a *toroidal* magnetic field, i.e., a magnetic field in the direction of the rotational Killing vector.[1] Thus, in order to study the magnetized neutron stars mentioned above, as well as convective relativistic bodies, we investigate the noncircular case in this paper. For that purpose, we develop what may be called a "(2+1)+1" formalism, which enables us to write the Einstein equations in a form well suited to numerical work, as well as to the discussion of the choice of coordinates. In this respect, we propose a class of coordinates which extends the coordinates adapted to the circular case described above. We argue that these coordinates are *global* ones, i.e., that they constitute a chart which covers the whole spacetime, this property being not trivial in the noncircular case (and even in the circular case, see Appendix A of Ref. [20]). Another interesting feature of these coordinates is that they satisfy the *maximal time slicing* condition widely used in gravitational collapse calculations, because of its singularity avoidance feature. This makes the solutions well suited to play the role of initial conditions for dynamical problems (for example gravitational collapse with strong magnetic field).

Within the proposed coordinates, the Einstein equations are reduced to a set of three scalar Poisson-like PDE's, one two-dimensional vector Poisson-like PDE and one three-dimensional vector Poisson-like PDE. All these equations are quasilinear and generalize the well-known equations of the circular case (as they are presented in Ref. [20]). Moreover all the equations are elliptic and can be solved by iterative numerical methods, for which the existence and uniqueness of a solution satisfying the boundary conditions of asymptotic flatness is guaranteed at each step of the iteration. As we show below, this is not the case with parabolic equations, for which a solution satisfying both the regularity conditions at the origin and that of asymptotic flatness cannot generally be found.

The plan of the paper is as follows: in Sec. II, we present the $3+1$ foliation of spacetime and the subsequent $2+1$ foliation of each 3-slice that we use. We also derive the corresponding orthogonal decomposition of the Einstein equation, by means of Gauss-Codazzi type relations that are established in Appendix A. In Sec. III we discuss the coordinate choice and introduce our privileged set of coordinates. The resulting equations are written in a two-dimensional covariant form. The complete system of PDE's for a specific coordinate subclass (isotropic meridional polar coordinates) is given in Appendix B in a form suitable for numerical integration. The equations governing the energy-momentum distribution of matter which ensures axisymmetry and stationarity are written in Sec. IV. Finally Sec. V contains the concluding discussion.

Notations and conventions

Greek indices $(\alpha,\beta,\mu,\nu,\ldots)$ range from 0 to 3, Latin indices alphabetically located after the letter i $(i,j,k,l,\ldots)$ range from 1 to 3, whereas Latin indices from the beginning of the alphabet $(a,b,c,\ldots)$ range from 1 to 2 only.

Three metric tensors are introduced throughout the paper: **g** on the whole spacetime, **h** and some hypersurfaces Σ_t, and **k** on some two-surfaces $\Sigma_{t\varphi}$. The corresponding covariant derivations are noted on the tensor indices by a semicolon for the 4-metric **g**; a single vertical stroke for the 3-metric **h**; a double vertical stroke for the 2-metric **k**. As usual, a partial derivative of a tensor component with respect to a given coordinate will be noted by a comma on the indices.

The signature of the 4-metric **g** is $(-,+,+,+)$ and the definition of the Riemann tensor follows the Misner-Thorne-Wheeler (MTW) sign convention [1].

Geometrized units, for which the gravitational constant G and the speed of light c are set equal to unity, are used throughout.

[1]The circular case corresponds instead to a purely *poloidal* magnetic field; we have presented the corresponding Einstein-Maxwell equations elsewhere [20].

II. (2+1)+1 DECOMPOSITION OF THE EINSTEIN EQUATION

We consider a spacetime $(\mathcal{E},\mathbf{g})$ which is *stationary, axisymmetric,* and *asymptotically flat. Stationarity* means that there exists a Killing vector field, ε, which is timelike at least at spatial infinity. This vector is defined up to a scale factor, which we fix by the requirement that the scalar product $\varepsilon_\alpha\varepsilon^\alpha$ tends to -1 at spatial infinity. *Axisymmetry* means there exists another Killing vector field, ξ, which vanishes on a timelike two-surface (called the *axis of symmetry),* is spacelike everywhere else and whose orbits are closed curves. ξ is normalized so that $(\xi_\alpha\xi^\alpha)_{;\beta}(\xi_\alpha\xi^\alpha)^{;\beta}/(4\xi_\alpha\xi^\alpha)$ tends to 1 on the axis of symmetry (this latter normalization ensures that the coordinate associated with ξ has the usual 2π periodicity).

We consider a coordinate system on $\mathcal{E}$, $(x^\alpha)=(t,x^1,x^2,\varphi)$, which is adapted to the spacetime symmetries in the sense that $x^0=t$ is an ignorable coordinate associated with the Killing vector ε and $x^3=\varphi$ is an ignorable coordinate associated with the Killing vector ξ:

$$\varepsilon=\frac{\partial}{\partial t} \text{ and } \xi=\frac{\partial}{\partial\varphi} . \tag{2.1}$$

Such a coordinate system is of course not unique, since any other set of coordinates (t',x'^1,x'^2,φ') deduced from the above one by the transformations

$$t'=t+\Psi(x^1,x^2) , \tag{2.2a}$$

$$\varphi'=\varphi+\Phi(x^1,x^2) , \tag{2.2b}$$

$$x'^1=f_1(x^1,x^2) , \tag{2.2c}$$

$$x'^2=f_2(x^1,x^2) , \tag{2.2d}$$

where Ψ, Φ, f_1, and f_2 are arbitrary smooth functions of (x^1,x^2), is still adapted to the spacetime symmetries. A specific coordinate choice, that will appear to be global, will be discussed in Sec. III.

A. 3+1 foliation of the whole spacetime

Let us designate by Σ_t the spacelike hypersurface of the spacetime $\mathcal{E}$ defined by $t=$const. $\mathcal{E}$ can be considered as being foliated by the family of Σ_t's, t ranging from $-\infty$ to $+\infty$. The decomposition of every tensor field between parts tangent to the Σ_t's, normal to them or mixed, and the corresponding decomposition of the Einstein field equation are the main features of the so-called *3+1 formalism* or *Cauchy problem* of general relativity. This well-known formulation arises mainly from the works of Lichnerowicz [21], Choquet-Bruhat [22], and Arnowitt, Deser, and Misner [23]. It is, for instance, extensively used in numerical studies of gravitational collapse. A modern and lucid account can be found in Ref. [24]. In the following, we recall briefly the main points of the 3+1 formalism in order primarily to fix the notations.

Let $\mathbf{n}$ be the unit timelike 4-vector orthogonal to Σ_t and oriented in the direction of increasing t:

$$n_\alpha = -N t_{,\alpha} \ . \tag{2.3}$$

The strictly positive coefficient N is called *the lapse function* and is determined by the requirement that $\mathbf{n}$ be normalized: $n_\alpha n^\alpha = -1$. Let $\mathbf{h}$ be the projection tensor orthogonally onto Σ_t:

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta \ . \tag{2.4}$$

Restricted to Σ_t, $\mathbf{h}$ defines the (positive definite) 3-metric induced by $\mathbf{g}$ on Σ_t. The associated covariant derivation, $^3\nabla$, is noted by a vertical stroke on the indices to distinguish from the covariant derivation $^4\nabla$ associated with the 4-metric $\mathbf{g}$, which is denoted by a semicolon. $^3\nabla$ can be deduced from $^4\nabla$ by the following formula valid for any tensor field $\mathbf{T}$ of type (p,q) on Σ_t:

$$T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q|\gamma} = h_{\mu_1}{}^{\alpha_1}\cdots h_{\mu_p}{}^{\alpha_p} h_{\beta_1}{}^{\nu_1}\cdots h_{\beta_q}{}^{\nu_q} h_\gamma{}^\sigma T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q;\sigma} \ . \tag{2.5}$$

We designate by $^3R_{\alpha\beta}$ the Ricci tensor of the 3-metric $\mathbf{h}$. Its relationship with the Ricci tensor of the 4-metric $\mathbf{g}$, $^4R_{\alpha\beta}$, is given by the Gauss and Codazzi equations recalled in Appendix A.

In general, the Killing vector $\partial/\partial t$ is not orthogonal to the hypersurface Σ_t; this leads to the definition of *shift vector* $\mathbf{N}$ as (minus) the orthogonal projection of $\partial/\partial t$ onto Σ_t:

$$N^\alpha = -h^\alpha{}_\sigma \left[\frac{\partial}{\partial t}\right]^\sigma \ . \tag{2.6}$$

The relationship between the vectors $\partial/\partial t$, $\mathbf{n}$, and $\mathbf{N}$ introduced above is

$$\left[\frac{\partial}{\partial t}\right]^\alpha = N n^\alpha - N^\alpha \ . \tag{2.7}$$

It follows that the components of the vector $\mathbf{n}$ with respect to the coordinates (x^α) are

$$n_\alpha = (-N,0,0,0) \text{ and } n^\alpha = \left[\frac{1}{N},\frac{N^1}{N},\frac{N^2}{N},\frac{N^3}{N}\right] , \tag{2.8}$$

where the N^i's are the contravariant components of the shift vector: $N^\alpha = (0,N^1,N^2,N^3)$.

The components $g_{\alpha\beta}$ and $g^{\alpha\beta}$ of the 4-metric $\mathbf{g}$ with respect to the coordinates (x^α) can be expressed in terms of the components of the 3-metric $\mathbf{h}$, those of the shift vector and the lapse function:

$$\begin{bmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{bmatrix} = \begin{bmatrix} N_k N^k - N^2 & -N_j \\ -N_i & h_{ij} \end{bmatrix} ; \tag{2.9a}$$

$$\begin{bmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{N^2} & -\dfrac{N^j}{N^2} \\ -\dfrac{N^i}{N^2} & h^{ij} - \dfrac{N^i N^j}{N^2} \end{bmatrix} . \tag{2.9b}$$

The acceleration 4-vector of $\mathbf{n}$, $\mathbf{a}$, is given by

$$a^\alpha = n^\sigma n^\alpha{}_{;\sigma} \ . \tag{2.10}$$

$\mathbf{a}$ is orthogonal to $\mathbf{n}$ (due to the normalization relation $n_\alpha n^\alpha = -1$), i.e., tangent to Σ_t. It can be expressed as the orthogonal projection onto Σ_t of the logarithmic gradient of the lapse:

$$a_\alpha = h_\alpha{}^\sigma (\ln N)_{,\sigma} = (\ln N)_{|\alpha} \ . \tag{2.11}$$

The imbedding of the surface Σ_t into $\mathcal{E}$ is characterized by the *extrinsic curvature tensor* $\mathbf{K}$, defined as

$$K_{\alpha\beta} = -\tfrac{1}{2}\mathcal{L}_{\mathbf{n}} h_{\alpha\beta} \ , \tag{2.12}$$

where $\mathcal{L}_{\mathbf{n}}$ denotes the Lie derivative along the vector field $\mathbf{n}$. It can easily be shown from Eqs. (2.4) and (2.3) that $K_{\alpha\beta}$ is symmetric (Weingarten identity) and that

$$K_{\alpha\beta} = -n_{\alpha;\beta} - a_\alpha n_\beta \ . \tag{2.13}$$

Alternative formulas for $K_{\alpha\beta}$ are

$$K_{\alpha\beta} = -h_\alpha{}^\mu h_\beta{}^\nu n_{\mu;\nu} = -h_\beta{}^\sigma n_{\alpha;\sigma} \ . \tag{2.14}$$

In particular, the trace K of $\mathbf{K}$ is linked to the 4-covariant divergence of $\mathbf{n}$ by

$$K = -n^\alpha{}_{;\alpha} \ . \tag{2.15}$$

The tensor $\mathbf{K}$ has the property of being tangent to the hypersurface Σ_t [cf. Eq. (2.14)]. Moreover, the Killing equation for $\partial/\partial t$, $(\partial/\partial t)_{(\alpha;\beta)} = 0$, once projected onto Σ_t, leads to the following relation between $\mathbf{K}$ and the 3-covariant derivatives of the shift vector:

$$K_{\alpha\beta} = -\frac{1}{2N}(N_{\alpha|\beta} + N_{\beta|\alpha}) \ . \tag{2.16}$$

In particular, the trace of $\mathbf{K}$ is related to the 3-covariant divergence of the shift vector by

$$K = -\frac{1}{N} N^\alpha{}_{|\alpha} \ . \tag{2.17}$$

B. 2+1 foliation of the t = const hypersurfaces

Let us define $\Sigma_{t\varphi}$ as the two-surface of intersection between the hypersurface t = const (i.e., Σ_t) and the hypersurface $\varphi=$const. Let $\mathbf{m}$ be the 4-vector field defined on $\mathcal{E}$ as the orthogonal projection onto Σ_t of the vector field $M\varphi_{,\alpha}$ (where $M\geq 0$) everywhere normal to the hypersurfaces $\varphi=$const and oriented in the direction of increasing φ:

$$m_\alpha = M h_\alpha{}^\sigma \varphi_{,\sigma} = M\varphi_{|\alpha} \ . \tag{2.18}$$

$\mathbf{m}$ is a vector field on $\mathcal{E}$ which is spacelike, by construction tangent to the Σ_t's, and normal to the two-surfaces $\Sigma_{t\varphi}$. Furthermore, we require that $\mathbf{m}$ be normalized:

$$m_\alpha m^\alpha = 1 \ . \tag{2.19}$$

This condition fixes the coefficient M of definition (2.18) as $M=(h^{\varphi\varphi})^{-1/2}$.

We can now introduce the projection tensor $\mathbf{k}$ orthogonally onto the two-surface $\Sigma_{t\varphi}$ as

$$\begin{aligned} k_{\alpha\beta} &= h_{\alpha\beta} - m_\alpha m_\beta \\ &= g_{\alpha\beta} + n_\alpha n_\beta - m_\alpha m_\beta \ . \end{aligned} \tag{2.20}$$

Restricted to $\Sigma_{t\varphi}$, $\mathbf{k}$ is the (positive definite) 2-metric induced by $\mathbf{g}$ on $\Sigma_{t\varphi}$. The associated covariant derivation, ${}^2\nabla$, is denoted by a double vertical stroke "$\|$" on the tensor indices to distinguish it from the two previously defined covariant derivations, ${}^4\nabla$ and ${}^3\nabla$, associated respectively, with the metrics $\mathbf{g}$ on $\mathcal{E}$ and $\mathbf{h}$ on Σ_t. ${}^2\nabla$ can be deduced from ${}^4\nabla$ by the following formula valid for any tensor field $\mathbf{T}$ of type (p,q) on $\Sigma_{t\varphi}$:

$$T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q\|\gamma} = k_{\mu_1}{}^{\alpha_1}\cdots k_{\mu_p}{}^{\alpha_p} k_{\beta_1}{}^{\nu_1}\cdots k_{\beta_q}{}^{\nu_q} k_\gamma{}^\sigma T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q;\sigma} \ . \tag{2.21}$$

${}^2\nabla$ can also be deduced from ${}^3\nabla$ by the same formula as above, replacing the 4-covariant derivative $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q;\sigma}$ by a 3-covariant one $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q|\sigma}$. We denote by ${}^2R_{\alpha\beta}$ the Ricci tensor of the 2-metric $\mathbf{k}$. Its relationship with the Ricci tensor of the 4-metric $\mathbf{g}$ and that of the 3-metric $\mathbf{k}$, ${}^4R_{\alpha\beta}$ and ${}^3R_{\alpha\beta}$, is given in Appendix A.

In general, the rotational Killing vector $\partial/\partial\varphi$ is not orthogonal to the two-surfaces $\Sigma_{t\varphi}$, and we define the 4-vector $\mathbf{M}$ as (minus) its orthogonal projection onto $\Sigma_{t\varphi}$:

$$M^\alpha = -k_\sigma{}^\alpha \left[\frac{\partial}{\partial\varphi}\right]^\sigma \ . \tag{2.22}$$

The orthogonal decomposition of $\partial/\partial\varphi$ with respect to the two-surfaces $\Sigma_{t\varphi}$ is then written

$$\left[\frac{\partial}{\partial\varphi}\right]^\alpha = M m^\alpha - M^\alpha \ . \tag{2.23}$$

It follows that the components of the vector $\mathbf{m}$ with respect to the coordinates $(x^\alpha)=(t,x^1,x^2,\varphi)$ are

$$m_\alpha = (-MN^\varphi,0,0,M) \text{ and } m^\alpha = \left[0,\frac{M^1}{M},\frac{M^2}{M},\frac{1}{M}\right] , \tag{2.24}$$

where the M^a's are the two nonvanishing contravariant components of the 4-vector $\mathbf{M}$: $M^\alpha=(0,M^1,M^2,0)$.

Similar to the relations (2.9) between the components of $\mathbf{g}$ and $\mathbf{h}$, we have the following relations between the components of $\mathbf{h}$ and $\mathbf{k}$:

$$\begin{bmatrix} h_{ab} & h_{a3} \\ h_{3b} & h_{33} \end{bmatrix} = \begin{bmatrix} k_{ab} & -M_a \\ -M_b & M^2+M_aM^a \end{bmatrix} , \tag{2.25a}$$

$$\begin{bmatrix} h^{ab} & h^{a3} \\ h^{3b} & h^{33} \end{bmatrix} = \begin{bmatrix} k^{ab}+\dfrac{M^aM^b}{M^2} & \dfrac{M^a}{M^2} \\ \dfrac{M^b}{M^2} & \dfrac{1}{M^2} \end{bmatrix} . \tag{2.25b}$$

Note that the sign differences between Eqs. (2.9) and (2.25) come from the one between Eqs. (2.4) and (2.20) and are due to the timelike character of $\mathbf{n}$ versus the spacelike character of $\mathbf{m}$. Combining Eqs. (2.9) and (2.25), we can express the total 4-metric $\mathbf{g}$ in terms of the 2-metric $\mathbf{k}$ on $\Sigma_{t\varphi}$, the shift $\mathbf{N}$, the vector $\mathbf{M}$, and the functions N and M as

$$\begin{aligned} g_{\alpha\beta}dx^\alpha dx^\beta = {} & -(N^2-N_iN^i)dt^2 - 2N_i dt dx^i \\ & + k_{ab}dx^a dx^b - 2M_a d\varphi dx^a \\ & + (M^2+M_aM^a)d\varphi^2 , \end{aligned} \tag{2.26}$$

where the functions N, N^i, M, M^a and k_{ab} depend only on the coordinates (x^1,x^2). Note that the determinant of $\mathbf{g}$ with respect to the coordinates (x^α) is linked to the determinants of $\mathbf{h}$ and $\mathbf{k}$ by the relations

$$\sqrt{-g} = N\sqrt{h} = NM\sqrt{k} \ . \tag{2.27}$$

Let us introduce the curvature vector $\mathbf{b}$ of the integral curves of $\mathbf{m}$ in the $(\Sigma_t,\mathbf{h})$ space:

$$b^\alpha = m^\sigma m^\alpha{}_{|\sigma} \ . \tag{2.28}$$

$\mathbf{b}$ is orthogonal to $\mathbf{m}$ (due to the normalization relation $m_\alpha m^\alpha=1$), i.e., tangent to the two-surfaces $\Sigma_{t\varphi}$. One may easily verify that $\mathbf{b}$ coincides with the orthogonal projection onto Σ_t of the acceleration of $\mathbf{m}$ in the $(\mathcal{E},\mathbf{g})$ space:

$$b^\alpha = h^\alpha{}_\mu m^\sigma m^\mu{}_{;\sigma} \ . \tag{2.29}$$

Similar to the relation between the acceleration $\mathbf{a}$ of $\mathbf{n}$ and the logarithmic gradient of N [Eq. (2.11)], $\mathbf{b}$ can be expressed as (minus) the orthogonal projection onto $\Sigma_{t\varphi}$ of the logarithmic gradient of the function M:

$$b_\alpha = -k_\alpha{}^\sigma(\ln M)_{,\sigma} = -(\ln M)_{\|\alpha} \ . \tag{2.30}$$

The imbedding of the two-surface $\Sigma_{t\varphi}$ into the 3-manifold Σ_t is characterized by its *extrinsic curvature tensor* $\mathbf{L}$, defined as

$$L_{\alpha\beta} = -\tfrac{1}{2}{}^3\mathcal{L}_{\mathbf{m}} k_{\alpha\beta} , \tag{2.31}$$

where ${}^3\mathcal{L}_{\mathbf{m}}$ stands for the Lie derivative along the vector field $\mathbf{m}$ *in the 3-manifold* Σ_t. It can easily be shown that ${}^3\mathcal{L}_{\mathbf{m}}k_{\alpha\beta}$ coincides with the orthogonal projection onto Σ_t of the Lie derivative $\mathcal{L}_{\mathbf{m}}k_{\alpha\beta}$ in the 4-manifold $\mathcal{E}$. From

Eqs. (2.20) and (2.18), $L_{\alpha\beta}$ is symmetric (Weingarten identity) and

$$L_{\alpha\beta}=-m_{\alpha|\beta}+b_\alpha m_\beta \ . \tag{2.32}$$

Alternative formulas for $L_{\alpha\beta}$ are

$$L_{\alpha\beta}=-k_\alpha{}^\mu k_\beta{}^\nu m_{\mu|\nu}=-k_\beta{}^\sigma m_{\alpha|\sigma} \ . \tag{2.33}$$

In particular, the trace L of $\mathbf{L}$ is linked to the 3-covariant divergence of $\mathbf{m}$ by

$$L=-m^\alpha{}_{|\alpha} \ . \tag{2.34}$$

The tensor $\mathbf{L}$ has the property of being tangent to the two-surface $\Sigma_{t\varphi}$ [cf. Eq. (2.33)]. Moreover, the Killing equation for $\partial/\partial\varphi$, $(\partial/\partial\varphi)_{(\alpha;\beta)}=0$, once projected onto $\Sigma_{t\varphi}$, leads to the following relation between $\mathbf{L}$ and the 2-covariant derivatives of the vector $\mathbf{M}$:

$$L_{\alpha\beta}=-\frac{1}{2M}(M_{\alpha\|\beta}+M_{\beta\|\alpha}) \ . \tag{2.35}$$

In particular, the trace of $\mathbf{L}$ is related to the 2-covariant divergence of the vector $\mathbf{M}$ by

$$L=-\frac{1}{M}M^\alpha{}_{\|\alpha} \ . \tag{2.36}$$

The shift vector $\mathbf{N}$ and the extrinsic curvature tensor of $\Sigma_t,\mathbf{K}$, are tangent to the hypersurface Σ_t; it is thus meaningful to consider their $2+1$ decomposition with respect to the two-surfaces $\Sigma_{t\varphi}$ (which are hypersurfaces of the manifold Σ_t). We write

$$N^\alpha=q^\alpha+\omega m^\alpha \ , \tag{2.37}$$

with

$$\omega=N^\mu m_\mu \tag{2.38a}$$

$$q^\alpha=k^\alpha{}_\mu N^\mu \tag{2.38b}$$

and

$$K_{\alpha\beta}=\kappa_{\alpha\beta}+m_\alpha\kappa_\beta+\kappa_\alpha m_\beta+\kappa m_\alpha m_\beta \ , \tag{2.39}$$

with

$$\kappa=K_{\mu\nu}m^\mu m^\nu \ , \tag{2.40a}$$

$$\kappa_\alpha=k_\alpha{}^\mu K_{\mu\nu}m^\nu \ , \tag{2.40b}$$

$$\kappa_{\alpha\beta}=k_\alpha{}^\mu k_\beta{}^\nu K_{\mu\nu} \ . \tag{2.40c}$$

The relation (2.16) between $\mathbf{K}$ and the 3-covariant derivatives of $\mathbf{N}$ can be translated by relations between κ, κ_α, $\kappa_{\alpha\beta}$ and the derivatives of q^α and ω as follows:

$$\kappa=-\frac{1}{MN}\left[\frac{M^\sigma}{M}\omega_{\|\sigma}+q^\sigma M_{\|\sigma}\right] , \tag{2.41a}$$

$$\kappa_\alpha=-\frac{1}{2N}\left[\frac{1}{M}[\mathbf{M},\mathbf{q}]_\alpha+M\left[\frac{\omega}{M}\right]_{\|\alpha}\right] , \tag{2.41b}$$

$$\kappa_{\alpha\beta}=-\frac{1}{2N}[q_{\alpha\|\beta}+q_{\beta\|\alpha}-2\omega L_{\alpha\beta}] \ , \tag{2.41c}$$

where $[\mathbf{M},\mathbf{q}]$ denotes the commutator of the vector fields $\mathbf{M}$ and $\mathbf{q}$: $[\mathbf{M},\mathbf{q}]^\alpha=M^\sigma q^\alpha{}_{\|\sigma}-q^\sigma M^\alpha{}_{\|\sigma}=M^\sigma q^\alpha{}_{,\sigma}-q^\sigma M^\alpha{}_{,\sigma}$ [cf. Eq. (A27)].

The point of view taken above amounts to considering the surfaces $\Sigma_{t\varphi}$ as hypersurfaces of the (sub)manifold Σ_t (hence the extrinsic curvature tensor $\mathbf{L}$). One may instead consider the $\Sigma_{t\varphi}$'s as two-dimensional submanifolds of the four-dimensional spacetime $\mathcal{E}$. The corresponding embedding is then naturally described by a type (1,2) tensor, called *the second fundamental tensor* of $\Sigma_{t\varphi}$ [the *first fundamental tensor* being the projection tensor $\mathbf{k}$ introduced in Eq. (2.20)], and defined by (see Ref. [25])

$$\mathcal{K}_{\alpha\beta}{}^\gamma=k_\alpha{}^\mu k_\beta{}^\nu k^\gamma{}_{\nu;\mu} \ . \tag{2.42}$$

We shall not use this tensor, but simply note that it is expressible in a simple way in terms of the various geometric objects introduced so far:

$$\mathcal{K}_{\alpha\beta}{}^\gamma=-\kappa_{\alpha\beta}n^\gamma+L_{\alpha\beta}m^\gamma \ . \tag{2.43}$$

C. Projections of the Einstein equation

The basic idea is to project the Einstein equation,

$$^4R_{\mu\nu}-\tfrac{1}{2}{}^4Rg_{\mu\nu}=8\pi T_{\mu\nu} \ , \tag{2.44}$$

onto the surfaces Σ_t and $\Sigma_{t\varphi}$ and along their respective normal vectors $\mathbf{n}$ and $\mathbf{m}$, in order to obtain equations that are covariant either for the 3-metric $\mathbf{h}$ or the 2-metric $\mathbf{k}$. These equations will be reduced rather easily to partial differential equations once a specific choice of the coordinates (x^1,x^2) is made.

We start by writing the $3+1$ decomposition of the stress-energy tensor $\mathbf{T}$ with respect to the hypersurfaces Σ_t:

$$T^{\alpha\beta}=S^{\alpha\beta}+n^\alpha J^\beta+J^\alpha n^\beta+En^\alpha n^\beta \ , \tag{2.45}$$

with

$$E=T^{\mu\nu}n_\mu n_\nu \ , \tag{2.46a}$$

$$J^\alpha=-h^\alpha{}_\mu T^{\mu\nu}n_\nu \ , \tag{2.46b}$$

$$S^{\alpha\beta}=h^\alpha{}_\mu h^\beta{}_\nu T^{\mu\nu} \ . \tag{2.46c}$$

The vector $\mathbf{J}$ and the tensor $\mathbf{S}$ defined by Eqs. (2.46b,c) are tangent to the hypersurface Σ_t; it is thus meaningful to consider their $2+1$ decomposition with respect to the two-surfaces $\Sigma_{t\varphi}$ and write

$$J^\alpha=j^\alpha+jm^\alpha \ , \tag{2.47}$$

with

$$j=J^\mu m_\mu \ , \tag{2.48a}$$

$$j^\alpha=k^\alpha{}_\mu J^\mu \ , \tag{2.48b}$$

and

$$S^{\alpha\beta}=s^{\alpha\beta}+m^\alpha s^\beta+s^\alpha m^\beta+sm^\alpha m^\beta \ , \tag{2.49}$$

with

$$s=S^{\mu\nu}m_\mu m_\nu=T^{\mu\nu}m_\mu m_\nu \ , \tag{2.50a}$$

$$s^\alpha=k^\alpha{}_\mu S^{\mu\nu}m_\nu=k^\alpha{}_\mu T^{\mu\nu}m_\nu \ , \tag{2.50b}$$

$$s^{\alpha\beta}=k^{\alpha}{}_{\mu}k^{\beta}{}_{\nu}S^{\mu\nu}=k^{\alpha}{}_{\mu}k^{\beta}{}_{\nu}T^{\mu\nu} . \quad (2.50c)$$

The various projections of the Einstein equation are made using the relations between the Ricci tensors ${}^4R_{\alpha\beta}$, ${}^3R_{\alpha\beta}$, ${}^2R_{\alpha\beta}$ of, respectively, **g**, **h**, and **k**, the extrinsic curvature tensors **K** and **L** of, respectively, Σ_t and $\Sigma_{t\varphi}$ and the acceleration vectors **a** and **b** of, respectively, **n** and **m**. All the relevant relations are derived in Appendix A from the Gauss and Codazzi equations.

First, by projecting the Einstein equation along **n** (i.e., by contracting Eq. (2.44) with $n^{\mu}n^{\nu}$) and using the contracted Gauss equation (A3) to substitute ${}^3R_{\mu\nu}$ and **K** for ${}^4R_{\mu\nu}$, we obtain the *Hamiltonian constraint equation* (see, e.g., Ref. [24]):

$$\tfrac{1}{2}({}^3R+K^2-K_{\mu\nu}K^{\mu\nu})=8\pi E . \quad (2.51)$$

Next, by projecting the Einstein equation onto the hypersurface Σ_t [i.e., by contracting Eq. (2.44) with $h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}$], using the Gauss equation under the form (A8), taking the trace and substituting Eq. (2.51) for ${}^3R$, we obtain the *lapse equation* (see, e.g., Ref. [24]):

$$N^{|i}{}_{|i}=N[4\pi(E+S_i^i)+K_{ij}K^{ij}]-N^iK_{,i} . \quad (2.52)$$

By contracting the Einstein equation (2.44) with $h^{i\mu}n^{\nu}$ and using the Codazzi equation (A2), we arrive at the *momentum constraint equation* (see, e.g., Ref. [24]):

$$K^{ij}{}_{|j}-K^{|i}=8\pi J^i . \quad (2.53)$$

Substituting for K^{ij} and K from Eqs. (2.16) and (2.17) yields

$$N^{i|j}{}_{|j}-N^j{}_{|j}{}^{|i}+{}^3R^i{}_jN^j=-16\pi NJ^i-2K^{ij}N_{,j}+2KN^{|i} . \quad (2.54)$$

Let us now write the 2+1 form of the Einstein equation in the hypersurfaces Σ_t. Projecting the Einstein equation onto $\Sigma_{t\phi}$ [i.e., contracting Eq. (2.44) with $k_{\alpha}{}^{\mu}k_{\beta}{}^{\nu}$] and taking the trace yields

$$k^{\mu\nu 4}R_{\mu\nu}-{}^4R=8\pi s^{\alpha}{}_{\alpha} . \quad (2.55)$$

We can reexpress the left-hand side of this equation by means of the relations (A21), (A18), and (A29) between ${}^4R_{\mu\nu}$ and ${}^2R_{\mu\nu}$, **L** and **K**. In this manner, we obtain an equation for the function MN:

$$(MN)^{\|a}{}_{\|a}+2M^a\left[\frac{M^b}{M}N_{\|b}\right]_{\|a}-M^a(LN)_{\|a}=8\pi MNs_a{}^a-2\kappa_a[\mathbf{M},\mathbf{q}]^a-M\left[q^a+\omega\frac{M^a}{M}\right](2\kappa+\kappa_b{}^b)_{\|a}$$
$$+MN(\kappa_{ab}\kappa^{ab}+2\kappa^2+\kappa\kappa_a{}^a-L_{ab}L^{ab}) . \quad (2.56)$$

Contracting the Einstein equation (2.44) with $k^{\alpha\mu}m^{\nu}$ gives

$$k^{\alpha\mu}m^{\nu 4}R_{\mu\nu}=8\pi s^{\alpha} . \quad (2.57)$$

The relations (A22) and (A30) then yield

$$-L^{ba}{}_{\|b}+L^{\|a}-\frac{1}{N}\left[\frac{M^b}{M}N_{\|b}\right]^{\|a}-L^{ab}\frac{N_{\|b}}{N}=8\pi s^a+\frac{1}{N}[\mathbf{q},\kappa]^a+\frac{\omega}{MN}[\mathbf{M},\kappa]^a$$
$$+(\kappa k^{ab}-\kappa^{ab})\frac{M}{N}\left[\frac{\omega}{M}\right]_{\|b}-2\kappa^a{}_b\kappa^b-\kappa_b{}^b\kappa^a . \quad (2.58)$$

By substituting for **L** its value in terms of the derivatives of **M** as given by the Killing equation (2.35) for $\partial/\partial\varphi$, we obtain an equation for the vector **M**:

$$M^{a\|b}{}_{\|b}-M^b{}_{\|b}{}^{\|a}+{}^2R^a{}_bM^b-\frac{2M}{N}\left[\frac{M^b}{M}N_{\|b}\right]^{\|a}+2L^{ab}\left[M_{\|b}-\frac{M}{N}N_{\|b}\right]-2LM^{\|a}$$
$$=16\pi Ms^a+\frac{2M}{N}[\mathbf{q},\kappa]^a+\frac{2\omega}{N}[\mathbf{M},\kappa]^a+2(\kappa k^{ab}-\kappa^{ab})\frac{M^2}{N}\left[\frac{\omega}{M}\right]_{\|b}-2M(2\kappa^a{}_b\kappa^b+\kappa_b{}^b\kappa^a) . \quad (2.59)$$

Finally, the Einstein equation projected along the vector **m** [i.e., Eq. (2.44) contracted with $m^{\mu}m^{\nu}$] gives

$${}^4R_{\mu\nu}m^{\mu}m^{\nu}-\tfrac{1}{2}{}^4R=8\pi s . \quad (2.60)$$

Making use of the relations (A19), (A18), and (A29) results in an equation involving the Laplacian of N with respect of the 2-metric **k** as well as the curvature scalar of **k**:

$$\frac{1}{N}N^{\|a}{}_{\|a}-\frac{1}{2}{}^2R-L\frac{M^a}{MN}N_{\|a}=8\pi s-\frac{1}{N}\left[q^a+\omega\frac{M^a}{M}\right]\kappa_b{}^b{}_{\|a}+\frac{2}{MN}\kappa_a[\mathbf{M},\mathbf{q}]^a+3\kappa_a\kappa^a+\frac{1}{2}[\kappa_{ab}\kappa^{ab}+(\kappa_b{}^b)^2+L_{ab}L^{ab}-L^2] . \quad (2.61)$$

Let us recall that, in the above equations, we use the convention on the index ranges stated at the beginning of the article.

D. Comparison with a previous (2+1)+1 formalism

Notice that the above treatment is different from the so-called "(2+1)+1 formalism" introduced by Maeda, Sasaki, Nakamura, and Miyama [26] and Sasaki [27] for spacetimes having a rotational Killing vector ξ. These authors are not interested by stationary axisymmetric configurations but rather by axisymmetric gravitational collapse. Their formalism is based on a work of Geroch [28], who introduced as three-dimensional space, the quotient space $\mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is the total spacetime and $\mathcal{S}$ is the collection of all orbits along the Killing vector field ξ. The $\mathcal{E}/\mathcal{S}$ space is sliced into spacelike hypersurfaces σ_t according to the usual $3+1$ (in this case $2+1$) formalism. This approach differs from ours in the following aspects: (i) we start by the time slicing and perform afterwards the azimuthal slicing, whereas Maeda *et al.* first make the azimuthal decomposition (by the quotient $\mathcal{E}/\mathcal{S}$) and then perform in a second step the time slicing; (ii) the metric on $\mathcal{E}/\mathcal{S}$ in $H_{\alpha\beta}=g_{\alpha\beta}-(\xi_\sigma\xi^\sigma)^{-2}\xi_\alpha\xi_\beta$; ξ being not hypersurface orthogonal, the resulting equations involve the twist of ξ and skew-symmetric tensors. As our privileged vectors are **n** and **m**, instead of ε and ξ, and are hypersurface orthogonal, our equations do not contain such terms and are very different from those of Maeda *et al.* In other words, our $(2+1)+1$ formalism is closer to the classical $3+1$ formalism spirit in the sense that each of the two decomposition steps introduces some hypersurfaces and their normal vector field, which is not the case of the first step of the $(2+1)+1$ formalism of Maeda *et al.*

III. CHOICE OF COORDINATES

A. Required coordinates properties

The above equations are fully covariant with respect to coordinates (x^1,x^2) (which span the subspace $\Sigma_{t\varphi}$) and have been derived under the assumption that the coordinates $x^0=t$ and $x^3=\varphi$ are adapted to the stationarity and axisymmetry of spacetime, i.e., are ignorable. As stated in the introduction of Sec. II, there is some freedom in the choice of such coordinates, which is expressed by the arbitrary nature of the functions Ψ, Φ, f_1, and f_2 in the transformation (2.2). We are going to take advantage of this freedom by choosing coordinates that fulfill the following requirements.

(i) The coordinate choice should lead to well-behaved elliptic operators in the above equations, so that their resolution can be reduced to an iterative procedure of solving Poisson-like equations at each step. In this manner, a solution that satisfies the boundary condition of asymptotic flatness is ensured to exist and to be unique *at each step* of the procedure and, if this latter converges, the final numerical solution will be an asymptotically flat solution of the Einstein equations. The introduced coordinates will then be *global* ones, i.e., they will constitute a chart which covers the whole spacetime (in the case of a starlike object) or at least the part from some horizon to infinity (in the case of a black hole).

(ii) In the circular limit, the coordinates should be the same as the usual ones (cf. Sec. I).

(iii) In order to consider noncircular axisymmetric stationary models as (unstable) initial conditions in a time evolution code for gravitational collapse, the coordinates should belong to one of the kind proposed for dynamical evolution (see, e.g., Ref. [29] for a review). This will avoid a cumbersome and numerically noisy change of coordinates at the beginning of the computation.

(iv) The coordinates should simplify as much as possible the field equations; in particular, they should minimize the number of second-order derivatives in the nonlinear terms (which will act as source terms in the Poisson-like equations that are to be solved numerically), since second-order derivatives are in general less precisely evaluated numerically than first order ones.

The requirement (i) is, from our point of view, the most crucial one. Indeed, global coordinates enable one to carry out the integration up to infinity, which is the only place where exact boundary conditions can be given, since it is the only place where the solution of the Einstein equation is known in advance (flat spacetime). In that spirit, we have recently devised a numerical scheme to integrate Poisson-like equations with a noncompactly supported (but asymptotically vanishing) source from the origin up to spatial infinity [20]. We want to point out that *algebraic* coordinates choices, made in order to simplify the components of the metric tensor by setting some of them to zero, do not generally fulfill the requirement (i). For example, the so-called *radial gauge* and *isothermal gauge,* (t,r,θ,φ), proposed by Bardeen and Piran [30] for axisymmetric spacetimes both demand that $M^r=0$ and $k_{r\theta}=0$ (within our notations). But, as recognized by these authors and others [31], this choice induces some irregularity in the equations for N^ϕ and M^θ, so that if one imposes that the solution be regular at the origin, it is no longer asymptotically flat. In other words, these coordinates are not global for asymptotically flat axisymmetric spacetimes (except in the nonrotating static case). The reason for this may be seen rather easily by considering Eq. (2.59) which determines the vector M^a. The principal linear terms on the left-hand side are the vector Laplacian minus the gradient of the divergence of M^a: $M^{a\|b}{}_{\|b}-M^b{}_{\|b}{}^{\|a}$. If the simplifying algebraic choice $M^r=0$ is made, the two-component equation (2.59) degenerates into one equation, involving the operator $M^{\theta\|b}{}_{\|b}-M^b{}_{\|b}{}^{\|\theta}$. Now, this operator is no longer elliptic, since the term $\partial^2M^\theta/\partial\theta^2$ arising from the gradient of the divergence cancels the corresponding one arising from the Laplacian. The operator is then merely parabolic and the resulting solution can fulfill only one of the conditions $M^a(r=0,\theta)=0$ (regularity at the origin) or $M^a(r=+\infty,\theta)=0$ (asymptotic flatness), since the integration has to proceed either inwards or outwards. Hence the radial and isothermal gauge lead to nonglobal coordinates. A convenient remedy is to require, as a coordinate choice, a divergence-free vector M^a: $M^b{}_{\|b}=0$. In this manner, the above operator becomes an elliptic vector Laplacian. As shown in Sec. III B, the solution will then be uniquely determined by the boundary condition $M^a(r=+\infty,\theta)=0$ and will automatically satisfy the regularity condition $M^a(r=0,\theta)=0$, provided that the source is well behaved (i.e., does not contain any monopolar term), which is the case of Eq. (2.59).

The requirement (ii) means that our choice of coordinates should allow that $N^r=N^\theta=0$ and $M^r=M^\theta=0$ in the circular limit.

Regarding the requirement (iii), we note that most gravitational collapse calculations make use of *maximal time slicing* coordinates for which the trace K of the extrinsic curvature tensor of the hypersurfaces Σ_t is identically zero [30,32,33]. The main reason for such a choice is that it gives asymptotically flat slices and has a very good singularity avoidance feature. Note that such a choice is compatible with the demand (ii) [20]. Note also that $K=0$ is equivalent to a divergence-free shift vector [cf. Eq. (2.17)], so that the operator in Eq. (2.54) for N^i becomes a well-behaved vector Laplacian, in full agreement with the demand (i), as we will see in Sec. III B.

The last requirement (iv) (simplification of the equations) has not to go against the requirement (i); for instance, we have seen that the simplifying choice $M^r=0$ is to be prohibited, whereas the choice $k_{12}=0$ seems to be allowed.

B. Maximal time slicing—conformally minimal azimuthal slicing

Following the above discussion, we consider the class of coordinates (t,x^1,x^2,φ) defined so that the associated foliations Σ_t and $\Sigma_{t\varphi}$ satisfy

$$n^\alpha{}_{;\alpha}=0 , \quad (3.1)$$

$$(Nm^\alpha)_{;\alpha}=0 . \quad (3.2)$$

The condition (3.1) means that the Σ_t's are *maximal* hypersurfaces of $(\mathscr{E},\mathbf{g})$. According to Eq. (2.15), it is equivalent to

$$K=\kappa_a{}^a+\kappa=0 . \quad (3.3)$$

As concerns the condition (3.2), it can be restated in terms of a covariant divergence in the $(\Sigma_t,\mathbf{h})$ space as

$$(N^2m^\alpha)_{|\alpha}=0 . \quad (3.4)$$

The exact analog for the two-surfaces $\Sigma_{t\varphi}$ of the maximal slicing choice (3.1) for Σ_t would have been instead $m^\alpha{}_{|\alpha}=0$: the $\Sigma_{t\varphi}$'s would then have been minimal[2] hypersurfaces of $(\Sigma_t,\mathbf{h})$. We prefer the slightly different choice (3.2) because it leads to greater simplifications in the gravitational field equations [requirement (iv) of Sec. III A]; in particular, it suppresses all the second-order derivatives of the lapse function N in Eqs. (2.56) for MN and (2.59) for M^a. In fact, the condition (3.2) can also be interpreted in terms of minimal slicing in the Σ_t space, provided that Σ_t is not supplied with the metric $\mathbf{h}$ induced by $\mathbf{g}$ but instead with the conformally related metric $\hat{\mathbf{h}}$ defined by

$$\hat{h}_{\alpha\beta}=N^2h_{\alpha\beta} . \quad (3.5)$$

Indeed, using (3.4), it can be easily shown that the condition (3.2) is equivalent to

$$\hat{m}^\alpha{}_{\uparrow\alpha}=0 , \quad (3.6)$$

where $\hat{m}^\alpha=(1/N)m^\alpha$ is the normal vector of the surfaces $\Sigma_{t\varphi}$ that has a unit norm for the conformal 3-metric $\hat{\mathbf{h}}$ and the symbol "$\uparrow$" denotes the covariant derivative with respect to $\hat{\mathbf{h}}$. Note that the condition (3.6) does mean the two-surfaces $\Sigma_{t\varphi}$ are *minimal* hypersurfaces of the space $(\Sigma_t,\hat{\mathbf{h}})$. Hence we propose to call the coordinates (t,x^1,x^2,φ) which satisfy (3.1) and (3.2), *maximal time slicing—conformally minimal azimuthal slicing* (MTCMA) coordinates.

It can be seen that any coordinate set adapted to axisymmetry and stationarity may be transformed, at least locally, to MTCMA coordinates by a suitable choice of the functions $\Psi(x^1,x^2)$ and $\Phi(x^1,x^2)$ in the transformation rules (2.2a) and (2.2b). Note that MTCMA coordinates define not a unique set but rather a class of coordinates since one has still the two degrees of freedom contained in the functions $f_1(x^1,x^2)$ and $f_2(x^1,x^2)$ that appear in Eqs. (2.2c) and (2.2d). In other words, the MTCMA choice fully determines the hypersurfaces Σ_t of $\mathscr{E}$ and the two-surfaces $\Sigma_{t\varphi}$ in Σ_t but let the coordinates (x^1,x^2) which spans $\Sigma_{t\varphi}$ unspecified. In Appendix B, we make the choice of *isotropic coordinates* $(x^1,x^2)=(r,\theta)$ such that the 2-metric $\mathbf{k}$ of $\Sigma_{t\varphi}$ satisfies $k_{r\theta}=0$ and $k_{\theta\theta}=r^2k_{rr}$, i.e.,

$$k_{ab}dx^a dx^b= A^2(r,\theta)[dr^2+r^2d\theta^2] . \quad (3.7)$$

Such a choice is always possible, at least locally, thanks to a suitable choice of the functions $f_1(x^1,x^2)$ and $f_2(x^1,x^2)$ in the transformation (2.2c) and (2.2d). But in the following, we keep the full covariance with respect to coordinates (x^1,x^2).

By means of Eqs. (2.8) and (2.24), the conditions (3.1) and (3.2) can be written as assumptions on the vectors $\mathbf{N}$ and $\mathbf{M}$, respectively:

$$N^i{}_{|i}=0 ; \quad (3.8)$$

$$(N^2M^a)_{\|a}=0 . \quad (3.9)$$

By developing Eq. (3.9) and making use of Eq. (2.36), we obtain an equivalent condition on the trace L of the extrinsic curvature tensor of the two-surfaces $\Sigma_{t\varphi}$ in the Riemannian space $(\Sigma_t,\mathbf{h})$:

$$L=\frac{2M^a}{MN}N_{\|a} . \quad (3.10)$$

[2] *Minimal* instead of *maximal* due to the signature $(+,+,+)$ of $\mathbf{h}$ instead of the signature $(-,+,+,+)$ of $\mathbf{g}$.

C. Gravitational field equations in MTCMA coordinates

Let us now write the gravitational field equations derived in Sec. II in MTCMA coordinates: Eqs. (2.52), (2.54), (2.56), (2.59), and (2.61) become, respectively,

$$N^{|i}{}_{|i}=N[4\pi(E+S_i{}^i)+K_{ij}K^{ij}] , \tag{3.11}$$

$$N^{i|j}{}_{|j}+{}^3R^i{}_jN^j=-16\pi NJ^i-2K^{ij}N_{,j} , \tag{3.12}$$

$$(MN)^{\|a}{}_{\|a}=8\pi MNs_a{}^a-2\kappa_a[\mathbf{M},\mathbf{q}]^a-M\left[q^a+\omega\frac{M^a}{M}\right]\kappa_{\|a}+MN(\kappa_{ab}\kappa^{ab}+\kappa^2-L_{ab}L^{ab}) , \tag{3.13}$$

$$M^{a\|b}{}_{\|b}+{}^2R^a{}_bM^b=16\pi Ms^a-2L^{ab}N\left[\frac{M}{N}\right]_{\|b}+\frac{L}{N}(MN)^{\|a}+\frac{2M}{N}[\mathbf{q},\kappa]^a$$
$$+\frac{2\omega}{N}[\mathbf{M},\kappa]^a+2(\kappa k^{ab}-\kappa^{ab})\frac{M^2}{N}\left[\frac{\omega}{M}\right]_{\|b}-2M(2\kappa^a{}_b\kappa^b-\kappa\kappa^a) , \tag{3.14}$$

$$\frac{1}{N}N^{\|a}{}_{\|a}-\frac{1}{2}{}^2R=8\pi s+\frac{1}{N}\left[q^a+\omega\frac{M^a}{M}\right]\kappa_{\|a}+\frac{2}{MN}\kappa_a[\mathbf{M},\mathbf{q}]^a+3\kappa_a\kappa^a+\frac{1}{2}(\kappa_{ab}\kappa^{ab}+\kappa^2+L_{ab}L^{ab}) . \tag{3.15}$$

Making the balance sheet for the number of equations and unknowns, one notes that the components of the system (3.11)–(3.15) involves eight equations. If the coordinate system is fully specified by, for example, a choice of the type (3.7) for (x^1,x^2), then one has also eight unknown functions: N, the three N^i's, M, the two M^a's and A (cf. Appendix B). Note that the Einstein equation has only $10-4=6$ independent components, thanks to the four Bianchi identities. We recover this fact by noticing that only six of our eight components are indeed independent, due to the coordinate choice which fixes the values of the divergence of N^i and M^a.

In order to discuss the existence and uniqueness of a solution of the system (3.11)–(3.15), let us investigate the properties of the various operators which appear on the equations' left-hand sides.

As concerns the scalar operators, they are merely scalar covariant Laplacians with respect to the 3-metric $\mathbf{h}$ for Eq. (3.11) or to the 2-metric $\mathbf{k}$ for Eq. (3.13) [and Eq. (3.15) if isotropic meridional coordinates are chosen, cf. Appendix B]. In general, the invertibility of the Laplacian in a noncompact Riemannian space is a delicate problem. However, in the case of asymptotically flat Riemannian spaces with the topology of $\mathbb{R}^3$ (which are relevant for "normal" stars without any horizon), a useful isomorphism theorem has been proved by Cantor [34,35] and can be applied to demonstrate the existence and uniqueness of a solution of the covariant Poisson equation (see Appendix B of Ref. [36]). Concerning Eq. (3.15), let us remark that if isotropic meridional coordinates (r,θ) are used, as suggested in Appendix B, this equation gives rise to an integral identity, which is a generalization of the virial theorem [37] and is very useful as a check of the accuracy of any numerical solution [20]. We will discuss this point in more detail in a forthcoming paper.

Let us now discuss the vector operators, which appear in Eqs. (3.12) and (3.14). They have the same form, except that one acts in the $(\Sigma_t,\mathbf{h})$ space and the other in the $(\Sigma_{t\varphi},\mathbf{k})$ space. Since $\mathbf{h}$ and $\mathbf{k}$ are both positive definite, we shall discuss Eq. (3.12) only; the conclusions will thus be valid for Eq. (3.14). The operator on the left-hand side of Eq. (3.12) is

$$(\tilde{\Delta}N)^i=N^{i|j}{}_{|j}+{}^3R^i{}_jN^j . \tag{3.16}$$

This operator is linear and very similar to those studied by York [38,39]:

$$(\Delta_KN)^i=N^{i|j}{}_{|j}+N^j{}_j{}^{|i}+{}^3R^i{}_jN^j , \tag{3.17a}$$

$$(\Delta_LN)^i=N^{i|j}{}_{|j}+\tfrac{1}{3}N^j{}_j{}^{|i}+{}^3R^i{}_jN^j . \tag{3.17b}$$

Moreover, since N^i is divergence-free, due to the maximal slicing condition (3.8), $(\tilde{\Delta}N)^i$, $(\Delta_KN)^i$, and $(\Delta_LN)^i$ are in fact identical. Let us consider the operator Δ_K; York [39] has shown that it is strongly elliptic and self-adjoint, and that in the case of an asymptotically flat space with the topology of $\mathbb{R}^3$, a solution N^i of $(\Delta_KN)^i=\sigma^i$ exists and is unique, provided that the source σ^i (here $\sigma^i=-16\pi NJ^i-2K^{ij}N_{,j}$) vanishes sufficiently fast at infinity (which is verified in physically relevant cases, see Appendix B of Ref. [36]). To apply this result to our operator $\tilde{\Delta}$, the crucial point is to notice that if one takes the divergence of $(\Delta_KN)^i=-16\pi NJ^i-2K^{ij}N_{,j}$ and makes use of the contracted Bianchi identities for ${}^3R_{ij}$, as well as the energy-momentum conservation equations written (with $K=0$) in Sec. IV, one is led to $N^i{}_{|i}{}^{|j}{}_{|j}=0$, which implies (in accordance to the above mentioned Cantor theorem) $N^i{}_{|i}=0$. Hence, the unique solution of $(\Delta_KN)^i=\sigma^i$ automatically satisfies the divergence-free condition $N^i{}_{|i}=0$. We therefore conclude that it is also the unique solution of $(\tilde{\Delta}N)^i=\sigma^i$, i.e., Eq. (3.12). In other words, provided that the matter source terms satisfy the energy-momentum conservation

[3]In the flat space case, this latter property corresponds to the well-known fact that the solution of a vector Poisson equation is divergence-free if, and only if, the source is divergence-free.

equations written under the hypothesis $K=0$ (cf. Sec. IV), there exists a unique solution N^i to Eq. (3.12), and this solution is automatically divergence-free[3], thanks to the Bianchi identities.

In summary, if the right-hand sides of the system of Eqs. (3.11)–(3.15) are held fixed, all the elliptic operators on the left-hand sides can be inverted to yield a unique asymptotically flat solution of the corresponding linear equations, at least in the case where Σ_t has the topology of $\mathbb{R}^3$. This strongly suggests that a complete solution of the system (3.11)–(3.15) may be found by an iterative method, which consists in inverting at each step the above linear elliptic operators. Of course, we have not demonstrated that such an iterative method will converge. All that we can assert is that, in the circular case, a similar method is seen to be numerically convergent [11,13,15,20].

From the numerical point of view, note that the iterative scheme may be slightly different as it is easier to invert at each iteration step the "flat space part" of the operators only, putting the "curvature part" with the source terms on the right-hand side. One is then led to invert flat space 3D and 2D scalar and vector Laplacians. In this respect, some techniques have been recently devised to solve Poisson equations with noncompactly supported source terms (which is the case here due to the contribution in the whole space of the quadratic terms) [18,20]. Basically, these methods use a change of variables of the type $u=1/r$ to map the infinite space external to the central object on a compact numerical grid. These methods also have the advantage of enabling one to impose exactly the boundary conditions of asymptotic flatness [cf. the discussion about the requirement (i) of Sec. III A].

D. Circular limit

In the circular limit, the 2-planes orthogonal to both the Killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$ are integrable in global two-surfaces [5]. We may choose naturally these two-surfaces as being $\Sigma_{t\varphi}$. The orthogonality of both $\partial/\partial t$ and $\partial/\partial\varphi$ with respect to $\Sigma_{t\varphi}$ is then expressed by [cf. Eqs. (2.6), (2.37), and (2.22)]

$$q^{\alpha}=0 \text{ and } M^{\alpha}=0 \ . \tag{3.18}$$

The immediate consequences of Eq. (3.18) are [cf. Eqs. (2.35) and (2.41a)]

$$L_{\alpha\beta}=0 \ , \tag{3.19a}$$

$$\kappa=0 \ , \tag{3.19b}$$

$$\kappa_{\alpha\beta}=0 \ , \tag{3.19c}$$

$$\kappa_{\alpha}=-\frac{M}{2N}\left[\frac{\omega}{M}\right]_{\|\alpha} . \tag{3.19d}$$

In particular, the relations $K=0$ [cf. Eq. (3.3)] and $(N^2M^a)_{\|a}=0$ are satisfied, so that the above choice for $\Sigma_{t\varphi}$ corresponds to MTCMA coordinates, in agreement with the requirement (ii) of Sec. III A.

As regards the stress-energy distribution in the circular case, Eqs. (3.18), (2.7), and (2.23) lead to $\varepsilon^{\alpha}=Nn^{\alpha}-\omega m^{\alpha}$ and $\xi^{\alpha}=Mm^{\alpha}$ so that one has, using Eqs. (2.45), (2.47), and (2.49),

$$\varepsilon_{\alpha}T^{\alpha[\beta}\varepsilon^{\gamma}\xi^{\delta]}=-NM(Nj^{[\beta}n^{\gamma}m^{\delta]}+\omega s^{[\beta}n^{\gamma}m^{\delta]}) \ , \tag{3.20a}$$

$$\xi_{\alpha}T^{\alpha[\beta}\varepsilon^{\gamma}\xi^{\delta]}=NM^2s^{[\beta}n^{\gamma}m^{\delta]} \ . \tag{3.20b}$$

By inserting Eqs. (3.20a) and (3.20b) into the circularity conditions (1.1a) and (1.1b), and contracting the result with $n_{\gamma}m_{\delta}$, we obtain the following constraints on the total stress-energy tensor:

$$j^{\alpha}=0 \ , \tag{3.21a}$$

$$s^{\alpha}=0 \ , \tag{3.21b}$$

which express the absence of meridional momentum, as well as anisotropic stress on any surface element in a meridional plane.

The metric equations of Sec. III B are reduced as follows. Equation (3.11) for N is left unchanged (in its 3-covariant form); the two first components of Eq. (3.12) are identically zero, due to Eqs. (3.18) and (3.21a); Eq. (3.13) for MN reduces to

$$(MN)^{\|a}{}_{\|a}=8\pi MNs_a{}^a \ , \tag{3.22}$$

the two components of Eq. (3.14) for M^a are identically zero, due to Eqs. (3.18) and (3.21b), and Eq. (3.15) is reduced to

$$\frac{1}{N}N^{\|a}{}_{\|a}-\tfrac{1}{2}{}^2R=8\pi s+3\kappa_a\kappa^a \ . \tag{3.23}$$

E. Relations with other coordinate systems

We have seen in the above section that, in the circular limit, MTCMA coordinates coincide with the usual coordinates adapted to orthogonal transitivity, and which have been used by all the authors mentioned in the introduction. Thus the requirement (ii) of Sec. III A is fulfilled.

As regards the requirement (iii) (compatibility with coordinate systems used in dynamical studies), we have already noticed (cf. Sec. III A) that the radial gauge and isothermal gauge used in axisymmetric gravitational collapse calculations [30,40,33] do not allow a global coverage of spacetime. MTCMA coordinates do not belong to this class of algebraic simplifying coordinates, having been introduced to remedy the difficulty of not being global. In fact, it can easily be shown that MTCMA coordinates belong to the class of *maximal time slicing-minimal distortion gauge* coordinates proposed by Smarr and York [36,24] for dynamical spacetimes and the generation of gravitational waves. Indeed, the minimal distortion gauge condition is equivalent to an elliptic equation for the shift vector **N** [Eq. (3.27) of Ref. [36]] which coincides with our Eq. (3.12) in the stationary case. Since the maximal time slicing-minimal distortion gauge seems to be well adapted to separate the "coordinate waves" from "pure gravitational waves," this property of MTCMA coordinates seems very attractive; it will enable us to use stationary unstable configurations constructed within MTCMA coordinates as initial conditions for gravitational collapse studies.

IV. ENERGY-MOMENTUM CONSERVATION

We give in this section the expression of the energy-momentum conservation equation,

$$T^{\mu\nu}{}_{;\nu}=0 , \tag{4.1}$$

in the stationary and axisymmetric case, in terms of the quantities E, j, j^α, s, s^α, and $s^{\alpha\beta}$ introduced in Sec. II C.

Projecting Eq. (4.1) onto $\mathbf{n}$ leads to the energy conservation equation

$$\frac{1}{N}\left[q^\sigma+\omega\frac{M^\sigma}{M}\right]E_{\|\sigma}+j^\sigma{}_{\|\sigma}+\frac{M^\sigma}{M}j_{\|\sigma}=\left[L-2\frac{M^\sigma}{MN}N_{\|\sigma}\right]j-\frac{1}{N^2M}(N^2M)_{\|\sigma}j^\sigma+\kappa_{\mu\nu}s^{\mu\nu}+2\kappa_\sigma s^\sigma+\kappa s+(\kappa+\kappa_\sigma{}^\sigma)E . \tag{4.2}$$

In MTCMA coordinates, this equation is simplified to

$$\frac{1}{N}\left[q^\sigma+\omega\frac{M^\sigma}{M}\right]E_{\|\sigma}+j^\sigma{}_{\|\sigma}+\frac{M^\sigma}{M}j_{\|\sigma}=-\frac{1}{N^2M}(N^2M)_{\|\sigma}j^\sigma+\kappa_{\mu\nu}s^{\mu\nu}+2\kappa_\sigma s^\sigma+\kappa s . \tag{4.3}$$

Note that in the circular limit (cf. Sec. III D), each side of Eq. (4.3) vanishes.

Projecting Eq. (4.1) onto $\mathbf{m}$ leads to the azimuthal momentum conservation equation

$$\frac{1}{N}\left[q^\sigma+\omega\frac{M^\sigma}{M}\right]j_{\|\sigma}+s^\sigma{}_{\|\sigma}+\frac{M^\sigma}{M}s_{\|\sigma}=-\frac{1}{NM^2}(NM^2)_{\|\sigma}s^\sigma$$
$$-\frac{j_\sigma}{MN}[\mathbf{M},\mathbf{q}]^\sigma-(E+s)\frac{M^\sigma}{MN}N_{\|\sigma}+Ls-L_{\mu\nu}s^{\mu\nu}+(2\kappa+k_\sigma{}^\sigma)j . \tag{4.4}$$

In MTCMA coordinates, this equation becomes

$$\frac{1}{N}\left[q^\sigma+\omega\frac{M^\sigma}{M}\right]j_{\|\sigma}+s^\sigma{}_{\|\sigma}+\frac{M^\sigma}{M}s_{\|\sigma}=-\frac{1}{NM^2}(NM^2)_{\|\sigma}s^\sigma-\frac{j_\sigma}{MN}[\mathbf{M},\mathbf{q}]^\sigma-(E-s)\frac{M^\sigma}{MN}N_{\|\sigma}-L_{\mu\nu}s^{\mu\nu}+\kappa j . \tag{4.5}$$

Note that in the circular limit (cf. Sec. III D), each side of Eq. (4.5) vanishes.

Finally, projecting Eq. (4.1) onto $\Sigma_{t\varphi}$ leads to the meridional momentum conservation equation

$$s_\alpha{}^\sigma{}_{\|\sigma}+\frac{1}{M}(M^\sigma s_{\alpha\|\sigma}+s^\sigma M_{\sigma\|\alpha})+\frac{1}{N}(q^\sigma j_{\alpha\|\sigma}+j^\sigma q_{\sigma\|\alpha})+\frac{\omega}{MN}(M^\sigma j_{\alpha\|\sigma}+j^\sigma M_{\sigma\|\alpha})$$
$$=-(Ek_\alpha{}^\sigma+s_\alpha{}^\sigma)\frac{1}{N}N_{\|\sigma}+(sk_\alpha{}^\sigma-s_\alpha{}^\sigma)\frac{1}{M}M_{\|\sigma}-j\frac{M}{N}\left[\frac{\omega}{M}\right]_{\|\alpha}+\left[L-\frac{M^\sigma}{MN}N_{\|\sigma}\right]s_\alpha-(\kappa+\kappa_\sigma{}^\sigma)j_\alpha . \tag{4.6}$$

In MTCMA coordinates, this equation becomes

$$s_\alpha{}^\sigma{}_{\|\sigma}+\frac{1}{M}(M^\sigma s_{\alpha\|\sigma}+s^\sigma M_{\sigma\|\alpha})+\frac{1}{N}(q^\sigma j_{\alpha\|\sigma}+j^\sigma q_{\sigma\|\alpha})+\frac{\omega}{MN}(M^\sigma j_{\alpha\|\sigma}+j^\sigma M_{\sigma\|\alpha})$$
$$=-(Ek_\alpha{}^\sigma+s_\alpha{}^\sigma)\frac{1}{N}N_{\|\sigma}+(sk_\alpha{}^\sigma-s_\alpha{}^\sigma)\frac{1}{M}M_{\|\sigma}-j\frac{M}{N}\left[\frac{\omega}{M}\right]_{\|\alpha}+\frac{M^\sigma}{MN}N_{\|\sigma}s_\alpha . \tag{4.7}$$

In the circular limit (cf. Sec. III D), Eq. (4.7) reduces to

$$s_\alpha{}^\sigma{}_{\|\sigma}=-(Ek_\alpha{}^\sigma+s_\alpha{}^\sigma)\frac{1}{N}N_{\|\sigma}+(sk_\alpha{}^\sigma-s_\alpha{}^\sigma)\frac{1}{M}M_{\|\sigma}-j\frac{M}{N}\left[\frac{\omega}{M}\right]_{\|\alpha} . \tag{4.8}$$

One may verify easily that it is the standard equation of stationary axisymmetric circular motion (compare with, e.g., Eq. (3.25) of Ref. [20]).

V. CONCLUSION

We have presented a formalism to treat axisymmetric stationary spacetimes in the most general case, when the stress-energy tensor is not assumed to be circular. Noncircularity in astrophysical objects applies to rotating neutron stars with a toroidal magnetic field or fluid convective motions. Our formulation is based on a $(2+1)+1$ slicing of spacetime and the corresponding projections of the Einstein equation. The foliation is associated with a coordinate choice of the type (t,x^1,x^2,φ), where t and φ are coordinates adapted, respectively, to stationarity and axisymmetry. The two other coordinates, (x^1,x^2), remains unspecified so that the formulation is fully covariant with respect to them.

The choice of the ignorable coordinates t and φ is not unique and the formalism presented here offers a suitable frame for discussing this choice. In particular, we have shown that algebraic coordinate choices, which consist of setting some of the components of the metric tensor (like $g_{r\varphi}$ or $g_{\theta\varphi}$) to zero in order to simplify the equations, do not yield global coordinate systems. We propose instead conditions of "fixed divergence" type, which are interpretable in terms of extremal slicing in the introduced $(2+1)+1$ foliations. This choice, namely *maximal time slicing-conformally minimal azimuthal slicing* (MTCMA), leads to well-behaved elliptic operators in the gravitational fields equations. These operators may be inverted at each step of an iterative procedure because of existence and uniqueness theorems, in the case of asymptotically flat and topologically euclidean spaces (as is the case for the spacetime generated by a rotating magnetized star). Consequently MTCMA coordinates are likely to be global if no horizon is present, i.e., to allow a complete coverage of the whole asymptotically flat spacetime. Let us stress that this global coverage is a desirable property when one wants to carry out an "exact" integration of the equations, since infinity is the only place where exact boundary conditions can be given, because it is the only place where the solution of the Einstein equation is known in advance (flat spacetime). Note as well that numerical methods exist that can carry out the integration from the interior of the central object to infinity [18,20], thanks to some compactification of the space external to the central object. Another interesting feature of MTCMA coordinates is that they belong to the *maximal slicing-minimal distortion gauge* class proposed for dynamical spacetimes and the generation of gravitational waves [36,24] so that one may use them to compute the (unstable) stationary initial conditions for a gravitational collapse calculation.

We have picked out a subclass of MTCMA coordinates, namely meridional isotropic MTCMA coordinates (t,r,θ,φ), in order to write explicitly the system of partial differential equations to be integrated numerically. The obtained system, though much longer than the circular one, still seems tractable given present day computers and will be implemented in the future to compute steady-state configurations of rotating neutrons stars with strong magnetic fields.

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APPENDIX A: RELATIONS BETWEEN THE VARIOUS RICCI AND EXTRINSIC CURVATURE TENSORS

1. ^4R in terms of $(^3R,K,a)$

The key equations are Gauss' equation

$$^3R^{\alpha}{}_{\beta\gamma\delta}=h^{\alpha}{}_{\mu}h^{\nu}{}_{\beta}h^{\rho}{}_{\gamma}h^{\sigma}{}_{\delta}{}^4R^{\mu}{}_{\nu\rho\sigma}-K^{\alpha}{}_{\gamma}K_{\beta\delta}+K^{\alpha}{}_{\delta}K_{\beta\gamma}\ , \tag{A1}$$

and Codazzi's equation

$$K^{\beta}{}_{\alpha|\beta}-K_{|\alpha}=-h^{\mu}{}_{\alpha}{}^4R_{\mu\nu}n^{\nu}\ . \tag{A2}$$

These relations result directly from the definition, via the Ricci identity, of the Riemann tensor $^3R^{\alpha}{}_{\beta\gamma\delta}$ of the 3-metric $\mathbf{h}$ and are established, for example, in Hawking and Ellis [41] [Eqs. (2.34) and (2.35); note that Hawking and Ellis' vector $\mathbf{n}$ and extrinsic curvature tensor $\mathbf{K}$ are both defined with a sign opposite from ours].

Contracting Eq. (A1) on the indices α and γ, and multiplying by $h^{\beta\gamma}$, yields a relation between the curvature scalars:

$$^3R={}^4R+2\,{}^4R_{\alpha\beta}n^{\alpha}n^{\beta}-K^2+K_{\alpha\beta}K^{\alpha\beta}\ . \tag{A3}$$

We can also deduce from Eq. (A1) a relation between the Ricci tensors. First, by contracting Eq. (A1) on the indices α and γ, we obtain

$$^3R_{\alpha\beta}=h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu\,4}R_{\mu\nu}-KK_{\alpha\beta}+K_{\alpha\sigma}K^{\sigma}{}_{\beta}+h_{\alpha}{}^{\nu}h_{\beta}{}^{\sigma}n^{\mu}n^{\rho\,4}R_{\mu\nu\rho\sigma}\ . \tag{A4}$$

Evaluating the last term of the right-hand side by means of the Ricci identity $^4R_{\mu\nu\rho\sigma}n^{\rho}=n_{\sigma;\mu\nu}-n_{\sigma;\nu\mu}$ and using Eq. (2.13) to express $n_{\mu;\nu}$ in terms of $K_{\mu\nu}$ and a_{μ} yields, after some rearrangement,

$$^3R_{\alpha\beta}=h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu\,4}R_{\mu\nu}+a_{\beta|\alpha}+a_{\alpha}a_{\beta}+n^{\sigma}K_{\alpha\beta;\sigma}-n_{\alpha}a^{\sigma}K_{\sigma\beta}-n_{\beta}a^{\sigma}K_{\sigma\alpha}-KK_{\alpha\beta}\ . \tag{A5}$$

We can replace the acceleration $\mathbf{a}$ by its expression (2.11) in terms of the lapse N and write Eq. (A5) as

$$^3R_{\alpha\beta}=h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu\,4}R_{\mu\nu}+\frac{1}{N}N_{|\alpha\beta}+n^{\sigma}K_{\alpha\beta;\sigma}-n_{\alpha}a^{\sigma}K_{\sigma\beta}-n_{\beta}a^{\sigma}K_{\sigma\alpha}-KK_{\alpha\beta}\ . \tag{A6}$$

The trace of the above equation gives another relation between the curvature scalars:

$$^3R={}^4R+{}^4R_{\mu\nu}n^{\mu}n^{\nu}+\frac{1}{N}N^{|\sigma}{}_{|\sigma}+n^{\sigma}K_{;\sigma}-K^2\ . \tag{A7}$$

By comparison with Eq. (A3), we obtain

$$^4R_{\mu\nu}n^{\mu}n^{\nu}=\frac{1}{N}N^{|\sigma}{}_{|\sigma}+n^{\sigma}K_{;\sigma}-K_{\mu\nu}K^{\mu\nu}\ , \tag{A8}$$

which is nothing more than the Raychaudhuri equation applied to the vector field **n** [recall that $K=-n^{\alpha}{}_{;\alpha}$, $(1/N)N^{|\sigma}{}_{|\sigma}=a^{\sigma}{}_{|\sigma}+a^{\sigma}a_{\sigma}=a^{\sigma}{}_{;\sigma}$ and $K_{\mu\nu}K^{\mu\nu}=n_{\mu;\nu}n^{\nu;\mu}$].

Substituting for ${}^4R_{\mu\nu}n^{\mu}n^{\nu}$ from Eq. (A8) into Eq. (A7), we obtain a relation between the curvature scalar that does not contain the Ricci tensor ${}^4R_{\mu\nu}$:

$$ {}^4R={}^3R-\frac{2}{N}N^{|\sigma}{}_{|\sigma}-2n^{\sigma}K_{;\sigma}+K^2+K_{\mu\nu}K^{\mu\nu} \; . \tag{A9}$$

2. ${}^3R$ in terms of $({}^2R,L,b)$

The relations are similar to the above ones, except that the Riemannian space under consideration is $(\Sigma_t,\mathbf{h})$, with the hypersurfaces $(\Sigma_{t\varphi},\mathbf{k})$—instead of $(\mathscr{E},\mathbf{g})$ with the hypersurfaces $(\Sigma_t,\mathbf{h})$—and that the vector **m** normal to the hypersurfaces $\Sigma_{t\varphi}$ has a scalar square equal to $+1$, whereas the vector **n** normal to the hypersurfaces Σ_t has a scalar square equal to -1. This latter difference results in some sign changes between the two families of formulas.

Gauss' equation now reads

$$ {}^2R^{\alpha}{}_{\beta\gamma\delta}=k^{\alpha}{}_{\mu}k^{\nu}{}_{\beta}k^{\rho}{}_{\gamma}k^{\sigma}{}_{\delta}{}^3R^{\mu}{}_{\nu\rho\sigma}+L^{\alpha}{}_{\gamma}L_{\beta\delta}-L^{\alpha}{}_{\delta}L_{\beta\gamma} \; , \tag{A10}$$

and Codazzi's equation

$$ L^{\beta}{}_{\alpha\|\beta}-L_{\|\alpha}=-k^{\mu}{}_{\alpha}{}^3R_{\mu\nu}m^{\nu} \; . \tag{A11}$$

By the same procedure as for Eqs. (A3) and (A5), we obtain, respectively,

$$ {}^2R={}^3R-2{}^3R_{\alpha\beta}m^{\alpha}m^{\beta}+L^2-L_{\alpha\beta}L^{\alpha\beta} \tag{A12}$$

and

$$ {}^2R_{\alpha\beta}=k_{\alpha}{}^{\mu}k_{\beta}{}^{\nu 3}R_{\mu\nu}-b_{\beta\|\alpha}+b_{\alpha}b_{\beta}-m^{\sigma}L_{\alpha\beta|\sigma} -m_{\alpha}b^{\sigma}L_{\sigma\beta}-m_{\beta}b^{\sigma}L_{\sigma\alpha}+LL_{\alpha\beta} \; . \tag{A13}$$

We can replace **b** by its expression (2.30) involving the function M and write Eq. (A13) as

$$ {}^2R_{\alpha\beta}=k_{\alpha}{}^{\mu}k_{\beta}{}^{\nu 3}R_{\mu\nu}+\frac{1}{M}M_{\|\alpha\beta}-m^{\sigma}L_{\alpha\beta|\sigma} -m_{\alpha}b^{\sigma}L_{\sigma\beta}-m_{\beta}b^{\sigma}L_{\sigma\alpha}+LL_{\alpha\beta} \; , \tag{A14}$$

the trace of which is

$$ {}^2R={}^3R-{}^3R_{\mu\nu}m^{\mu}m^{\nu}+\frac{1}{M}M^{\|\sigma}{}_{\|\sigma}-m^{\sigma}L_{|\sigma}+L^2 \; . \tag{A15}$$

By comparison with Eq. (A12), we obtain the Raychaudhuri equation for the vector field **m**:

$$ {}^3R_{\mu\nu}m^{\mu}m^{\nu}=-\frac{1}{M}M^{\|\sigma}{}_{\|\sigma}+m^{\sigma}L_{|\sigma}-L_{\mu\nu}L^{\mu\nu} \; . \tag{A16}$$

Substituting for ${}^3R_{\mu\nu}m^{\mu}m^{\nu}$ from Eq. (A16) into Eq. (A15), we obtain a relation between the curvature scalars that does not contain the Ricci tensor ${}^3R_{\mu\nu}$:

$$ {}^3R={}^2R-\frac{2}{M}M^{\|\sigma}{}_{\|\sigma}+2m^{\sigma}L_{|\sigma}-L^2-L_{\mu\nu}L^{\mu\nu} \; . \tag{A17}$$

3. ${}^4R$ in terms of $({}^2R,K,L,a,b)$

Substituting for ${}^3R$ from Eq. (A17) into Eq. (A9) yields

$$ {}^4R={}^2R-\frac{2}{N}N^{\|\sigma}{}_{\|\sigma}-\frac{2}{M}M^{\|\sigma}{}_{\|\sigma}-\frac{2m^{\nu}}{N}(m^{\mu}N_{|\mu})_{|\nu}-\frac{2}{MN}M^{\|\sigma}N_{\|\sigma}+2L\frac{m^{\sigma}}{N}N_{|\sigma} -2n^{\sigma}K_{;\sigma}+2m^{\sigma}L_{|\sigma}+K^2+K_{\mu\nu}K^{\mu\nu}-L^2-L_{\mu\nu}L^{\mu\nu} \; . \tag{A18}$$

Multiplying Eq. (A6) by $m^{\alpha}m^{\beta}$ and using Eq. (A16) gives

$$ {}^4R_{\mu\nu}m^{\mu}m^{\nu}=-\frac{1}{M}M^{\|\sigma}{}_{\|\sigma}-\frac{m^{\nu}}{N}(m^{\mu}N_{|\mu})_{|\nu}-\frac{1}{MN}M^{\|\sigma}N_{\|\sigma}+m^{\sigma}L_{|\sigma}-m^{\mu}m^{\nu}(n^{\sigma}K_{\mu\nu;\sigma}-KK_{\mu\nu})-L_{\mu\nu}L^{\mu\nu} \; . \tag{A19}$$

Substituting for ${}^3R_{\mu\nu}$ from Eq. (A6) into Eq. (A14) yields

$$ k_{\alpha}{}^{\mu}k_{\beta}{}^{\nu 4}R_{\mu\nu}={}^2R_{\alpha\beta}-\frac{1}{N}N_{\|\alpha\beta}-\frac{1}{M}M_{\|\alpha\beta}+L_{\alpha\beta}\frac{m^{\sigma}}{N}N_{|\sigma}+m^{\sigma}L_{\alpha\beta|\sigma}-LL_{\alpha\beta}-k_{\alpha}{}^{\mu}k_{\beta}{}^{\nu}(n^{\sigma}K_{\mu\nu;\sigma}-KK_{\mu\nu}) +m_{\alpha}b^{\sigma}L_{\sigma\beta}+m_{\beta}b^{\sigma}L_{\sigma\alpha} \; . \tag{A20}$$

Taking the trace (with respect of the 2-metric **k**) of the above equation gives

$$ k^{\mu\nu 4}R_{\mu\nu}={}^2R-\frac{1}{N}N^{\|\sigma}{}_{\|\sigma}-\frac{1}{M}M^{\|\sigma}{}_{\|\sigma}+L\frac{m^{\sigma}}{N}N_{|\sigma}+m^{\sigma}L_{|\sigma}-n^{\sigma}K_{;\sigma}+m^{\mu}m^{\nu}(n^{\sigma}K_{\mu\nu;\sigma}-KK_{\mu\nu})+K^2-L^2 \; . \tag{A21}$$

By multiplying Eq. (A6) by $h_{\gamma}{}^{\alpha}m^{\beta}$ and inserting the result into Codazzi's equation (A11), one obtains

$$ k_{\alpha}{}^{\mu}m^{\nu 4}R_{\mu\nu}=-L^{\sigma}{}_{\alpha\|\sigma}+L_{\|\alpha}-\frac{1}{N}(m^{\sigma}N_{|\sigma})_{\|\alpha}-\frac{1}{N}L^{\sigma}_{\alpha}N_{\|\sigma}-k_{\alpha}{}^{\mu}m^{\nu}(n^{\sigma}K_{\mu\nu;\sigma}-KK_{\mu\nu}) \; . \tag{A22}$$

4. Simplifications in the stationary axisymmetric case

The above equations are general and do not use the fact that the spacetime under consideration is stationary and axisymmetric. In this latter case, derivatives along the vectors **n** and **m** can be simplified as follows:

(i) For any scalar field A which respects the stationarity and the axisymmetry, we have (the notations are those of Sec. II)

$$n^\sigma A_{;\sigma} = \frac{1}{N}\left[q^\sigma + \omega \frac{M^\sigma}{M} \right] A_{\|\sigma} , \qquad (A23)$$

$$m^\sigma A_{|\sigma} = \frac{M^\sigma}{M} A_{\|\sigma} . \qquad (A24)$$

(ii) For any vector field **v** that is tangent to Σ_t and respects the stationarity, we have

$$n^\sigma v^\alpha{}_{;\sigma} = \frac{1}{N}[\mathbf{N},\mathbf{v}]^\alpha - K^\alpha{}_\sigma v^\sigma + v^\sigma a_\sigma n^\alpha . \qquad (A25)$$

(iii) For any vector field **w** that is tangent to $\Sigma_{t\varphi}$ and respects the axisymmetry, we have

$$m^\sigma w^\alpha{}_{|\sigma} = \frac{1}{M}[\mathbf{M},\mathbf{w}]^\alpha - L^\alpha{}_\sigma w^\sigma - w^\sigma b_\sigma m^\alpha . \qquad (A26)$$

In these equations appear the commutators

$$[\mathbf{N},\mathbf{v}]^\alpha = N^\sigma v^\alpha{}_{|\sigma} - v^\sigma N^\alpha{}_{|\sigma} , \qquad (A27a)$$

$$[\mathbf{M},\mathbf{w}]^\alpha = M^\sigma w^\alpha{}_{\|\sigma} - w^\sigma M^\alpha{}_{\|\sigma} . \qquad (A27b)$$

They take an alternative form in which the covariant derivatives can be replaced by partial derivatives of the components, which is useful from a computational point of view.

Using these relations, we obtain

$$n^\sigma m^\alpha{}_{;\sigma} = -\frac{1}{NM}[\mathbf{M},\mathbf{q}]^\alpha - \kappa^\alpha + m^\sigma a_\sigma n^\alpha . \qquad (A28)$$

By introducing the 2+1 decomposition of **K** (cf. Sec. II), we can express the contractions of $n^\sigma K_{\mu\nu;\sigma}$ with $m^\mu m^\nu$ and $k_\alpha{}^\mu m^\nu$, which appear in the relations between ${}^4R_{\alpha\beta}$ and ${}^2R_{\alpha\beta}$ above, as

$$m^\mu m^\nu (n^\sigma K_{\mu\nu;\sigma} - K K_{\mu\nu}) = \frac{1}{N}\left[q^\sigma + \omega \frac{M^\sigma}{M} \right] \kappa_{\|\sigma} + \frac{2}{NM}\kappa_\sigma [\mathbf{M},\mathbf{q}]^\sigma + 2\kappa_\sigma \kappa^\sigma - \kappa^2 - \kappa \kappa_\sigma{}^\sigma \qquad (A29)$$

and

$$k^{\alpha\mu} m^\nu (n^\sigma K_{\mu\nu;\sigma} - K K_{\mu\nu}) = \frac{1}{N}[\mathbf{q},\kappa]^\alpha + \frac{\omega}{MN}[\mathbf{M},\kappa]^\alpha + (\kappa k^{\alpha\sigma} - \kappa^{\alpha\sigma}) \frac{M}{N} \left[\frac{\omega}{M} \right]_{\|\sigma} - 2\kappa^\alpha{}_\sigma \kappa^\sigma - \kappa_\sigma{}^\sigma \kappa^\alpha . \qquad (A30)$$

Note that in the circular limit (cf. Sec. III D), the expressions (A29) and (A30) reduce, respectively, to $2\kappa_\sigma\kappa^\sigma$ and 0.

APPENDIX B: THE EQUATIONS IN MTCMA ISOTROPIC COORDINATES

In this appendix, we specify fully the coordinate system by choosing the MTCMA slicing introduced in Sec. III B as well as taking *isotropic polar coordinates* $(x^1,x^2)=(r,\theta)$ on the two-surfaces $\Sigma_{t\varphi}$. These coordinates take their values in $[0,+\infty[\times[0,\pi]$, the axis of symmetry containing the point $r=0$ and being characterized by $\theta=0$ and $\theta=\pi$, and are such that the 2-metric **k** of $\Sigma_{t\varphi}$ reads

$$k_{ab}dx^a dx^b = A^2(r,\theta)[dr^2 + r^2 d\theta^2] . \qquad (B1)$$

Thanks to the allowed coordinate transform (2.2c) and (2.2d), it is always possible to find, at least locally, such a coordinate system.

This choice being made, the gravitational field equations (3.11)–(3.15) can be written as a system of eight coupled partial differential equations for the eight functions $N(r,\theta)$, $N^r(r,\theta)$, $N^\theta(r,\theta)$, $N^\varphi(r,\theta)$, $M(r,\theta)$, $M^r(r,\theta)$, $M^\theta(r,\theta)$, and $A(r,\theta)$. For computational convenience, the following quantities are introduced:

$$\nu(r,\theta) = \ln N(r,\theta) , \qquad (B2a)$$

$$\alpha(r,\theta) = \ln A(r,\theta) , \qquad (B2b)$$

$$B(r,\theta) = M(r,\theta)/(r \sin\theta) , \qquad (B2c)$$

$$\beta(r,\theta) = \ln B(r,\theta) , \qquad (B2d)$$

$$\mu(r,\theta) = \ln M(r,\theta) = \ln(r\sin\theta) + \beta(r,\theta) . \qquad (B2e)$$

The function B is introduced in order to single out the vanishing behavior (in $r\sin\theta$) of M on the axis of symmetry; in the flat space limit, $B=1$. Note that the function μ is not defined on the axis of symmetry.

With the above coordinates, the 2-metric **k** has the conformally flat expression (B1). Consequently, the 2-covariant operators which appear in Eqs. (3.13) and (3.14) take a simple form [in accordance with the requirement (iv) of Sec. III A]. For example, the scalar Laplacian with respect to **k** reduces to the flat space Laplacian expressed in polar coordinates, up to a scale factor $1/A^2$ (see, e.g., Appendix D of Ref. [42]). Then, the only complex expressions to evaluate are the vector Laplacian with respect to the 3-metric **h** and the Ricci tensor of **h** which both appear in the shift equation (3.12). We performed this tedious calculus with help of a Mathematica [43] algebraic computing code, which is presented, as well as the various tests it has passed, in Ref. [20].

The equation (3.11) for the lapse function $N=\exp(\nu)$ becomes

$$A^{-2}\{\nu_{,rr}+(1/r+\mu_{,r})\nu_{,r}+\nu_{,\theta\theta}/r^2+\mu_{,\theta}\nu_{,\theta}/r^2\}+[A^{-2}+(m^r)^2](\nu_{,r})^2+[(rA)^{-2}+(m^\theta)^2](\nu_{,\theta})^2+(m^r)^2\nu_{,rr}+2m^r m^\theta \nu_{,r\theta}$$
$$+(m^\theta)^2\nu_{,\theta\theta}+(m^r m^r_{,r}+m^\theta m^r_{,\theta})\nu_{,r}+(m^r m^\theta_{,r}+m^\theta m^\theta_{,\theta})\nu_{,\theta}=4\pi(E+S^i_i)+K_{ij}K^{ij}+L^2/2\ . \quad \text{(B3)}$$

In the above equation, partial derivatives are noted in subscripts as commas. Use has been made of Eqs. (2.36) and (3.10) to let the term L^2 appear. Note that, if μ is replaced by its expression (B2e), one recognizes inside the curly brackets the 3D flat space Laplacian of ν [expressed in spherical coordinates (r,θ,φ)] plus the corrective quadratic term $\beta_{,r}\nu_{,r}+\beta_{,\theta}\nu_{,\theta}/r^2$. Besides, recall that m^r and m^θ are linked to M^r, M^θ, and M by Eq. (2.24): $m^r=M^r/M$ and $m^\theta=M^\theta/M$.

The three components of Eq. (3.12) for the shift (N^r,N^θ,N^φ) turn out to be

r component:

$$[A^{-2}+(m^r)^2](N^r_{,rr}+(r^{-1}+\mu_{,r})N^r_{,r}-[r^{-2}+(\mu_{,r})^2]N^r)+[(rA)^{-2}+(m^\theta)^2]N^r_{,\theta\theta}+[(rA)^{-2}-(m^\theta)^2]\mu_{,\theta}N^r_{,\theta}$$
$$-(2/r)[A^{-2}+(m^r)^2]N^\theta_{,\theta}-\{[A^{-2}-(m^r)^2]M_{,r\theta}/M+[A^{-2}+(m^r)^2]\mu_{,r}\mu_{,\theta}\}N^\theta+2m^r m^\theta N^r_{,r\theta}$$
$$+N^r_{,r}\{2[A^{-2}+(m^r)^2]\alpha_{,r}+m^r m^\theta[2\alpha_{,\theta}-\mu_{,\theta}]+m^r M^\theta_{,\theta}/M+m^\theta M^r_{,\theta}/M\}$$
$$+N^r_{,\theta}\{2[(rA)^{-2}+2(m^\theta)^2]\alpha_{,\theta}+m^r m^\theta[\mu_{,r}+4\alpha_{,r}+1/r]-(m^r/r^2)M^r_{,\theta}/M+(m^\theta/M)[M^r_{,r}+2M^\theta_{,\theta}]\}$$
$$+N^\theta_{,r}\{2A^{-2}\alpha_{,\theta}+2(m^r)^2\mu_{,\theta}-2(m^r/M)M^r_{,\theta}\}$$
$$+N^\theta_{,\theta}\{-2[A^{-2}+(m^r)^2]\alpha_{,r}+2m^r m^\theta[\mu_{,\theta}-\alpha_{,\theta}]-2m^r M^\theta_{,\theta}/M+(m^\theta/M)[r^2M^\theta_{,r}-M^r_{,\theta}]\}$$
$$+N^\varphi_{,\theta}\{2M^r/(Ar)^2[\mu_{,\theta}-\alpha_{,\theta}]+2(M^\theta/A^2)[\alpha_{,r}-\mu_{,r}+1/r+(Am^r)^2/r]+2m^r m^\theta[M^\theta_{,\theta}-M^r_{,r}]$$
$$-[(rA)^{-2}-(m^r/r)^2+(m^\theta)^2]M^r_{,\theta}+[A^{-2}+(m^r)^2-(rm^\theta)^2]M^\theta_{,r}\}$$
$$+N^r\{2[A^{-2}+2(m^r)^2]\alpha_{,r}[\mu_{,r}-1/r]-[A^{-2}-(m^r)^2]M_{,rr}/M$$
$$+m^r m^\theta[-\mu_{,r}\mu_{,\theta}+4\alpha_{,\theta}\mu_{,r}-2\alpha_{,\theta}/r-2\alpha_{,r}\alpha_{,\theta}+M_{,r\theta}/M-2A_{,r\theta}/A]+(m^r)^2[2\mu_{,r}/r-2(\alpha_{,r})^2-2A_{,rr}/A]$$
$$+(m^r/M)[M^r_{,r}(2\mu_{,r}-8\alpha_{,r}-2/r)-M^\theta_{,r}(\mu_{,\theta}+2\alpha_{,\theta})+M^\theta_{,\theta}(2\mu_{,r}-2\alpha_{,r}-2/r)-2M^r_{,rr}-M^\theta_{,r\theta}]$$
$$+(m^\theta/M)[M^r_{,r}(\mu_{,\theta}-4\alpha_{,\theta})-M^r_{,r\theta}]-(1/M^2)[2(M^r_{,r})^2+M^r_{,r}M^\theta_{,\theta}+r^2(M^\theta_{,r})^2]\}$$
$$+N^\theta\{(2/A^2)\alpha_{,\theta}\mu_{,r}+m^r m^\theta[4\alpha_{,\theta}\mu_{,\theta}-2(\alpha_{,\theta})^2-(\mu_{,\theta})^2+M_{,\theta\theta}/M-2A_{,\theta\theta}/A]$$
$$+2(m^r)^2[\mu_{,\theta}/r-\alpha_{,\theta}/r-\alpha_{,r}\alpha_{,\theta}+2\mu_{,\theta}\alpha_{,r}-A_{,r\theta}/A]$$
$$+(m^r/M)[2M^r_{,r}(\mu_{,\theta}-\alpha_{,\theta})-2M^r_{,\theta}(3\alpha_{,r}+1/r)+M^\theta_{,\theta}(\mu_{,\theta}-4\alpha_{,\theta})-M^\theta_{,\theta\theta}-2M^r_{,r\theta}]$$
$$+(m^\theta/M)[M^r_{,\theta}(\mu_{,\theta}-4\alpha_{,\theta})-M^r_{,\theta\theta}/M]-(1/M^2)[M^r_{,\theta}M^\theta_{,\theta}+2M^r_{,r}M^r_{,\theta}+r^2M^\theta_{,r}M^\theta_{,\theta}]\}$$
$$=-16\pi NJ^r-2K^{rr}N_{,r}-2K^{r\theta}N_{,\theta}\ . \quad \text{(B4a)}$$

θ component:

$$
\begin{aligned}
&[A^{-2}+(m^r)^2]N^\theta_{,rr}+[(3/r+\mu_{,r})/A^2+(m^r)^2(3/r-\mu_{,r})]N^\theta_{,r}+[(rA)^{-2}+(m^\theta)^2][N^\theta_{,\theta\theta}+\mu_{,\theta}N^\theta_{,\theta}]\\
&\quad -N^\theta([(rA)^{-2}+(m^\theta)^2](\mu_{,\theta})^2+[(rA)^{-2}-(m^\theta)^2]M_{,\theta\theta}/M)+2/r^3A^{-2}N^r_{,\theta}+2m^rm^\theta N^\theta_{,r\theta}\\
&\quad +N^r_{,r}\{-2[(rA)^{-2}+(m^\theta)^2]\alpha_{,\theta}+2m^rm^\theta[\mu_{,r}-\alpha_{,r}]+(1/M)[m^r(M^r_{,\theta}/r^2-M^\theta_{,r})-2m^\theta M^r_{,r}]\}\\
&\quad +N^r_{,\theta}\{2/(rA)^2\alpha_{,r}+2(m^\theta)^2\mu_{,r}-2m^\theta M^\theta_{,r}/M\}\\
&\quad +N^\theta_{,r}\{2[A^{-2}+2(m^r)^2]\alpha_{,r}+m^rm^\theta[\mu_{,\theta}+4\alpha_{,\theta}]+(1/M)[m^r(2M^r_{,r}+M^\theta_{,\theta})-m^\theta r^2M^\theta_{,r}]\}\\
&\quad +N^\theta_{,\theta}\{2[(rA)^{-2}+(m^\theta)^2]\alpha_{,\theta}+m^rm^\theta(2\alpha_{,r}-\mu_{,r}+1/r)+(1/M)[m^rM^\theta_{,r}+m^\theta M^r_{,r}]\}\\
&\quad +N^\varphi_{,r}\{2M^r(rA)^{-2}[\alpha_{,\theta}-\mu_{,\theta}]+2(M^\theta/A^2)[\mu_{,r}-\alpha_{,r}-1/r-(Am^r)^2/r]+2m^rm^\theta[M^r_{,r}-M^\theta_{,\theta}]\\
&\qquad +(1/r^2)[A^{-2}-(m^r)^2+(rm^\theta)^2]M^r_{,\theta}-[A^{-2}+(m^r)^2-(rm^\theta)^2]M^\theta_{,r}\}\\
&\quad +N^r\{((2/r^3)A^{-2}-[(rA)^{-2}+(m^\theta)^2]\mu_{,r})\mu_{,\theta}-[(rA)^{-2}-(m^\theta)^2]M_{,r\theta}/M+2/r[(Ar)^{-2}-(m^\theta)^2]\alpha_{,\theta}\\
&\qquad +(2/(rA)^2)\alpha_{,r}\mu_{,\theta}+m^rm^\theta[2\mu_{,r}/r-4\alpha_{,r}/r-1/r^2+4\alpha_{,r}\mu_{,r}-2(\alpha_{,r})^2-(\mu_{,r})^2+M_{,rr}/M-2A_{,rr}/A]\\
&\qquad +(m^\theta)^2[(4\mu_{,r}-2\alpha_{,r})\alpha_{,\theta}-2A_{,r\theta}/A]+m^r/M[(\mu_{,r}-4\alpha_{,r}-3/r)M^\theta_{,r}-M^\theta_{,rr}]\\
&\qquad +(m^\theta/M)[M^r_{,r}(\mu_{,r}-4\alpha_{,r}-1/r)-6M^\theta_{,r}\alpha_{,\theta}-2M^\theta_{,\theta}(-\mu_{,r}+\alpha_{,r}+1/r)-M^r_{,rr}-2M^\theta_{,r\theta}]\\
&\qquad -(1/M^2)[M^r_{,r}M^r_{,\theta}/r^2+M^r_{,r}M^\theta_{,r}+2M^\theta_{,r}M^\theta_{,\theta}]\}\\
&\quad +N^\theta\{2[(Ar)^{-2}+2(m^\theta)^2]\alpha_{,\theta}\mu_{,\theta}+m^rm^\theta[2\mu_{,\theta}/r-2\alpha_{,\theta}/r-\mu_{,r}\mu_{,\theta}+4\alpha_{,r}\mu_{,\theta}-2\alpha_{,r}\alpha_{,\theta}+M_{,r\theta}/M-2A_{,r\theta}/A]\\
&\qquad -2(m^\theta)^2[(\alpha_{,\theta})^2+A_{,\theta\theta}/A]+(m^r/M)[M^\theta_{,\theta}(\mu_{,r}-4\alpha_{,r}-3/r)-M^\theta_{,r\theta}]\\
&\qquad +(m^\theta/M)[2M^r_{,r}(\mu_{,\theta}-\alpha_{,\theta})-M^r_{,\theta}(\mu_{,r}+2\alpha_{,r}+1/r)+2M^\theta_{,\theta}(\mu_{,\theta}-4\alpha_{,\theta})-M^r_{,r\theta}/M-2M^\theta_{,\theta\theta}/M-2A_{,\theta\theta}/A]\\
&\qquad -(1/M^2)[(M^r_{,\theta}/r)^2+2(M^\theta_{,\theta})^2+M^r_{,r}M^\theta_{,\theta}]\}=-16\pi NJ^\theta-2K^{\theta r}N_{,r}-2K^{\theta\theta}N_{,\theta}\ . \qquad \text{(B4b)}
\end{aligned}
$$

φ component:

$$
\begin{aligned}
&[A^{-2}+(m^r)^2]N^\varphi_{,rr}+((1/r)[A^{-2}+(m^r)^2]+[3A^{-2}-(m^r)^2]\mu_{,r})N^\varphi_{,r}+[(rA)^{-2}+(m^\theta)^2]N^\varphi_{,\theta\theta}\\
&\quad +[3(rA)^{-2}-(m^\theta)^2]\mu_{,\theta}N^\varphi_{,\theta}+2m^rm^\theta N^\varphi_{,r\theta}+2(N^r_{,r}/M)\{m^r[\mu_{,r}-\alpha_{,r}]-m^\theta\alpha_{,\theta}-M^r_{,r}/M\}\\
&\quad +(N^r_{,\theta}/M)\{2m^\theta\mu_{,r}-(1/M)[M^\theta_{,r}+M^r_{,\theta}/r^2]\}\\
&\quad +(N^\theta_{,r}/M)\{2m^r\mu_{,\theta}-(1/M)[r^2M^\theta_{,r}+M^r_{,\theta}]\}+2(N^\theta_{,\theta}/M)\{m^\theta[\mu_{,\theta}-\alpha_{,\theta}]-m^r[\alpha_{,r}+1/r]-M^\theta_{,\theta}/M\}\\
&\quad +N^\varphi_{,r}\{-m^rm^\theta\mu_{,\theta}+4m^r[m^r\alpha_{,r}+m^\theta\alpha_{,\theta}]+(m^r/M)[4M^r_{,r}+M^\theta_{,\theta}]+(m^\theta/M)[2M^r_{,\theta}+r^2M^\theta_{,r}]\}\\
&\quad +N^\varphi_{,\theta}\{-m^rm^\theta\mu_{,r}+4m^\theta[m^r\alpha_{,r}+m^\theta\alpha_{,\theta}]+3m^rm^\theta/r+(m^r/M)[M^r_{,\theta}/r^2+2M^\theta_{,r}]+(m^\theta/M)[M^r_{,r}+4M^\theta_{,\theta}]\}\\
&\quad +(N^r/M)\{m^r[2\mu_{,r}/r-4\alpha_{,r}/r-1/r^2-(\mu_{,r})^2-2(\alpha_{,r})^2+4\alpha_{,r}\mu_{,r}+M_{,rr}/M-2A_{,rr}/A]\\
&\qquad\quad +m^\theta[-\mu_{,r}\mu_{,\theta}+4\alpha_{,\theta}\mu_{,r}-2\alpha_{,\theta}/r-2\alpha_{,r}\alpha_{,\theta}+M_{,r\theta}/M-2A_{,r\theta}/A]\\
&\qquad\quad +(1/M)[M^r_{,r}(\mu_{,r}-4\alpha_{,r}-1/r)-M^\theta_{,r}(\mu_{,\theta}+2\alpha_{,\theta})+2M^\theta_{,\theta}(\mu_{,r}-\alpha_{,r}-1/r)-M^r_{,rr}-M^\theta_{,r\theta}]\}\\
&\quad +(N^\theta/M)\{m^r[-\mu_{,r}\mu_{,\theta}+(4\alpha_{,r}+2/r)\mu_{,\theta}-2\alpha_{,\theta}/r-2\alpha_{,r}\alpha_{,\theta}+M_{,r\theta}/M-2A_{,r\theta}/A]\\
&\qquad\quad +m^\theta[-(\mu_{,\theta})^2-2(\alpha_{,\theta})^2+4\alpha_{,\theta}\mu_{,\theta}+M_{,\theta\theta}/M-2A_{,\theta\theta}/A]\\
&\qquad\quad +(1/M)[2M^r_{,r}(\mu_{,\theta}-\alpha_{,\theta})-M^r_{,\theta}(\mu_{,r}+2\alpha_{,r}+1/r)+M^\theta_{,\theta}(\mu_{,\theta}-4\alpha_{,\theta})-M^r_{,r\theta}-M^\theta_{,\theta\theta}]\}\\
&\qquad\qquad =-16\pi NJ^\varphi-2K^{\varphi r}N_{,r}-2K^{\varphi\theta}N_{,\theta}\ . \qquad \text{(B4c)}
\end{aligned}
$$

Equation (3.13) for the product MN becomes

$$
\begin{aligned}
A^{-2}\{(MN)_{,rr}+(MN)_{,r}/r+(MN)_{,\theta\theta}/r^2\}&=8\pi MNs_a{}^a-2\kappa_r[\mathbf{M},\mathbf{q}]^r-2\kappa_\theta[\mathbf{M},\mathbf{q}]^\theta-M(q^r+\omega M^r/M)\kappa_{,r}\\
&\quad -M(q^\theta+\omega M^\theta/M)\kappa_{,\theta}+MN(\kappa_{ab}\kappa^{ab}+\kappa^2-L_{ab}L^{ab})\ . \qquad \text{(B5)}
\end{aligned}
$$

In the above equation, the commutator $[\mathbf{M},\mathbf{q}]$ has not been written explicitly: one should replace it by its expression (A27b), with partial derivatives. Note that the term inside the curly brackets is nothing else but the 2D flat space Laplacian of MN [expressed in polar coordinates (r,θ)].

r component:

$$A^{-2}\{M^r_{,rr}+(1/r+2\alpha_{,r})M^r_{,r}+M^r_{,\theta\theta}/r^2+(2/r^2)\alpha_{,\theta}M^r_{,\theta}-(1/r^2+2\alpha_{,r}/r)M^r+2\alpha_{,\theta}M^\theta_{,r}-2(1/r+\alpha_{,r})M^\theta_{,\theta}\}$$
$$=16\pi Ms^r-2L^{rr}N(M/N)_{,r}-2L^{r\theta}N(M/N)_{,\theta}+(L/NA^2)(MN)_{,r}+2(M/N)[\mathbf{q},\kappa]^r+2(\omega/N)[\mathbf{M},\kappa]^r$$
$$+2(\kappa A^{-2}-\kappa^{rr})(M^2/N)(\omega/M)_{,r}-2\kappa^{r\theta}(M^2/N)(\omega/M)_{,\theta}-2M(2\kappa^r{}_{,r}\kappa^r+2\kappa^r{}_{,\theta}\kappa^\theta-\kappa\kappa^r)\ . \quad \text{(B6a)}$$

θ component:

$$A^{-2}\{M^\theta_{,rr}+(3/r+2\alpha_{,r})M^\theta_{,r}+M^\theta_{,\theta\theta}/r^2+(2/r^2)\alpha_{,\theta}M^\theta_{,\theta}-(2/r^2)\alpha_{,\theta}M^r_{,r}+(2/r^3+2/r^2\alpha_{,r})M^r_{,\theta}+(2/r^3)\alpha_{,\theta}M^r\}$$
$$=16\pi Ms^\theta-2L^{\theta r}N(M/N)_{,r}-2L^{\theta\theta}N(M/N)_{,\theta}+(L/r^2NA^2)(MN)_{,\theta}+2(M/N)[\mathbf{q},\kappa]^\theta+2(\omega/N)[\mathbf{M},\kappa]^\theta$$
$$-2\kappa^{\theta r}(M^2/N)(\omega/M)_{,r}+[\kappa(rA)^{-2}-\kappa^{\theta\theta}](M^2/N)(\omega/M)_{,\theta}-2M(2\kappa^\theta{}_{,r}\kappa^r+2\kappa^\theta{}_{,\theta}\kappa^\theta-\kappa\kappa^\theta)\ . \quad \text{(B6b)}$$

Finally, Eq. (3.15) gives an equation for the sum $\alpha+\nu=\ln(AN)$, and thus determines the conformal factor A [ν being determined by Eq. (B3)]:

$$A^{-2}\{(\alpha+\nu)_{,rr}+(\alpha+\nu)_{,r}/r+(\alpha+\nu)_{,\theta\theta}/r^2\}$$
$$=8\pi s-A^{-2}[(\nu_{,r})^2+(\nu_{,\theta})^2/r^2]+(1/N)(q^r+\omega M^r/M)\kappa_{,r}+(1/N)(q^\theta+\omega M^\theta/M)\kappa_{,\theta}+(2/MN)\kappa_r[\mathbf{M},\mathbf{q}]^r$$
$$+(2/MN)\kappa_\theta[\mathbf{M},\mathbf{q}]^\theta+3\kappa_a\kappa^a+1/2(\kappa_{ab}\kappa^{ab}+\kappa^2+L_{ab}L^{ab})\ . \quad \text{(B7)}$$

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