On self-dual gravity

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We study the Ashtekar-Jacobson-Smolin equations that characterize four-dimensional complex metrics with self-dual Riemann tensor. We find that we can characterize any self-dual metric by a function that satisfies a nonlinear evolution equation, to which the general solution can be found iteratively. This formal solution depends on two arbitrary functions of three coordinates. We study the symmetry algebra of these equations and find that they admit a generalized $W_\infty \oplus W_\infty$ algebra. We then find the associated conserved quantities which are found to have vanishing Poisson brackets (up to surface terms). We construct explicitly some families of solutions that depend on two free functions of two coordinates, included in which are the multi-center metrics of Gibbons and Hawking. Finally, in an appendix, we show how our formulation of self-dual gravity is equivalent to that of Plebañski.

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I. INTRODUCTION

In four dimensions the Hodge duality operation takes two-forms to two-forms. Given a four-dimensional metric, the most important two-form associated with it is the curvature two-form $R^a{}_b$. It is therefore natural to be interested in four-dimensional metrics whose curvature form obeys the self-duality relation

$$R^a{}_b = {}^*R^a{}_b, \tag{1}$$

where * is the Hodge duality operator. We will refer to such metrics as "self-dual." Such metrics automatically have vanishing Ricci tensor, and so satisfy the vacuum Einstein equations with a vanishing cosmological constant. Unfortunately, the only real Lorentzian self-dual metric is flat Minkowski space, so we choose instead to work with metrics with four *complex* dimensions.

Physically, these metrics may be of interest in attempts to quantize gravity. It has been suggested that we may be able to interpret self-dual metrics as "one-particle states" in a quantized gravity theory [1]. Alternatively, in the path-integral approach to Euclidean quantum gravity, such metrics will make large contributions to a path integral over metrics, since they are saddle points of the classical Einstein-Hilbert action [2]. Also, by analogy with Yang-Mills theory, we can look for "instanton" solutions — complete, nonsingular solutions with curvature that dies away at large distances [3,4]. Solutions that are asymptotically flat at spatial infinity, but periodic in time, then contribute to a thermal canonical ensemble [5]. Solutions that are asymptotically flat in the four-dimensional sense (asymptotically locally Euclidean) can be interpreted as tunneling amplitudes between inequivalent gravitational vacua.

From a purely mathematical point of view these metrics are interesting since they are "hyper-Kähler." Hyper-Kähler manifolds are Riemannian dimensional manifolds that admit three automorphisms $\mathbf{J}^i$ of the tangent bundle which obey the quaternion algebra and are covariantly constant [6]. In other words,

$$\nabla \mathbf{J}^i = 0, \quad \mathbf{J}^i \mathbf{J}^j = -\delta_{ij} + \epsilon_{ijk} \mathbf{J}^k, \tag{2}$$

where ∇ is the covariant derivative with respect to the Levi-Civita connection. In four dimensions, it turns out that for a metric $\mathbf{g}$ to be hyper-Kähler it must have either self-dual, or anti-self-dual, curvature tensor [7].

The problem of constructing metrics with self-dual curvature tensor has been tackled in several ways. The most direct approach is to formulate the problem in terms of partial differential equations [8,9]. An alternative approach is Penrose's "nonlinear graviton" technique [1]. Here, the task of solving partial differential equations is replaced by that of constructing deformed twistor spaces, and holomorphic lines on them. In practice this turns out to be just as difficult as solving partial differential equations, but in principle one can construct the general self-dual metric in this way.

Here we concentrate on partial differential equations. In Sec. II we find a formulation which is similar to Plebañski's first heavenly equation [8], but which can be viewed as simply an evolution equation. This means that the free functions in our solution are just a field and its time derivative on some initial hypersurface, i.e., two free functions of three coordinates. We construct, in a somewhat formal manner, the general solution to this equation in Sec. III. Section IV is devoted to an analysis of the symmetries and conservation laws associated with the system. The conclusion is that the system admits a symmetry group which is a generalization of $W_\infty \oplus W_\infty$ and we find two infinite-dimensional families of conserved quantities which have vanishing Poisson

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brackets. This could be looked on as an explicit verification of the fact that self-dual gravity is integrable. In Sec. V we construct explicitly some infinite-dimensional families of solutions to our equations with triholomorphic Killing vectors. The analysis of the holomorphic Killing vector case is, however, incomplete. Finally, in an Appendix, we show how this formulation of self-dual gravity is equivalent to Plebański's.

II. CONSTRUCTION OF SELF-DUALITY CONDITION

In [10] the equations for complex self-dual metrics were reformulated in terms of the new Hamiltonian variables for general relativity introduced in [11]. By fixing the four manifold to be of the form $\mathcal{M} = \Sigma \times R$ and using the coordinate T to foliate the manifold, they reduced the problem of finding self-dual metrics to that of finding a triad of complex vectors $\{\mathbf{V}_i : i = 1,2,3\}$ that satisfy the equations

$$\mathrm{Div}\,\mathbf{V}_i = 0, \tag{3}$$

$$\frac{\partial}{\partial T}\mathbf{V}_i = \frac{1}{2}\epsilon_{ijk}[\mathbf{V}_j, \mathbf{V}_k]. \tag{4}$$

Defining the densitized inverse three-metric

$$\hat{q}^{ab} = V_i^a V_j^b \delta_{ij}, \tag{5}$$

we recover the undensitized inverse three-metric q^{ab} by the relation $q^{ab} = \hat{q}\,\hat{q}^{ab}$, where $\hat{q} = \det \hat{q}_{ab} = (\det \hat{q}^{ab})^{-1}$. If we now define the lapse function N by $N = (\det q_{ab})^{1/2}$, then we find that the metric defined by the line element

$$ds^2 = N^2\,dT^2 + q_{ab}\,dx^a\,dx^b \tag{6}$$

is self-dual.

Later, it was found that this triad of vectors could be related to the complex structures $\mathbf{J}^i$ that hyper-Kähler metrics admit [12]. Given a self-dual metric, we choose local coordinates (T, x^a) to put the line element in the form of Eq. (6). If we define the triad of vectors $\mathbf{V}_i = -\mathbf{J}^i(*, \partial_T)$, then these vectors will satisfy (3) and (4).

Here we will concentrate on the problem of finding *local* solutions to Eqs. (3) and (4). We thus introduce a local coordinate chart (X, Y, Z) on the three-surface Σ with its natural flat metric and connection. Thus (3) becomes just

$$\frac{\partial}{\partial x^a} V_i^a = 0. \tag{7}$$

The crucial step is to realize that we can write Eq. (4) as

$$\left[\frac{\partial}{\partial T}, \mathbf{V}_i\right] = \frac{1}{2}\,\epsilon_{ijk}\,[\mathbf{V}_j, \mathbf{V}_k]. \tag{8}$$

If we consider only Euclidean metrics, then we take the $\mathbf{V}_i$ to be real. In this case we define two complex vectors $\mathbf{A}, \mathbf{B}$ by

$$\mathbf{A} = \frac{\partial}{\partial T} + i\,\mathbf{V}_1, \quad \mathbf{B} = \mathbf{V}_2 - i\,\mathbf{V}_3, \tag{9}$$

which, by virtue of (8), obey the Lie bracket algebra

$$[\mathbf{A},\mathbf{B}] = 0, \quad [\bar{\mathbf{A}},\bar{\mathbf{B}}] = 0, \quad [\mathbf{A},\bar{\mathbf{A}}] + [\mathbf{B},\bar{\mathbf{B}}] = 0, \tag{10}$$

where the overbar denotes complex conjugate. We can generalize these equations by considering four complex vectors $\mathbf{U}, \mathbf{V}, \mathbf{W}$, and $\mathbf{X}$ that satisfy the relations

$$[\mathbf{U},\mathbf{V}] = 0, \tag{11a}$$

$$[\mathbf{W},\mathbf{X}] = 0, \tag{11b}$$

$$[\mathbf{U},\mathbf{W}] + [\mathbf{V},\mathbf{X}] = 0. \tag{11c}$$

Here we are thinking of $\mathbf{W}$ and $\mathbf{X}$ as "generalized complex conjugates" of $\mathbf{U}$ and $\mathbf{V}$, respectively. By Frobenius' theorem, we can use (11a) to define a set of coordinates (t, x) on the two (complex)-dimensional surface defined by vectors $\mathbf{U}$ and $\mathbf{V}$, and take $\mathbf{U}$ and $\mathbf{V}$ to be

$$\mathbf{U} = \frac{\partial}{\partial t}, \quad \mathbf{V} = \frac{\partial}{\partial x}. \tag{12}$$

We can now foliate our whole space using the coordinates (t, x, y, z). Equation (11c) then becomes $\partial_t \mathbf{W} + \partial_x \mathbf{X} = 0$. This means there exists a vector field $\mathbf{Y}$ such that $\mathbf{W} = \partial_x \mathbf{Y}, \mathbf{X} = -\partial_t \mathbf{Y}$. Thus we are only left with the problem of solving for vectors $\mathbf{Y}$ that satisfy $[\partial_t \mathbf{Y}, \partial_x \mathbf{Y}] = 0$ [13]. We expand $\mathbf{W}$ and $\mathbf{X}$ as

$$\mathbf{W} = \partial_t + f_x \partial_y + g_x \partial_z, \tag{13}$$

$$\mathbf{X} = -f_t \partial_y - g_t \partial_z, \tag{14}$$

where subscripts denote partial derivatives. (The reason for the ∂_t term in $\mathbf{W}$ is, as alluded to above, that we are thinking of $\mathbf{W}$ as a sort of complex conjugate of $\mathbf{U} = \partial_t$. Although this argument only seems sensible for t a real coordinate, we are still perfectly at liberty to expand $\mathbf{W}$ in this way if t is complex.) If, by analogy with (3), we impose $\frac{\partial}{\partial x^a} W^a = \frac{\partial}{\partial x^a} X^a = 0$, then we find that there exists a function $h(t, x, y, z)$ such that $f = h_z, g = -h_y$. Imposing (11b), we find that there exists a function $\alpha(t, x)$ such that

$$h_{tt} + h_{xz} h_{ty} - h_{xy} h_{tz} = \alpha(t, x). \tag{15}$$

We can absorb the arbitrary function α into the function h, and conclude that we can form a self-dual metric for any function h that satisfies

$$h_{tt} + h_{xz} h_{ty} - h_{xy} h_{tz} = 0. \tag{16}$$

This is just an evolution equation. Thus we can arbitrarily specify data h and h_t on a $t = \text{const}$ hypersurface and propagate it throughout the space according to (16) to get a solution. For example, if we expand h around the $t = 0$ hypersurface, and insist that h is regular on this surface, then h is of the form

$$h = a_0\,(x,y,z) \;+\; a_1\,(x,y,z)\,t \;+\; a_2\,(x,y,z)\,\frac{t^2}{2!} + a_3\,(x,y,z)\,\frac{t^3}{3!} \;+\cdots . \qquad (17)$$

Substituting this into (16) shows that a_0 and a_1 are arbitrary functions of x, y, and z. $a_2, a_3, \ldots$ are then completely determined for chosen a_0 and a_1 by

$$a_2 \;=\; a_{0xy}\,a_{1z} \;-\; a_{0xz}\,a_{1y}, \qquad (18)$$

$$a_3 \;=\; a_{0xy}\,a_{2z} \;-\; a_{0xz}\,a_{2y} \;+\; a_{1xy}\,a_{1z} \;-\; a_{1xz}\,a_{1y}, \quad (19)$$

and so on. Thus, in principle, we have a solution that depends on two arbitrary functions of three coordinates. It is interesting to compare our Eq. (16) with Plebański's first heavenly equation

$$\Omega_{p\tilde{q}}\,\Omega_{\tilde{p}q} \;-\; \Omega_{p\tilde{p}}\,\Omega_{q\tilde{q}} \;=\; 1. \qquad (20)$$

Here it is not so obvious what our free functions are, and an expansion along the lines of (17) does not work. [It is shown in the appendix how to get Eq. (20) from our equation, showing that the two approaches are equivalent. Thus for *any* self-dual metric there will exist a corresponding function h that satisfies (16).]

From the work of [14] we know that the vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{X}$ are proportional to a null tetrad that determines a self-dual metric. Indeed the tetrad is given by $\sigma_a \;=\; f^{-1}\mathbf{V}_a$, where $\mathbf{V}_a \;=\; (\mathbf{U},\,\mathbf{V},\,\mathbf{W},\,\mathbf{X})$ for $a =$ 0, 1, 2, 3 and $f^2 \;=\; \epsilon\,(\mathbf{U},\,\mathbf{V},\,\mathbf{W},\,\mathbf{X})$, for ϵ the four-dimensional volume form $dt \wedge dx \wedge dy \wedge dz$. In our case, $f^2 = -h_{tt}$ and our line element is

$$ds^2 \;=\; dt\,(h_{ty}\,dy \;+\; h_{tz}\,dz) \;+\; dx\,(h_{xy}\,dy \;+\; h_{xz}\,dz) + \frac{1}{h_{tt}}\,(h_{ty}\,dy \;+\; h_{tz}\,dz)^2. \qquad (21)$$

III. THE FORMAL SOLUTION

We now construct, at least formally, the general solution to (16). Instead of working with this equation directly, it is helpful to define two functions $A = h_t$, $B = h_x$, and rewrite (16) in the equivalent form

$$A_t + A_y B_z - A_z B_y = 0, \qquad (22)$$

$$A_x = B_t. \qquad (23)$$

If we just view B as some arbitrary function, then the solution to (22) is

$$A(t,x,y,z) \;=\; \exp\left[\int_0^t dt_1\,[B_y(t_1,x,y,z)\,\partial_z \;-\; B_z(t_1,x,y,z)\,\partial_y]\right]\,a_1(x,y,z), \qquad (24)$$

where $a_1(x,y,z)$ is the value of A at $t = 0$ as in (17). The exponential here is defined by its power series with the nth term in this series being

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\,[\{B_y(t_1)\,\partial_z \;-\; B_z(t_1)\,\partial_y\}\cdots\{B_y(t_n)\,\partial_z \;-\; B_z(t_n)\,\partial_y\}]a_1(x,y,z). \qquad (25)$$

We now must impose (23) as a consistency condition on this solution. This gives us

$$A(t,x,y,z) \;=\; \exp\left[\int_0^t dt_1 \int_0^{t_1} dt_2\,[A_{xy}(t_2,x,y,z)\,\partial_z - A_{xz}(t_2,x,y,z)\,\partial_y]\right]\,a_1(x,y,z). \qquad (26)$$

Formally, this equation can now be solved iteratively. We can make successive approximations

$$A^{(0)} \;=\; a_1(x,y,z), \qquad (27)$$

$$A^{(1)} \;=\; \exp\left[\left(t\,a_{0xy} \;+\; \frac{t^2}{2}a_{1xy}\right)\,\partial_z - \left(t\,a_{0xz} \;+\; \frac{t^2}{2}a_{1xz}\right)\,\partial_y\right]\,a_1(x,y,z), \qquad (28)$$

$$A^{(n+1)} \;=\; \exp\left[\int_0^t dt_1 \int_0^{t_1} dt_2\,(A^{(n)}_{xy}\,\partial_z \;-\; A^{(n)}_{xz}\,\partial_z\,)\right]\,a_1(x,y,z), \qquad (29)$$

for $n \geq 1$. Then defining $A \;=\; \lim_{n\to\infty} A^{(n)}$ gives the formal solution for A [15]. Integrating A with respect to t and imposing $h(t = 0) = a_0(x,y,z)$ then gives us a solution of (16).

Finally, we note that (22) means that the quantity $A(t,x,\tilde{y},\tilde{z})$ is t independent, where $\tilde{y}$ and $\tilde{z}$ are defined implicitly by

$$\tilde{y}(t) \;=\; y \;+\; \int_0^t dt_1\,B_z(t_1,x,\tilde{y}(t_1),\tilde{z}(t_1)), \qquad (30)$$

$$\tilde{z}(t) = z - \int_0^t dt_1\, B_y(t_1, x, \tilde{y}(t_1), \tilde{z}(t_1)). \quad (31)$$

(This may be important if we were to look for action angle variables for the system.) This implies that $A(t,x,y,z) = a_1(x, y', z')$, where the coordinates y', z' are defined by $\tilde{y}(t, x, y', z') = y, \tilde{z}(t, x, y', z') = z$. Thus the dynamics are characterized by a coordinate transformation in the y, z plane.

IV. GROUP METHODS

Several powerful techniques have been developed for the study of partial differential equations [16]. One of the most powerful is that of group analysis [17,18]. By studying the Lie algebra under which a given system of partial differential equations is invariant, we can hopefully find new solutions to these equations. One method of doing so is to look for similarity solutions which are left invariant by the action of some subalgebra of this symmetry algebra. This will reduce the number of independent variables present in the equation, possibly reducing a partial differential equation to an ordinary differential equation. However, such similarity solutions, by construction, will have some symmetries imposed upon them, so this method is not very useful if one is looking for the general solution to a system of equations.

A more powerful method of obtaining solutions is to exponentiate the infinitesimal action of the Lie algebra into a group action, which takes one solution of the equation to another. However, even if this is possible, it is unlikely that the group action can be used to find the general solution to the equation from any given solution.

Instead of attempting to find the symmetry algebra of (16), it is easier to work with the equivalent system (22) and (23). Using the standard methods [17,18], we find that (22) and (23) admit a symmetry group defined by the infinitesimal generators

$$\xi_1 = f_A\, \partial_t - f_x\, \partial_B, \quad (32)$$

$$\xi_2 = (t\, g_x + B\, g_A)_A\, \partial_t + g_A\, \partial_x - g_x \partial_A - (t\, g_x + B\, g_A)_x\, \partial_B, \quad (33)$$

$$\xi_3 = k\, t\, \partial_t + k\, x\, \partial_x + k\, y\, \partial_y, \quad (34)$$

$$\xi_4 = l_z\, \partial_y - l_y\, \partial_z, \quad (35)$$

where f and g are arbitrary functions of x and A, l is an arbitrary function of y and z, and k is an arbitrary constant [19]. ξ_3 just generates dilations, whereas ξ_4 generates area preserving diffeomorphisms in the $y-z$ plane. Although ξ_4 gives a representation of W_∞ (modulo cocycle terms) [20], they are really only coordinate transformations, so are not too interesting. However, we have two interesting symmetries, generated by ξ_1 and ξ_2.

It is possible to exponentiate the action of ξ_1 directly for an arbitrary function f. We find that if $A(t,x,y,z)$ and $B(t,x,y,z)$ are a solution of the system (22) and (23) then we can implicitly define a new solution $\tilde{A}$ and $\tilde{B}$ by

$$\tilde{A} = A(t + f_A(x, \tilde{A}), x, y, z),$$
$$\tilde{B} = B(t + f_A(x, \tilde{A}), x, y, z) + f_x(x, \tilde{A}), \quad (36)$$

for any function $f(x, A)$. Using this implicit form we can solve iteratively for the functions $\tilde{A}$ and $\tilde{B}$ given functions A, B and f. This means that given one solution of (16), we can form an infinite-dimensional family of solutions depending on that solution. For a given function g we can also exponentiate the action of ξ_2, although its action cannot be exponentiated directly for a general function g.

Although both (34) and (35) give rise to infinite-dimensional families of solutions from any given solution, they are not enough to derive a solution with arbitrary initial data from any given solution.

If we compute the commutators of generators $\xi_1(f_i)$ and $\xi_2(g_j)$ for arbitrary functions f_i and g_j we find that they obey the algebra

$$[\xi_1(f_1), \xi_1(f_2)] = 0, \quad (37a)$$

$$[\xi_1(f), \xi_2(g)] = \xi_1(f_A\, g_x - f_x\, g_A), \quad (37b)$$

$$[\xi_2(g_1), \xi_2(g_2)] = \xi_2(g_{1A}\, g_{2x} - g_{1x}\, g_{2A}). \quad (37c)$$

Defining a basis for transformations

$${}^{(\alpha)}T_i^m = \xi_\alpha(x^{i+1}\, A^{m+1}) \quad (38)$$

for $\alpha = 1, 2$, where m and i are integers, this algebra becomes

$$[{}^{(1)}T_i^m, {}^{(1)}T_j^n] = 0, \quad (39a)$$

$$[{}^{(1)}T_i^m, {}^{(2)}T_j^n] = [(m+1)(j+1) - (n+1)(i+1)]\, {}^{(1)}T_{i+j}^{m+n}, \quad (39b)$$

$$[{}^{(2)}T_i^m, {}^{(2)}T_j^n] = [(m+1)(j+1) - (n+1)(i+1)]\, {}^{(2)}T_{i+j}^{m+n}. \quad (39c)$$

The algebra (39c) is again just the extended conformal algebra W_∞ [20]. Thus (39) represents some generalization of W_∞. These are similar results to those found in [21,22].

We now note that Eq. (16) can be derived from the Lagrangian

$$S = \int d^4x \{ \tfrac{1}{2} h_t^2 + \tfrac{1}{3} h_t\, (h_y\, h_{xz} - h_z\, h_{xy}) \}. \quad (40)$$

The Hamiltonian is then

$$H = \frac{1}{2}\int_\Sigma d^3x\,[\pi - \tfrac{1}{3}(h_y h_{xz} - h_z h_{xy})]^2, \qquad (41)$$

where $\pi = h_t + \frac{1}{3}(h_y h_{xz} - h_z h_{xy})$ is the momentum canonically conjugate to h. We now define the Poisson brackets of functionals of h and π by

$$\{\alpha,\beta\} = \int_\Sigma d^3x\left(\frac{\delta\alpha}{\delta h}\frac{\delta\beta}{\delta\pi} - \frac{\delta\alpha}{\delta\pi}\frac{\delta\beta}{\delta h}\right). \qquad (42)$$

The algebra (39) now reflects the fact that, ignoring surface terms, we have two infinite families of conserved quantities of the form

$$I_1(f(x,A)) = \int_\Sigma f(x,A)\,d^3x, \qquad (43)$$

$$I_2(g(x,A)) = \int_\Sigma [t g_x(x,A) + B\,g_A(x,A)]d^3x. \qquad (44)$$

The time independence of quantities follows from the conservation equations

$$\partial_t(f) + \partial_y(f B_z) - \partial_z(f B_y) = 0, \qquad (45)$$

and

$$\partial_t(t g_x + B g_A) + \partial_x(g) - \partial_y(t g_{xA} B_z + g_A B B_z) + \partial_z(t g_{xA} B_y + g_A B B_y) = 0, \qquad (46)$$

derived from (22) and (23). Again these are similar to results in [21,23] where an infinite heirarchy of conservation laws were constructed for the system. Up to surface terms, the quantities in (43) and (44) have vanishing Poisson brackets; i.e., they are in involution. This could be looked on as a proof that self-dual gravity is classically integrable. It also seems to be the relationship between this formalism and the twistor approach to the problem [21–24].

V. SOLUTIONS

We begin by looking for solutions that admit a *triholomorphic* Killing vector ξ. This means the three complex structures $\mathbf{J}^i$ are invariant under the action of ξ, i.e., $\mathcal{L}_\xi \mathbf{J}^i = 0$, where $\mathcal{L}$ is the Lie derivative. Using the relationship between the complex structures and the vectors $\mathbf{V}_i$ given in Sec. II and the fact that ξ is a Killing vector, we see that we require $\mathcal{L}_\xi \mathbf{V}_i = 0$.

If ∂_x is a triholomorphic Killing vector, this means that $\partial_x \mathbf{X} = \partial_x \mathbf{W} = 0$, where $\mathbf{X}$ and $\mathbf{W}$ are as in (13) and (14). This means h is of the form $a(t,y,z) + x\,b(y,z)$ for some functions a and b. In terms of functions $A = h_t$ and $B = h_x$ this means that $A = A(t,y,z)$, $B = B(y,z)$, so (23) is automatically satisfied. If we take $A(t=0) = a_1(y,z)$ and $B(t=0) = \phi(y,z)$, it is straightforward to show that the solution to (22) is then

$$A(t,y,z) = \exp\{t(\phi_y\partial_z - \phi_z\partial_y)\}a_1(y,z),$$
$$B(y,z) = \phi(y,z). \qquad (47)$$

For given functions ϕ and a_1 it is straightforward to do the exponentiation [18], giving A explicitly. Using the exponentiated form of (32) and (33), we could now use these solutions to generate new solutions which had some restricted x-dependent initial data as well.

We can also consider metrics with a triholomorphic Killing vector ∂_z. This means we require $\partial_z \mathbf{V}_i = 0$. In this case, we take $h = -t\,z + g(t,x,y)$. We then recover the result [4,25] that g must satisfy the three-dimensional Laplace equation $g_{tt} + g_{xy} = 0$. The general solution to this is known [26], and can be written in terms of two arbitrary functions $a_0(x,y)$ and $a_1(x,y)$. An almost identical reduction occurs if we take ∂_y as a triholomorphic Killing vector. Again, using the symmetries (32) and (33), we can generate infinite-dimensional families of new solutions, that in general have no Killing vectors.

We note in passing that the solution corresponding to the multicenter Eguchi-Hansen metric [4] is

$$A = -z + \alpha\sum_{i=1}^{s}\operatorname{arcsinh}\left(\frac{(t-t_i)}{2\sqrt{(x-x_i)(y-y_i)}}\right), \qquad (48)$$

$$B = -\frac{\alpha}{2}\sum_{i=1}^{s}\frac{\sqrt{(t-t_i)^2 + 4(x-x_i)(y-y_i)}}{(x-x_i)}, \qquad (49)$$

where α is a constant. This is the only metric with a triholomorphic Killing vector that has a nonsingular real (Euclidean) section [27].

It would be interesting now to study the case of a holomorphic Killing vector ζ. In terms of the complex structures this is characterized by

$$\mathcal{L}_\zeta \mathbf{J}^1 = 0,\quad \mathcal{L}_\zeta \mathbf{J}^2 = \mathbf{J}^3,\quad \mathcal{L}_\zeta \mathbf{J}^3 = -\mathbf{J}^2. \qquad (50)$$

In terms of the vectors $\mathbf{V}_i$ this means that

$$\mathcal{L}_\zeta \mathbf{V}_1 = 0,\quad \mathcal{L}_\zeta \mathbf{V}_2 = \mathbf{V}_3,\quad \mathcal{L}_\zeta \mathbf{V}_3 = -\mathbf{V}_2. \qquad (51)$$

It should be possible to relate this to the known results on such metrics [28,29]. It should also be noted that since we have an initial value formulation of the self-duality problem, we can systematically generate multicenter generalizations of a given metric. For example if a solution has initial data $a_0(x,y,z), a_1(x,y,z)$, then the multicenter generalization will have initial data of the form

$$a_0'(x,y,z) = \sum_{i=1}^{s} a_0(x-x_i, y-y_i, z-z_i),$$
$$a_1'(x,y,z) = \sum_{i=1}^{s} a_1(x-x_i, y-y_i, z-z_i), \qquad (52)$$

for any points $\{(x_i,y_i,z_i) : i = 1,\ldots,s\}$. It would therefore be of interest to study the Atiyah-Hitchin metric

[30] in this formalism, since it would give a systematic way of generating a multi-Atiyah-Hitchin solution which, though known to exist, has not been constructed explicitly.

VI. CONCLUSION

We have shown how, at least formally, to construct the general complex metric with self-dual Riemann tensor. We have also studied the symmetry algebra of the system, which turns out to be a generalized version of $W_\infty \oplus W_\infty$, and found two infinite-dimensional families of conserved quantities that have vanishing Poisson brackets. This should be of interest if we were to try and quantize the system [31]. Finally, although we have managed to characterize metrics with a triholomorphic Killing vector, it remains to relate the holomorphic Killing vector case to known results.

It should be emphasized that all the considerations here have been inherently *local* in nature, and we have imposed no sorts of boundary conditions on our solutions. If we were to look for metrics that are well defined globally, this would lead us to cohomological problems [32], which appear to be best tackled using the twistor formalism [1].

Note added. Since this work was completed, the ideas developed here have been extended by Plebański *et al.* to reduce the second heavenly equation, the holomorphic Killing vector equation, and special cases of the hyper-heavenly equation to Cauchy-Kovalevski form and to write down formal solutions to these equations [33]. I would like to thank Jerzy Plebański for sending me a copy of this work before its publication.

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APPENDIX: THE FIRST HEAVENLY EQUATION

We now show how our formalism is related to Plebański's [8]. Starting with (22) and (23), instead of looking on A as a function of t, x, y, and z we take A as a coordinate and look on $f \equiv t$ and $g \equiv B$ as functions of $p \equiv A$, $q \equiv x$, $r \equiv y$, $s \equiv z$. This transformation is well defined as long as $A_t \neq 0$. Inverting (22) and (23) gives

$$f_q = -g_p, \tag{A1}$$

$$f_r\, g_s - f_s\, g_r = 1. \tag{A2}$$

Equation (A1) means we can introduce a function $\Omega(p,q,r,s)$ such that $f = -\Omega_p$, $g = \Omega_q$. Equation (A2) then means that Ω must satisfy $\Omega_{ps}\,\Omega_{qr} - \Omega_{pr}\,\Omega_{qs} = 1$. Carrying out the same transformation on the line element (21), we find it becomes $ds^2 = \Omega_{pr}\,dp\,dr + \Omega_{ps}\,dp\,ds + \Omega_{qr}\,dq\,dr + \Omega_{qs}\,dq\,ds$. Thus we have recovered the Plebański formalism. It would be interesting to see if a similar transformation can be used to turn the problem of conformally self-dual metrics with nonzero cosmological constant into an initial value problem.

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