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Miens of the three-dimensional black hole

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Some aspects of the rotating three-dimensional Einstein–anti-de Sitter black-hole solution constructed recently by Banados, Teitelboim, and Zanelli are discussed. It is shown explicitly that this black hole represents the most general black-hole-type solution of the Einstein–anti-de Sitter theory. The interpretation of one of the integrals of motion as the spin is discussed. Its physics relies on the topological structure of the black-hole manifold, and the notion of simultaneity of spacelike-separated intervals. The relationship of the black-hole solution to string theory on a $(2+1)$-dimensional target space is examined, and it is shown that the black hole can be understood as a part of the full axion-dilaton gravity, realized as a WZWN σ model. In conclusion, the pertinence of this solution to four-dimensional black strings and topologically massive gravity is pointed out.

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I. INTRODUCTION

The black-hole conundrum has long been one of the most outstanding problems of modern physics. It has remained in focus as one of the potential testing grounds for quantum-gravitational phenomena for a long time. The formal difficulties of four-dimensional gravity, however, have often made the study of black holes inherently more complicated. To surmount some of these difficulties many researchers have resorted to models of gravity in dimensions lower than four, in the hope that the essential properties of black holes in lower dimensions will model reasonably accurately those of four-dimensional solutions. One such attempt has resulted recently in the construction of the Einstein–anti-de Sitter rotating black hole in three dimensions by Banados, Teitelboim, and Zanelli (BTZ) [1]. Their solution has attracted further attention as it was later shown how it could be obtained by restricting a four-dimensional Minkowski manifold of signature zero on a coset [2,3], followed by the one-point compactification of one of the coordinates to a circle. Furthermore, the conditions under which such black holes can form in a collapse of matter in conventional general relativity have been investigated in [4]. The purpose of this work is to demonstrate that their solution can be easily incorporated in the framework of string theory with some minor extensions [5]. Namely, the BTZ solution with half the initial cosmological constant can be extended with the inclusion of the antisymmetric Kalb-Ramond axion field carrying the other half of the cosmological constant, and then reinterpreted as either an ungauged or extremely gauged Wess-Zumino-Witten-Novikov (WZWN) σ model derived from the group SL$(2,R)$ [6–11]. The one-point compactification of one of the coordinates can be accomplished either by factoring out a discrete group in the ungauged construction, or requiring that the model lives on a coset $\mathrm{SL}(2,R)\times R/R$.

This paper is organized as follows. In Sec. II, I will derive the solution by solving the equations of motion, by employing the Kaluza-Klein dimensional reduction from three dimensions to one [12], and show that the solution of Ref. [1] is the unique solution of three-dimensional Einstein gravity with a negative cosmological constant which features horizons. In Sec. III, I will comment on the interpretation of one of the constants of motion as the spin of the black hole, and show that this stems from interweaving the topological structure of the manifold with the requirement of global simultaneity of spacelike intervals. Section IV concentrates on the stringy interpretation of the solution and demonstrates how the solution is realized as a WZWN σ model. Last, I will comment on the relationship of this solution to topologically massive gravity [13] and cosmic strings.

II. CLASSICAL THEORY

The classical theory is defined with the Einstein-Hilbert action in three dimensions:

$$S=\int d^3x\sqrt{g}\,\left(\frac{1}{2\kappa^2}R+\Lambda\right), \tag{1}$$

where R is the Ricci scalar and Λ the cosmological constant. The conventions employed here are that the metric is of signature $+2$, the Riemann tensor is defined according to $R^{\mu}_{\nu\lambda\sigma}=\partial_\lambda\Gamma^{\mu}_{\nu\sigma}-\cdots$, and the cosmological constant is defined with the opposite sign from the more usual conventions: here, $\Lambda>0$ denotes a negative cosmological constant. In the remainder of this paper, I will work in Planck mass units: $\kappa^2=1$.

The Einstein equations associated with this theory, in the absence of other sources, yield the locally trivial solution $R^{\mu}_{\nu\lambda\sigma}=-\Lambda(\delta^{\mu}_{\lambda}g_{\nu\sigma}-\delta^{\mu}_{\sigma}g_{\nu\lambda})$ which suggests that the unique solution is the anti-de Sitter space in three dimensions. However, there appear nontrivial configurations in association with the global structure of the manifold described with the above curvature tensor. It is interesting to note that all the metric solutions have well-defined curvature, except possibly at a point later to be identified with black-hole singularity [2,3].

Therefore, to inspect all possible solutions one should resort to a closer scrutiny of the problem at hand. The investigation of [1–3] demonstrates how nontrivial black-hole solutions can be obtained by factorization and topological identification in the anti-de Sitter manifold. However, a particularly simple procedure can be followed, where one solves the differential equations derived from (1) and investigates allowed values for the integration constants. In addition, this procedure yields further information regarding whether all possibilities for the construction of nontrivial solutions are exhausted by the above-mentioned identifications.

Instead of writing out explicitly Einstein's equations, I will here work in the action, as this approach offers an especially simple way to find the solutions. The background *Ansatz* is that of a stationary axially symmetric metric:

$$ds^2 = \mu^2 \, dr^2 + G_{jk}(r) \, dx^j dx^k, \tag{2}$$

where the 2×2 matrix $G_{jk}(r)$ is of signature 0 as the metric (2) is Lorentzian and one of the coordinates $\{x^k\}$ is timelike. The "lapse" function μ^2 is kept arbitrary as its variation in (1) yields the constraint equation. The cross terms $dr dx^k$ corresponding to the "shift" functions can be removed by coordinate transformations $x^k \to x^k + F^k(r)$.

The metric above clearly has two toroidal coordinates $\{x^k\}$ which are dynamically unessential. Hence the problem is effectively one dimensional. The Kaluza-Klein reduction, with rescaling of the action (1) according to $S_{\rm eff} = 2S/\int d^2x$ yields

$$S_{\rm eff} = \int dr \, \mu e^{-\phi} \left(\frac{1}{\mu^2} \phi'^2 + 2\Lambda + \frac{1}{4\mu^2} \, {\rm Tr} G'^{-1} G' \right) \tag{3}$$

with the "dilaton" field ϕ being constrained (rather, defined) by $\exp(-2\phi) = -\det G$. The minus sign here follows from the fact that $\mathrm{sgn}(G) = 0$, i.e., $\det G < 0$. The prime denotes a derivative with respect to r. Hence, the problem is reducible to a simple mechanical system describing the "motion" of the matrix G with several rheonomic solvable constraints. As such, the "dilaton" constraint above can be solved for ϕ, which then may be completely eliminated from the action. However, it is instructive to keep the explicit dilaton in (3) and enforce the above constraint with the help of an additional Lagrange multiplier λ. Furthermore, there is an additional simplification coming from properties of 2×2 matrices. In the above equations for the action and the "dilaton" the inverse and determinant of G figure explicitly, thus giving the problem in question the appearance of a highly nonlinear one. The 2×2 magic comes to the rescue: it is possible to reexpress the action in Gaussian form in terms of G only. From

$$\begin{aligned} \det G &= -\frac{1}{2} {\rm Tr}(\epsilon G)^2, \\ G^{-1} &= -\frac{1}{\det G} \epsilon G \epsilon \end{aligned} \tag{4}$$

with $\epsilon = i\sigma_2$ being the two-dimensional antisymmetric symbol, the action (3) can be rewritten as

$$S_{\rm eff} = \int dr \left(2\Lambda \mu e^{-\phi} + \frac{e^{\phi}}{4\mu} \left[{\rm Tr}(\epsilon G')^2 + \lambda \, {\rm Tr}(\epsilon G)^2 \right] - \frac{\lambda e^{-\phi}}{2\mu} \right). \tag{5}$$

Clearly, the theory has three Lagrange multipliers, μ, ϕ, and λ, which all propagate according to algebraic equations. Thus the associated equations of motion are very simple. Indeed, the standard variational procedure leads to

$$\begin{aligned} &2\Lambda \mu e^{-\phi} - \frac{e^{\phi}}{4\mu} {\rm Tr}(\epsilon G')^2 = 0, \\ &\frac{e^{\phi}}{4\mu} {\rm Tr}(\epsilon G)^2 - \frac{e^{-\phi}}{2\mu} = 0, \\ &\frac{\lambda e^{-\phi}}{2\mu} = 0, \\ &\left(\frac{e^{\phi}}{\mu} G' \right)' = \frac{\lambda e^{\phi}}{\mu} G. \end{aligned} \tag{6}$$

Obviously, $\lambda = 0$. The system of equations above simplifies to

$$\begin{aligned} &2\Lambda \mu e^{-\phi} - \frac{e^{\phi}}{4\mu} {\rm Tr}(\epsilon G')^2 = 0, \\ &{\rm Tr}(\epsilon G)^2 - 2e^{-2\phi} = 0, \\ &\left(\frac{e^{\phi}}{\mu} G' \right)' = 0, \end{aligned} \tag{7}$$

and by using the gauge freedom expressed by the arbitrary "lapse" μ and fixing the gauge to $\mu \exp(-\phi) = 1$, the solution is easy to find. It is just

$$\begin{aligned} \mu^2 &= -\frac{1}{\det G}, \\ G &= Cr + D, \end{aligned} \tag{8}$$

and C, D are constant symmetric matrices determined from the initial conditions, and the constraint $\det C = -4\Lambda$. The minus sign in (8) is precisely the same one discussed following Eq. (3).

What remains is to analyze the values of the integration constants. To begin with, the metric can be rewritten as

$$ds^2 = -\frac{dr^2}{\det(Cr + D)} + (Cr + D)_{jk} \, dx^j dx^k. \tag{9}$$

Since C is symmetric and nonsingular ($\det C = -4\Lambda \neq 0$), it can be diagonalized with an orthogonal transformation. So, $C = O^T C_d O$. From the metric (9) such a transformation is just a coordinate transformation of the $\{x^k\}$ part of the metric, $x^k \to O^k_j x^j$. Hence C could have been assumed to be diagonal from the beginning. Furthermore, its eigenvalues c_1, c_2 can be set equal to ± 1 by a scale transformation $x^k \to x^k / \mid c_k \mid$. Thus C is just the $1+1$ Minkowski metric, $C = \eta = \mathrm{diag}(1,-1)$. At this point one could object that the rescaling can introduce a nontrivial deficit angle if the coordinate x^k is compact. This can be restored later by changing the period of compactification. Moreover, the diagonalization of C can also be accomplished with a shift of the spacelike coordinate by a linear function of time. Therefore, the above discussion is fully justified.

The next step is the matrix D, which only has to be symmetric. None of the above manipulations with coordinates in order to reduce C to the $1+1$ Minkowski metric affects the general structure of the matrix D. Therefore, the 2×2 metric can be written as

$$G = \eta r + D = \begin{pmatrix} r + d_{11} & d_{12} \\ d_{12} & -r + d_{22} \end{pmatrix} \tag{10}$$

and evidently, one of the diagonal elements of D can be removed by a shift in r. This indicates an additional requirement which ought to be imposed on D. Since one is interested in a black-hole-type solution, with physical horizons defined as the hypersurfaces where the timelike Killing vector outside the black hole has a vanishing norm, and flips into spacelike after passing through the horizon, it must be $d_{22} \geq -d_{11}$. Then, one can simply set $d_{11} = 0$. Lastly, if the spacelike coordinate θ is to be interpreted as an angle, the identification $\theta \cong \theta + 2\pi$ must be made. Thus, the final solution is

$$ds^2 = \frac{dr^2}{4\Lambda[r(r-d_{22}) + d_{12}^2]} + (d\theta, dt) \begin{pmatrix} r & d_{12} \\ d_{12} & -r + d_{22} \end{pmatrix} \begin{pmatrix} d\theta \\ dt \end{pmatrix}. \tag{11}$$

Equation (11) is precisely the solution of Ref. [1] as can be seen after a coordinate transformation. The integration constants can be rearranged by introducing the mass $M = d_{22}\sqrt{\Lambda}$ and the spin $J = -2d_{12}\sqrt{\Lambda}$, as well as the parameter measuring the position of the horizon in the new coordinates: $\rho_+^2 = M[1-(J/M)^2]^{1/2}$. With the definitions $R^2 = r = (\sqrt{\Lambda}/2)(\rho^2 + M - \rho_+^2)$ and $N^\theta = -J/2R^2$, the metric (11) can be put in the BTZ form:

$$ds^2 = \frac{d\rho^2}{\Lambda(\rho^2 - \rho_+^2)} + R^2(d\theta + N^\theta dt)^2 - \frac{\rho^2}{R^2}\frac{\rho^2 - \rho_+^2}{\Lambda}dt^2. \tag{12}$$

From the formulas above one finds that physical black holes should also satisfy the constraint $\mid J \mid \leq M$. If this were not satisfied, one would end up with a singular structure, manifest by the appearance of closed timelike curves in the manifold accessible to an external observer, crossing the point $R = 0$. Such a voyage has been investigated in [5] for the spinless case, and also in [9] for the vacuum. Moreover, it has been argued that, although solution (12) does not have curvature singularities, they can develop if the metric is slightly perturbed by a matter distribution [2,3]. Thus, the singularities are hidden by a horizon if the spin is bounded above by the mass. Thermodynamics of (12) has been analyzed in [1,2], where the Hawking temperature has been calculated. The solution with $J = M$ is understood as the extremal black hole, and $J = M = 0$ serves the role of the vacuum. These two solutions actually appear to have similar local properties, as will be discussed in the next section. The anti-de Sitter metric is recovered with $J = 0, M = -1$ [1–3].

III. SPIN AND SIMULTANEITY

There still remains to determine the physical nature of the spin J. It has been so interpreted by the careful examination of the boundary terms in the action which appear in the Arnowitt-Deser-Misner (ADM) formulation of general relativity in three dimensions [1,2]. Yet an interesting observation is in place here. The r-dependent part of the metric G is an SO(1,1) invariant, being the $1+1$ Minkowski metric. Then one can ask if the matrix D can be diagonalized by a coordinate transformation. Indeed, the transformation $x'^k = \tilde{O}^k_j x^j$ where

$$\tilde{O} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \tag{13}$$

and

$$\sinh\beta = \mathrm{sgn}(J)\frac{1}{\sqrt{2}}\left(\frac{1-\sqrt{1-(J/M)^2}}{\sqrt{1-(J/M)^2}}\right)^{1/2} \tag{14}$$

removes the cross term $d\theta dt$ from the metric, and is clearly valid for all physical black holes with $\mid J \mid \leq M$. In terms of the new coordinates the metric (12) can be rewritten as

$$ds^2 = \frac{d\rho^2}{\Lambda(\rho^2 - \rho_+^2)} + \rho^2 d\theta'^2 - \frac{\rho^2 - \rho_+^2}{\Lambda}dt'^2. \tag{15}$$

This solution describes a black hole of spin $J' = 0$ and mass $M' = M\left(1-(J/M)^2\right)^{1/2}$.

The question one should ask is if transformation (13) is globally defined. If the answer is positive, then the angular momentum in the metric would be spurious. What can be seen immediately is that with the help of (13), which corresponds to a "boost" in the azimuthal direction, a comoving observer can be found who will not be able to discern the influence of the angular momentum by any local experiment. Hence (12) and (15) are completely equivalent locally.

The answer is that due to the identification $\theta \cong \theta + 2\pi$ the global structure of the manifold with the metric (12) is not invariant under a coordinate transformation generated by a "boost" (13). This can be seen as follows. The manifold can be foliated by cylinders $R \times S^1$ corresponding to constant r (or ρ). The cylinders in the frame where the identification has been made (and the spin J has been defined) can be represented as rectangular patches in the t-θ plane with edges at $\theta = 0$ and $\theta = 2\pi$ identified along the congruence $t = 0$. After the boost has been performed, in the new coordinates the manifold is represented with patches tipped with respect to the t' axis by angle $\arccos\cosh\beta$ and identification now goes along the congruence $t' = \cosh\beta\theta'$. Hence, the global simultaneity of spacelike events is lost. If one goes around the Universe, with a clock which remembers the initial point, upon the return to it the clock reading exibits a discrete jump. Therefore, to measure the spin of the black hole, one can build a stroboscopic device by measuring the discrepancy of the arrival time of light rays sent around the black hole in opposite directions. Boosting in the azimuthal direction, observers can bring themselves to the frame where the light signals arrive back simultaneously, and then measure the spin. It is interesting to note that precisely the same phenomenon can be found in special relativity on a cylinder with a flat metric.

The discussion above gives an interesting connection between the vacuum $J = M = 0$ and an extremal black hole $J = M$. If one takes the limit $M \to J + 0^+$ in the boosted coordinates (15), the metric reduces to the vacuum solution $J' = M' = 0$, but the coordinate transformation (13) is ill defined in the limit, since $\tanh\beta \to 1$. But this corresponds to boosting up to the speed of light in special relativity, where in the $1+1$ case the spacetime tends to a degenerate case for such an observer. Thus one can think of the extremal black hole as the maximally boosted vacuum, up to the global structure.

The physical interpretation of spin is therefore derived from the global properties of the manifold. Essentially, the spin is introduced by choosing a special observer who is granted the judgment of how to perform the identification. More elaborate, but similar properties have been found by Misner in connection with Taub-NUT (Newman-Unti-Tamburino) space-times [14]. These conclusions are in perfect agreement with the constructions of Refs. [1–3].

IV. WESS-ZUMINO-WITTEN-NOVIKOV σ MODEL APPROACH

In this section I will discuss the relationship of solution (12) to string theory and show how it can be extended to represent an exact string solution too. In order to do it, an elementary review of the WZWN σ model approach is provided first. The dynamics of string theory on the world sheet is defined by the tree-level Polyakov action:

$$S_\sigma = \frac{1}{\pi}\int d^2\sigma\left(G_{\mu\nu} + 2\sqrt{\frac{2}{3}}B_{\mu\nu}\right)\partial_+X^\mu\partial_-X^\nu, \quad (16)$$

where $G_{\mu\nu}$ and $B_{\mu\nu}$ are the world-sheet target metric and the Kalb-Ramond antisymmetric field. The rather unusual factor $2\sqrt{2/3}$ in Eq. (1) is introduced following the normalization convention in earlier work, where the wedge product of two forms is defined by $\alpha\wedge\beta = \mathrm{Alt}\,(\alpha\otimes\beta)$ as opposed to the other usual convention, $\alpha\wedge\beta = \frac{(p+q)!}{p!q!}\mathrm{Alt}\,(\alpha\otimes\beta)$. Action (16) in general also includes the dilaton, but it can be computed in the semiclassical approach from the associated effective field theory on target space. Its effective action is, in the world-sheet frame and to order $O(\alpha'^0)$,

$$S = \int d^3x\sqrt{G}e^{-\sqrt{2}\kappa\Phi}\left(\frac{1}{2\kappa^2}R - H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \partial_\mu\Phi\partial^\mu\Phi + \Lambda\right). \quad (17)$$

Here $H_{\mu\nu\lambda} = \partial_{[\lambda}B_{\mu\nu]}$ is the field strength associated with the Kalb-Ramond field $B_{\mu\nu}$ and Φ is the dilaton field, which appears naturally in the string sector and whose dynamics guarantee the conformal anomaly cancellation. The brackets denote antisymmetrization over enclosed indices. The cosmological constant has been included to represent the central charge deficit $\Lambda = \frac{2}{3}\delta c_T = \frac{2}{3}(c_T - 3) \geq 0$. It arises as the difference of the internal theory central charge and the total central charge for a conformally invariant theory $c_{\mathrm{tot}} = 26$ [6–11,15].

The WZWN approach starts with the construction of the field theory on the world sheet defined by the WZWN σ model action on level k:

$$S_\sigma = \frac{k}{4\pi}\int d^2\sigma\,\mathrm{Tr}\left(g^{-1}\partial_+g\,g^{-1}\partial_-g\right) - \frac{k}{12\pi}\int_M d^3\zeta\,\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right), \quad (18)$$

where g is an element of some group G. The action above has a very big global invariance, the continuous part of which is $G\times G$. One way to construct the string solutions of this theory, which can be put in form (16), is choosing a group G, the parameter space of which represents the target manifold, and maintaining conformal invariance. The other may be to identify a part of the parameter manifold by factoring out locally a subgroup of the global invariance group $G\times G$. This is accomplished by choosing an anomaly-free subgroup $H\subset G\times G$ and gauging it with stationary gauge fields. Either way, after the group has been parametrized, (18) can be rewritten in terms of the parameters in form (16) and the metric and the axion are just simply read off from the resulting expressions. The dilaton then can be computed from the effective action (17), as has been mentioned above. It appears because of the requirement of conformal invariance. In the remainder of this section I will demonstrate how the solution (12) arises in this approach as the gravitational sector of the WZWN constructions in two different ways.

I will first demonstrate that the theory described by (18) with the group $G = \mathrm{SL}(2,R)/P$ contains solution (12). The discrete group P will be specified later. The central charge of the target for this model for level k is $c_T = 3k/(k-2)$. Thus the central charge deficit, by the formulas above, will be given by

$$\delta c_T = \frac{6}{k-2} \simeq \frac{6}{k} \tag{19}$$

in the semiclassical limit $k \to \infty$, where the theory is most reliable. Therefore, the cosmological constant is $\Lambda = 4/k$. The group SL$(2,R)$ can be parametrized according to

$$g = \begin{pmatrix} e^{\sqrt{\frac{2}{k}}q\theta'}\cosh\vartheta & e^{\sqrt{\frac{2}{k}}qt'}\sinh\vartheta \\ e^{-\sqrt{\frac{2}{k}}qt'}\sinh\vartheta & e^{-\sqrt{\frac{2}{k}}q\theta'}\cosh\vartheta \end{pmatrix}, \tag{20}$$

where q is an arbitrary constant. In terms of these parameters, the action (18) can be rewritten as

$$S_\sigma = \frac{1}{\pi}\int d^2\sigma\Big(\frac{k}{2}\partial_+\vartheta\partial_-\vartheta + q^2\cosh^2\vartheta\partial_+\theta'\partial_-\theta' - q^2\sinh^2\vartheta\partial_+t'\partial_-t'\Big) + \frac{\sqrt{2k}}{\pi}\int d^2\sigma q\Big(\sqrt{\frac{2}{k}}qt' + \ln\sinh\vartheta\Big)\sinh\vartheta\cosh\vartheta\Big(\partial_+\theta'\partial_-\vartheta - \partial_-\theta'\partial_+\vartheta\Big). \tag{21}$$

Comparing with (16), one deduces

$$G_{\mu\nu} = \begin{pmatrix} \frac{k}{2} & 0 & 0 \\ 0 & q^2\cosh^2\vartheta & 0 \\ 0 & 0 & -q^2\sinh^2\vartheta \end{pmatrix},$$
$$B_{\mu\nu} = \sqrt{\frac{3k}{4}}q\Big(\sqrt{\frac{2}{k}}qt' + \ln\sinh\vartheta\Big)\sinh\vartheta\cosh\vartheta\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

The metric $G_{\mu\nu}$ is exactly the canonical metric on SL$(2,R)$, induced by the map of the Cartan-Killing form in the neighborhood of unity in the Lie algebra $\mathrm{Tr}\big(g^{-1}dg\big)^2$. This is no surprise, since this is exactly the σ model part of the action. The axion is induced completely by the Wess-Zumino term. The dilaton for this solution actually is constant, as can be readily verified from the effective action (17). The dilaton equation of motion is

$$\sqrt{2}\,\nabla^2\phi - (\nabla\phi)^2 + (\tfrac{1}{2}R - H^2 + \Lambda) = 0 \tag{23}$$

and the substitution of solution (22) in (23) yields

$$\Lambda = \frac{Q^2}{3}e^{2\sqrt{2}\Phi_0}, \tag{24}$$

where Q is the three-form cohomology charge of the axion field $H = dB$, defined by the "Gauss law," which, since $^*H$ is a zero form, is just $Q = \frac{1}{2\pi}e^{-\sqrt{2}\Phi_0}\ {}^*H =$ const. In order to make contact with solution (12), a change of coordinates and the compactification of the spatial coordinate θ' need to be made. To do this, I will show that solution (22) is equivalent to (11). Namely, the "radial" coordinate is introduced by $r = q^2\cosh^2\vartheta$. Then, the metric can be rewritten as

$$ds^2 = \frac{1}{2\Lambda}\frac{dr^2}{r(r-q^2)} + r\ d\theta'^2 - (r-q^2)\ dt'^2. \tag{25}$$

This is almost precisely solution (11), with $d_{12} = 0$ and $d_{22} = q^2$. The only difference is the cosmological constant in (11) is half that in (25). The reason for this discrepancy is that the presence of the axion introduces an extra contribution to the cosmological constant, which just cancels one-half of it, since the dilaton is constant. A careful examination of duality transformations of action (17) confirms this. The axion field can be rewritten as $B = \sqrt{3/8}\ \{t' + \sqrt{k/2q^2}\ln[(r/q^2) - 1]\}\ d\theta' \wedge dr$, in form notation, and after the above transformation of coordinates. The axion is apparently time dependent, which can be remedied by recalling the gauge invariance of the axion: B and $B' = B + d\Upsilon$ both describe the same physics. Then, if $\Upsilon = \sqrt{3/8}\Big\{t'r + \sqrt{k/2q^2}(r - q^2)\{\ln[(r/q^2)^2 - 1] - 1\}\Big\}d\theta'$, the gauge transformed axion is

$$B = -\sqrt{\frac{3}{8}}\ r\ d\theta' \wedge dt'. \tag{26}$$

Solutions (25) and (26) have already been found previously in Ref. [5]. At this point, one can perform the identification. Normally, it can be accomplished by identifying $\theta' \cong \theta' + 2\pi$. This would correspond to factoring out the discrete group $P' = \exp(2n\pi\xi')$, with $\xi' = \partial/\partial\theta'$ the translational Killing vector generating motions along θ' and n integers, from SL$(2,R)$. The resulting metric would have zero angular momentum, $J = 0$. However, as was discussed in Sec. III, one can arbitrarily choose to identify along any boosted translational Killing vector which is spatial outside the horizon. Hence, in order to get solution (12) with mass M and angular momentum J, one can boost back the coordinates θ', t' by (13) and identify points by factoring the subgroup $P = \exp(2n\pi\xi)$ out of SL$(2,R)$, where $\xi = \partial/\partial\theta = \cosh\beta\partial/\partial\theta' - \sinh\beta\partial/\partial t'$, and $\sinh\beta$ is given in Eq. (14). The resulting configuration is the metric (12) extended with the axion (26).

There is a minor subtlety here. In order to complete the identification, the axion solution has been gauge transformed by a gauge transformation which involves explicitly the compactified variables (t') and therefore is not single valued on the covering space of the manifold. In this respect, it can be treated as a "large" gauge transformation. Furthermore, since the target has been compactified along θ, in general when a closed string moves on such a world sheet there appear the winding modes associated with the compact directions of the target [16]. The winding modes in principle can interact with the axion field before and after the gauge transformation differently, and thus distinguish between solutions (22), (25), and (26). This can be avoided if one resorts to a different way of constructing the black-hole solution (25) and (26). The reason for the appearance of the gauge transformation Υ was that the axion field has arisen from the Wess-Zumino term in (18). A simple remedy is to gauge the WZWN σ model on $SL(2,R)\times R$ by factoring out the axial vector subgroup of $SL(2,R)\times SL(2,R)$ mixed with translations along R, and taking the extremal limit where all the gauging is along R [9]. This amounts to taking for the target the coset $SL(2,R)\times R/R$. (A similar procedure has been performed in [17], where a closed Bianchi-type I cosmology was constructed.) The central charge of this target for level k is $c_T = 3k/(k-2)+1-1$, where ± 1 corresponds to the free boson and the gauging, respectively. Hence, $c_T = 3k/(k-2)$ and the cosmological constant is still $\Lambda = 4/k$. The resulting solution is exactly (25) and (26), as can be easily verified. The group $SL(2,R)\times R$ can now be parametrized as

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} e^{\frac{q}{\sqrt{k}}\theta'}, \tag{27}$$

where $ab+uv=1$, the explicit form of the ungauged σ model of Eq. (18) is

$$S_\sigma = -\frac{k}{4\pi}\int d^2\sigma\Big(\partial_+ u\partial_- v + \partial_- u\partial_+ v + \partial_+ a\partial_- b + \partial_- a\partial_+ b\Big)$$
$$+\frac{k}{2\pi}\int d^2\sigma \ln u\Big(\partial_+ a\partial_- b - \partial_- a\partial_+ b\Big) + \frac{q^2}{2\pi}\int d^2\sigma\partial_+\theta'\partial_-\theta'. \tag{28}$$

The gauge transformations corresponding to the axial subgroup of $SL(2,R)\times SL(2,R)$ mixed with translations along the free boson are

$$\delta a = 2\epsilon a,\quad \delta b = -2\epsilon b,\quad \delta u = \delta v = 0,$$
$$\delta\theta' = \frac{2\sqrt{2}}{q}\epsilon c,\quad \delta A_j = -\partial_j\epsilon, \tag{29}$$

and the gauged form of the σ model (6) is

$$S_\sigma(g,A) = S_\sigma(g)+\frac{k}{2\pi}\int d^2\sigma A_+\Big(b\partial_- a - a\partial_- b - u\partial_- v + v\partial_- u + \frac{4qc}{\sqrt{2}k}\partial_-\theta'\Big)$$
$$+\frac{k}{2\pi}\int d^2\sigma A_-\Big(b\partial_+ a - a\partial_+ b - v\partial_+ u + u\partial_+ v + \frac{4qc}{\sqrt{2}k}\partial_+\theta'\Big)$$
$$+\frac{k}{2\pi}\int d^2\sigma 4A_+A_-\Big(1+\frac{2c^2}{k} - uv\Big). \tag{30}$$

The remaining steps of the procedure for obtaining the solution are to integrate out the gauge fields, fix the gauge of the group choosing $b=\pm a$ so that the anomaly cancels (removing the need for the "large" gauge transformation Υ as argued above), rescale $\theta' \to (2c/\sqrt{k})\,\theta'$, and take the limit $c\to\infty$ which effectively decouples the $SL(2,R)$ part from the gauge fields. The resulting Polyakov σ model action can be rewritten as

$$S_{\sigma\text{eff}} = -\frac{k}{8\pi}\int d^2\sigma\frac{v^2\partial_+ u\partial_- u + u^2\partial_- v\partial_+ v + (2-uv)\big(\partial_+ u\partial_- v + \partial_- u\partial_+ v\big)}{1-uv}$$
$$+\frac{q^2}{2\pi}\int d^2\sigma\Big(2(1-uv)\partial_+\theta'\partial_-\theta'\Big)$$
$$+\frac{q\sqrt{k}}{2\sqrt{2}\pi}\int d^2\sigma\Big(\big(u\partial_- v - v\partial_- u\big)\partial_+\theta' + \big(v\partial_+ u - u\partial_+ v\big)\partial_-\theta'\Big). \tag{31}$$

A transformation of coordinates

$$u = e^{\sqrt{\frac{2}{k}}qt'}\sqrt{\frac{r}{q^2}-1}, \quad v = -e^{-\sqrt{\frac{2}{k}}qt'}\sqrt{\frac{r}{q^2}-1} \quad (32)$$

reproduces solutions (25) and (26). The dilaton can be found either from the associated effective action, as was discussed before, or from a careful computation of the Jacobian determinant arising from integrating out the gauge fields [8]. Since the metric-axion solution is exactly (25) and (26), by the arguments before the dilaton must be a constant, $\Phi = \Phi_0$. The Jacobian matrix method confirms this. Its inspection before the limit $c \to \infty$ is taken shows that it is $J \propto 1/[1 + (2c^2/k) - uv] = (k/2c^2)/\big(1 + (k/2c^2)(1-uv)\big)$ (see Refs. [8,17]). As $c \to \infty$ the nonconstant terms decouple and do not contribute to the dilaton. Then, the identification procedure can be carried out along the lines elaborated following Eq. (26).

If one compares the method of obtaining (25) and (26) to the constructions of Refs. [1–3], one might object that the identification has invoked a somewhat arbitrary step involving the choice of the vector ξ along which the factorization of $\mathrm{SL}(2,R)$ has been performed. The reason for this arbitrariness lies in the asymptotic properties of solutions (25) and (26). As $r \to \infty$, the metric (25) approaches the vacuum solution with $J = M = 0$. The axion (26) is independent of the mass and spin, and already in the "vacuum" form, and so invariant under boosts (13). Therefore, infinitely far away from the black hole the boost generator is approximately Killing, and hence the metrics with different spins become locally indistinguishable. This, on the other hand, justifies the interpretation of J as a spin, as discussed in Sec. III, and the factorization procedure outlined above. However, it is possible to generate the spin directly in the effective action of the type of Eq. (30), and always compactify along the free boson which appears in the definition of the group. This can be done if one starts with the group $\mathrm{SL}(2,R) \times R^2$, and performs a double gauging[1] down to a coset $\mathrm{SL}(2,R) \times R^2/R^2$ with two different vector fields. The subgroups of $\mathrm{SL}(2,R) \times \mathrm{SL}(2,R)$ are axial *and* vector, mixed with the translations along the two bosons in such way that the anomaly still cancels. This will necessarily introduce a constraint on the gauge charges of the two bosons, but it can be satisfied. I hope to address this issue in a forthcoming paper [18].

V. CONCLUSIONS AND FUTURE INTERESTS

It is evident that the three-dimensional anti-de Sitter black hole of Banados, Teitelboim, and Zanelli represents a very interesting addition to the growing family of black objects. It is not only a nice example of a black hole in three dimensions, but also exhibits a surprisingly rich structure. Moreover, it is also a solution of many different theories of gravity in three dimensions, in growing order of complexity: general relativity, topologically massive gravity (TMG), and string theory. In this paper, only GR and string theory have been explicitly investigated. However, it is not difficult to see that solution (12) also represents a solution of TMG with a negative cosmological constant.

The reason for this is that TMG differs from GR in the presence of the Lorentz Chern-Simons form in the action. This changes Einstein's equations by adding the Cotton tensor to the Einstein equation. Yet both these terms vanish for solution (12). This can be seen as follows. The Cotton tensor is constructed from the covariant derivatives of the Ricci tensor. When the constructions encountered in this paper are present, the Riemann tensor is covariantly conserved and so are all the other curvature invariants. Hence the Cotton tensor is zero. Furthermore, since by the boost (13) solution (12) can be brought locally in the diagonal form (15), and the Lorentz Chern-Simons form of (15) is trivially zero, it also vanishes for (12). Therefore, as claimed, solution (12) also solves the equations of motion of TMG in a trivial way.

One other interesting feature of (12) is that it can be used for constructing a black string configuration in four dimensions. Effectively, all one needs to do is to tensor (12) with a flat direction R. Related considerations have been investigated previously in [5]. There remains a subtlety regarding the interpretation of such solutions. I hope to return to this question elsewhere [18].

In summary, the three-dimensional black hole has shown great promise. It is a truly multifaceted configuration, which possesses rich geometrical structure, and appears to relate different formulations of gravity by being a solution of all of them. One cannot resist the temptation, that perhaps this is not an accident, but rather a beacon pointing at a certain, more fundamental, interconnectedness of these theories of gravity.

Note added. Upon the completion of this work I have received Ref. [19] which overlaps in some length the subject of this work, and where similar results were found.

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[1]Double gauging has been employed previously on different groups in the last of Refs. [7,8] as well as in [11].

[1] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[2] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[3] D. Cangemi, M. Leblanc, and R.B. Mann, Phys. Rev. D (to be published).
[4] S.F. Ross and R.B. Mann, Phys. Rev. D **47**, 3319 (1993).
[5] N. Kaloper, Phys. Rev. D **47**, 2403 (1993).
[6] E. Witten, Phys. Rev. D **44**, 314 (1991).
[7] N. Ishibashi, M. Li, and A.R. Steif, Phys. Rev. Lett. **67**, 3336 (1991); P. Ginsparg and F. Quevedo, Nucl. Phys. **B385**, 527 (1992); I. Bars and K. Sfetsos, Phys. Rev. D **46**, 4495 (1993); **46**, 4510 (1992); K. Sfetsos, Nucl. Phys. **B389**, 424 (1993); I. Bars and K. Sfetsos, Phys. Lett. B **301**, 183 (1993).
[8] P. Horava, Phys. Lett. B **278**, 101 (1992); E.B. Kiritisis, Mod. Phys. Lett. A **6**, 2871 (1991); D. Gershon, Tel Aviv University Report No. TAUP-1937-91, 1991 (unpublished).
[9] J.H. Horne and G.T. Horowitz, Nucl. Phys. **B368**, 444 (1992).
[10] E. Raiten, Fermilab Report No. FERMILAB-PUB-91-338-T 1991 (unpublished).
[11] C.R. Nappi and E. Witten, Phys. Lett. B **293**, 309 (1992).
[12] E. Cremmer, in *Supergravity 81*, edited by S. Ferrara and J.G. Taylor (Cambridge University Press, Cambridge, England, 1982), p. 313.
[13] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[14] C. Misner, in *Relativity Theory and Astrophysics*, edited by J. Ehlers (American Mathematical Society, Providence, RI, 1967).
[15] J. Antoniades, C. Bachas, J. Ellis, and D. Nanopoulos, Phys. Lett. B **221**, 393 (1988); J. Antoniades, C. Bachas, and A. Sagnotti, *ibid.* **235**, 255 (1990).
[16] For a review, see L. Alvarez-Gaume and M.A. Vasquez-Mozo, CERN Report No. CERN-TH-6736-92, 1992 (unpublished).
[17] N. Kaloper, Mod. Phys. Lett. A **8**, 421 (1993).
[18] N. Kaloper (unpublished).
[19] G.T. Horowitz and D.L. Welch, Phys. Rev. Lett. **71**, 328 (1993).