Closed strings with low harmonics and kinks

R. W. Brown, M. E. Convery, S. A. Hotes, M. G. Knepley, and L. S. Petropoulos
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
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Low-harmonic formulas for closed relativistic strings are given. General parametrizations are presented for the addition of second- and third-harmonic waves to the fundamental wave. The method of determination of the parametrizations is based upon a product representation found for the finite Fourier series of string motion in which the constraints are automatically satisfied. The construction of strings with kinks is discussed, including examples. A procedure is laid out for the representation of kinks that arise from self-intersection, and subsequent intercommutation, for harmonically parametrized cosmic strings.

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I. INTRODUCTION

The relativistic string model has been at the heart of much of theoretical elementary particle physics for the past decade, both as a model of elementary particles and as a description of cosmic string defects postulated to have been produced in the early Universe. The cosmic string hypothesis [1] in particular continues to attract interest in attempts to understand the large-scale structure observed in galaxy distributions. Recently it has been argued that cosmic strings are consistent with the nonuniformities observed in the background radiation [2]. In addition, the string model has proven to be a wellspring of discoveries in connecting different mathematics to physics, and as a vehicle for exploring new mathematics and new mathematical techniques.

In the study of closed cosmic strings, one subject of interest has been the harmonic solutions to the relativistic string equations in flat space. Whenever there are damping mechanisms at work, one expects that higher harmonics would get relatively suppressed. One can then consider a finite Fourier series—a series of a finite number of harmonics—and derive the Fourier coefficients required by the constraint equations in a given gauge. Although an infinite number of harmonics is considered at the outset in the study of a quantized string, in order to have a complete basis for distributions and to preserve locality, certain fundamental string issues may be conveniently studied with strings containing a finite number of harmonics.

There is a specific mechanism that regularly dumps power into the higher harmonics of cosmic strings, in the midst of the damping. As a result of the intercommutation of intersecting strings, infinite and closed, scars develop in the form of distinct kinks [3–5]. The kinks present in generic cosmic loops at early stages of the universe do eventually decay away, not by radiation alone [6], but by back reaction to that radiation [7].

In the course of cosmic string studies concerning radiation, self-intersection, and black-hole formation, it has proven useful to construct loop solutions for a few low harmonics [8–11]. A systematic investigation of more and more general parametrizations of the low-lying harmonics has been undertaken by our group over the past few years [12–14]. This has led to a general solution for any closed string with an arbitrarily large number N for the largest harmonic to be included [15–17]. The result corresponds to finding the most general Fourier series of a unit vector, given an arbitrary but finite number of harmonics. Consistent with what we have come to expect from string theory, we have a new mathematics tool useful for other applications.

In this paper we put the new methodology to work. We construct general $N \leq 3$ solutions for closed loops. A catalog of previous solutions is presented in terms of the product representation parameters, and the inclusion of kinks is considered (see Refs. [18,19] for earlier discussions). A construction algorithm for string solutions containing a single left- or right-moving kink is illustrated. We analyze the fact that when kinks are created through intercommutation, the kink will split into right- and left-traveling pieces. A general procedure is provided for analytically describing these resulting equations of motion, and an example is given.

The flat-space string equations and their Fourier analysis comprise Secs. II and III, respectively. Section IV concentrates upon the exact solutions for the general case of strings with first, second, and third harmonics using the special rotation form described in Sec. III. Strings previously developed by others are rewritten within the framework of the rotation form in Sec. V, and shown to be a subset of this more general procedure. Figures are used to illustrate loop motion for various parametrizations. We next introduce the concept of a kinked string. In Sec. VI we show how single kink strings may be introduced by fine-tuning the parametrization of the right- (or left-) traveling wave as it traverses the Kibble-Turok sphere. Section VII describes the process of intercommutation and an example, employing the results of the previous section. Section VIII contains some concluding remarks.

II. CLOSED STRING REVIEW

In an orthonormal gauge [20], the string position $\mathbf{r}(\sigma, t)$ satisfies the wave equation $\mathbf{r}_{tt} - \mathbf{r}_{\sigma\sigma} = 0$ ($f_t \equiv \partial f/\partial t$, etc.), and the constraints are transverse motion $\mathbf{r}_t \cdot \mathbf{r}_\sigma = 0$ and unit energy density $r_t^2 + r_\sigma^2 = 1$. Within this Lorentz frame, the string parameter σ has the same units as the time t, and we scale to the interval $0 \le \sigma \le 2\pi$. The general "right-going" ($u = \sigma - t$) plus "left-going" ($v = \sigma + t$) wave solution is

$$\mathbf{r} = \frac{1}{2}[\mathbf{a}(u) + \mathbf{b}(v)], \tag{1}$$

in what has become rather standard notation. The constraints imply

$$\mathbf{a}'^{\,2} = \mathbf{b}'^{\,2} = 1; \tag{2}$$

that is, $\mathbf{a}'$ and $\mathbf{b}'$ traverse the Kibble-Turok sphere [8]. We see that we must have both left- and right-going waves, and they must be "equally weighted" in the sense that both derivatives have unit magnitude.

The overall spatial periodicity of a closed loop of string, $\mathbf{r}(\sigma + 2\pi, t) = \mathbf{r}(\sigma, t)$ or

$$\mathbf{a}(u + 2\pi) + \mathbf{b}(v + 2\pi) = \mathbf{a}(u) + \mathbf{b}(v), \tag{3}$$

holds true up to linear (c.m.) terms for $\mathbf{a}$ and $\mathbf{b}$ as well. Consider two values of u, but with v fixed. Equation (3) yields

$$\mathbf{a}(u_1 + 2\pi) - \mathbf{a}(u_2 + 2\pi) = \mathbf{a}(u_1) - \mathbf{a}(u_2), \tag{4}$$

and,

$$\mathbf{a}'(u + 2\pi) = \mathbf{a}'(u). \tag{5}$$

Similarly,

$$\mathbf{b}'(v + 2\pi) = \mathbf{b}'(v). \tag{6}$$

That is, the unit vectors of Eq. (2) are periodic. From Eqs. (3), (5), and (6),

$$\mathbf{a}(u + 2\pi) = \mathbf{a}(u) + \mathbf{k}, \tag{7}$$

$$\mathbf{b}(v + 2\pi) = \mathbf{b}(v) - \mathbf{k}. \tag{8}$$

We can write

$$\mathbf{a}(u) \equiv \mathbf{a}_p(u) + \mathbf{k}u/2\pi, \tag{9}$$

$$\mathbf{b}(v) \equiv \mathbf{b}_p(v) - \mathbf{k}v/2\pi, \tag{10}$$

where $\mathbf{a}_p$ and $\mathbf{b}_p$ are periodic functions of their arguments, with period 2π. In a Fourier series, $\mathbf{a}_p'$ and $\mathbf{b}_p'$ have no zero harmonic terms. The derivatives,

$$\mathbf{a}' = \mathbf{a}_p' + \mathbf{k}/2\pi, \tag{11}$$

$$\mathbf{b}' = \mathbf{b}_p' - \mathbf{k}/2\pi, \tag{12}$$

are indeed periodic, and $\pm\mathbf{k}/2\pi$ are identified as the respective zero harmonics.

Finally, we have

$$\mathbf{r}(\sigma, t) = \mathbf{r}_p(\sigma, t) + \mathbf{K}t, \tag{13}$$

with

$$\mathbf{r}_p \equiv \frac{1}{2}[\mathbf{a}_p(u) + \mathbf{b}_p(v)], \tag{14}$$

$$\mathbf{K} \equiv -\mathbf{k}/\pi. \tag{15}$$

Equation (13) is the generic form for closed loops: periodic left-going and right-going superimposed on uniform (c.m.) motion.

Now we address temporal periodicity. Modulo the c.m. motion, the string certainly repeats in 2π time intervals. From (4),

$$\mathbf{r}_p(\sigma, t + 2\pi) = \mathbf{r}_p(\sigma, t). \tag{16}$$

But in fact the effective time period is half of this [8] because under $\sigma \to \sigma + \pi$, $t \to t + \pi$, we have $u \to u$, $v \to v + 2\pi$,

$$\mathbf{r}_p(\sigma + \pi, t + \pi) = \mathbf{r}_p(\sigma, t). \tag{17}$$

The string "looks" exactly the same every time interval T,

$$T = \pi. \tag{18}$$

The first half of the string ($0 < \sigma < 2\pi$) switches places with the second half ($\pi < \sigma < 2\pi$) every time T. The individual unit vectors Eqs. (9) and (10), however, have period 2π.

III. FOURIER ANALYSIS AND PRODUCT REPRESENTATION

We are thus led to consider the Fourier analysis of $\mathbf{a}'$ and $\mathbf{b}'$, the integration of which will give us the string configuration. We restrict ourselves to a finite number of harmonics according to the discussion in the introduction. The problem of finding the harmonic coefficients in the finite Fourier series for a periodic vector whose magnitude is fixed has just recently been solved. The product representation of Ref. [15] automatically satisfies the magnitude constraint, exhibits the correct degrees of freedom, and gives the general solution.

Consider a periodic unit vector $\hat{u}_N(s) = \hat{u}_N(s + 2\pi)$ defined by the N-harmonic real series,

$$\hat{u}_N(s) = \mathbf{Z} + \sum_{n=1}^{N}(\mathbf{A}_n \cos ns + \mathbf{B}_n \sin ns), \tag{19}$$

with N arbitrary. The constraint $(\hat{u}_N)^2 = 1$ requires a set of nonlinear relations among the vector coefficients in

Eq. (19). In terms of the real basis, we have the $4N+1$ real equations

$$\sum_{n=m-N}^{N} (\boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_{m-n} - \boldsymbol{\beta}_n \cdot \boldsymbol{\beta}_{m-n}) = 4\delta_{m0}, \qquad m = 0, 1, \ldots, 2N, \quad (20)$$

$$\sum_{n=m-N}^{N} (\boldsymbol{\alpha}_n \cdot \boldsymbol{\beta}_{m-n} + \boldsymbol{\beta}_n \cdot \boldsymbol{\alpha}_{m-n}) = 0, \quad m = 1, \ldots, 2N. \quad (21)$$

Here,

$$\boldsymbol{\alpha}_n = \boldsymbol{\alpha}_{-n} = \mathbf{A}_n, \quad \boldsymbol{\beta}_n = -\boldsymbol{\beta}_{-n} = \mathbf{B}_n, \qquad n \neq 0, \quad (22)$$

$$\boldsymbol{\alpha}_0 = 2\mathbf{Z}, \quad \boldsymbol{\beta}_0 = 0. \quad (23)$$

The product representation which solves Eq. (19) can be written in terms of standard matrices such as $R_\xi(\psi)$, which refers to a rotation of a vector through the angle ψ about the ξ-axis (or of the coordinate system through $-\psi$). Making a specific choice of a constant unit vector which we rotate in succession, we have

$$\hat{u}_N(s) = r_N R_z(s) \rho_N R_z(s) \rho_{N-1} \cdots R_z(s) \rho_1 \hat{z}. \quad (24)$$

Here,

$$\rho_i = R_z(-\theta_i) R_x(\phi_i) R_z(\theta_i). \quad (25)$$

We can let $\theta_1 = \pi/2$, or

$$\rho_1 = R_y(-\phi_1). \quad (26)$$

Also, there is the overall orientation freedom

$$r_N = \begin{cases} R_z(\alpha)R_x(\beta)R_z(\gamma), & N \geq 1, \\ R_z(\alpha)R_x(\beta), & N = 0. \end{cases} \quad (27)$$

where α, β, and γ are additional constants (angle parameters).

Let us consider how Eq. (24) gives the desired properties. The rotations preserve the vector magnitude, satisfying the original constraint, and the N factors of $R_z(s)$ generate N harmonics. In general, this is a complete and independent representation, although one needs to look at the detailed proof [15,16] to see this. The $2N+2$ independent degrees of freedom $[(6N+3)-(4N+1)]$ are unrestricted angles, θ_i and ϕ_i, whose ranges are independent of each other ($0 \leq \theta_i \leq \pi$, $0 \leq \phi_i \leq 2\pi$). (In examples, there may exist reflection symmetries, such as $\phi_i \to -\phi_i$ or $\pi - \phi_i$ for some of the angles, reducing the overall range accordingly.) Because the signs can be a source of confusion in the derivation of the results in the next section, we list the matrix conventions for active rotations

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{pmatrix}, \quad R_y(\theta) = \begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix}, \quad R_z(\theta) = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (28)$$

with $s\theta \equiv \sin\theta$, $c\theta \equiv \cos\theta$.

To complete the groundwork for a procedure for constructing closed strings for a given N (zeroth plus first N harmonics), we note that the overall rotation r_N can be omitted in the product representation (24) of the unit vector $\mathbf{a}$,

$$\mathbf{a}'_N(u, \theta_i, \phi_i) = \prod_1^N [R_z(u)\rho_i]\hat{\mathbf{z}}. \quad (29)$$

Here, $f(\theta_i) \equiv f(\{\theta_i\})$, etc. In place of r_N, we consider the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ basis as arbitrarily oriented. To obtain final polynomial expressions in $\sin\theta_i$, $\cos\theta_i$, etc., it is convenient to use the iterative property

$$\mathbf{a}'_N = R_z(u-\theta_N) R_x(\phi_N) R_z(\theta_N) \mathbf{a}'_{N-1}. \quad (30)$$

Afterwards, it is necessary to separate out the zero harmonic piece in Eq. (11), defined to be $\boldsymbol{\alpha}_N$,

$$\mathbf{a}'_N(u, \theta_i, \phi_i) = \mathbf{a}'_{p,N}(u, \theta_i, \phi_i) + \boldsymbol{\alpha}_N(\theta_i, \phi_i). \quad (31)$$

We can immediately write down the other unit vector and its zero harmonic component β_N

$$\begin{aligned} \mathbf{b}'_N(v, \theta'_i, \phi'_i) &= \mathbf{b}'_{p,N}(v, \theta'_i, \phi'_i) + \boldsymbol{\beta}_N(\theta'_i, \phi'_i) \\ &= \mathbf{a}'_N(v, \theta'_i, \phi'_i)|_{\hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}} \to \hat{\mathbf{x}}'\hat{\mathbf{y}}'\hat{\mathbf{z}}'}. \end{aligned} \quad (32)$$

That is, $u \to v$, and prime everything else. Finally, we must have the periodicity condition (15),

$$\boldsymbol{\alpha}_N(\theta_i, \phi_i) = -\beta_N(\theta'_i, \phi'_i) = -\frac{1}{2}\mathbf{K}_N. \quad (33)$$

This last condition can be met in two ways. The first we call the traveling string case where all θ'_i, ϕ'_i, $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, $\hat{\mathbf{z}}'$ are found so that (33) is satisfied. The second we refer to as the c.m. frame, where $\mathbf{K}_N = 0$. We find all θ_i, ϕ_i such that

$$\boldsymbol{\alpha}_N = 0, \quad (34)$$

and θ'_i, ϕ'_i such that

$$\boldsymbol{\beta}_N = 0. \quad (35)$$

IV. GENERAL PARAMETRIZATIONS OF N=0,1,2,3 CLOSED STRINGS

We begin with a zeroth harmonic and then add in sequence a first, second, and finally a third harmonic. In each case we describe two different solutions corresponding to moving strings and c.m. strings, as described in the previous section.

A. $N=0$

The $N = 0$ trivial zero-harmonic case for Eq. (30) is included if only to highlight the freedom to choose an overall z-axis direction. We have $\mathbf{a}' = -\hat{\mathbf{z}}$, $\mathbf{b}' = \hat{\mathbf{z}}'$, with the periodicity of $\mathbf{r}$ in σ forcing $\hat{\mathbf{z}}' = \hat{\mathbf{z}}$ [recall Eq. (17)]. The result is a point moving at the speed of light,

$$\mathbf{r}_0 = \hat{\mathbf{z}}t. \tag{36}$$

B. $N = 1$

The general combination of zeroth plus first harmonic is the $N = 1$ case in Eq. (30). In simpler notation ($\phi_1 \equiv -\theta, -\theta'$ for $\mathbf{a}', \mathbf{b}'$, respectively), matrix multiplication leads to the intermediate answers

$$\mathbf{a}' = \sin\theta\cos u\,\hat{\mathbf{x}} + \sin\theta\sin u\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \tag{37}$$

$$\mathbf{b}' = \sin\theta'\cos v\,\hat{\mathbf{x}}' \pm \sin\theta'\sin v\,\hat{\mathbf{y}}' + \cos\theta'\,\hat{\mathbf{z}}'. \tag{38}$$

In addition to the z freedom, we have chosen a particular handedness for the time rotation of $\mathbf{a}'$ for a given θ. But then $\mathbf{b}'$ can have both right- and left-handed circulation.

1. Traveling string

For the "moving string" procedure of satisfying σ periodicity, $\hat{\mathbf{z}}' = \hat{\mathbf{z}}$ and $\theta' = \theta + \pi$. Integration gives the traveling-loop solution for both circulations,

$$\mathbf{r}^{\pm} = \frac{1}{2}\sin\theta[(\sin u - \sin v)\hat{\mathbf{x}} + (-\cos u \pm \cos v)\hat{\mathbf{y}}] - \cos\theta\hat{\mathbf{z}}t. \tag{39}$$

[In such deliberations, any rotation by an angle χ of $\hat{\mathbf{x}}' - \hat{\mathbf{y}}'$ relative to $\hat{\mathbf{x}} - \hat{\mathbf{y}}$ can be absorbed into redefinitions (shifts) of σ and t: $\hat{\mathbf{x}}'\cos v + \hat{\mathbf{y}}'\sin v = \hat{\mathbf{x}}\cos(v+\chi) + \hat{\mathbf{y}}\sin(v+\chi) \to \hat{\mathbf{x}}\cos v + \hat{\mathbf{y}}\sin v$, for $\sigma \to \sigma - \chi/2$, $t \to t - \chi/2$.] Equation (39) refers to two simple planar strings with constant c.m. motion that is perpendicular to the planes. Rewriting, we see these are, respectively, circles with oscillating radii,

$$\mathbf{r}^{+} = -\sin\theta\sin t\hat{\boldsymbol{\rho}}(\sigma) - \cos\theta\hat{\mathbf{z}}t, \tag{40}$$

and uniformly rotating sticks of fixed length,

$$\mathbf{r}^{-} = -\sin\theta\cos\sigma\hat{\boldsymbol{\phi}}(-t) - \cos\theta\hat{\mathbf{z}}t. \tag{41}$$

Here we have used the cylindrical unit vectors

$$\hat{\boldsymbol{\rho}}(A) = \hat{\mathbf{x}}\cos A + \hat{\mathbf{y}}\sin A, \qquad \hat{\boldsymbol{\phi}}(A) = -\hat{\mathbf{x}}\sin A + \hat{\mathbf{y}}\cos A. \tag{42}$$

The speed of light is reached at the ends of the sticks or when the circles pass through zero radius. The limits $\theta = 0, \pi$ give the trivial case presented in Eq. (36).

2. c.m. string

In the "c.m." procedure, we eliminate the zero harmonics in Eqs. (37) and (38) separately and hence the c.m. motion, by the requirement that $\theta = \theta' = \pi/2$. Visualizing the most general intersection of two great circles on the Kibble-Turok sphere, we let $\hat{\mathbf{y}}' = \hat{\mathbf{y}}$ and $\hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos\psi + \hat{\mathbf{z}}\sin\psi$:

$$\mathbf{r}^{\pm} = \frac{1}{2}[(\sin u + \sin v\cos\psi)\hat{\mathbf{x}} - (\cos u \pm \cos v)\hat{\mathbf{y}} + \sin v\sin\psi\hat{\mathbf{z}}]. \tag{43}$$

These strings look like rotating ellipses that oscillate in size, collapsing periodically to sticks. They are natural interpolations between the circle and stick results in Eqs. (40), (41), forms to which they reduce in the $\psi = 0, \pi$ limits but without any overall motion. See Fig. 1 for an illustration. It is interesting, however, that we cannot get the general c.m. solution by simple limits on the traveling string solution.

C. $N = 2$

The combination of the three harmonics—zeroth, first, and second—illustrates an interesting point. In the c.m. string limit where one eliminates the zeroth harmonic, it is seen that no solution is possible for that particular "half" of the string ($\mathbf{a}$ or $\mathbf{b}$). A first harmonic and a second harmonic cannot coexist. In fact, the only pair of nonzero harmonics that can coexist must be in the ratio of three to one [13].

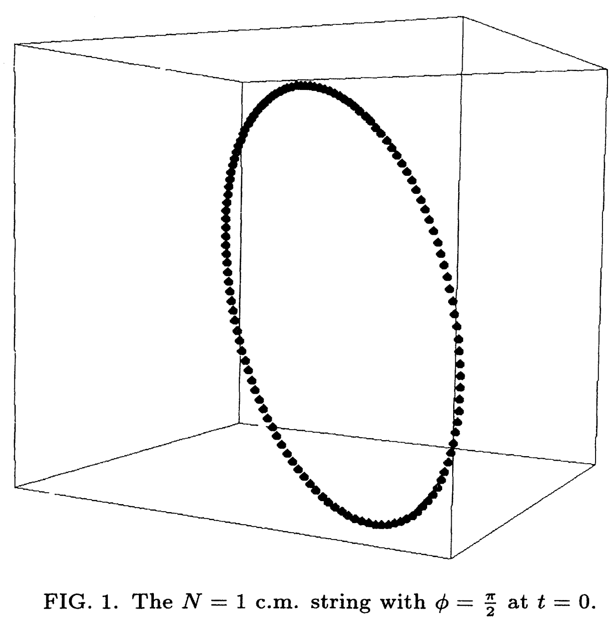

FIG. 1. The $N = 1$ c.m. string with $\phi = \frac{\pi}{2}$ at $t = 0$.

1. *Traveling string*

For a nonzero zero-harmonic contribution, all three harmonics coexist. In the "moving string" procedure of dealing with the zero harmonics of $\mathbf{a}'$ and $\mathbf{b}'$, the σ terms cancel when the angles are related by

$$\phi_2' = \pm\phi_2,\ \phi_1' = \phi_1 + \pi,\ \theta_2' = \theta_2, \tag{44}$$

where the unprimed and primed angles are parameters of $\mathbf{a}$ and $\mathbf{b}$, respectively. This gives four strings due to the two possible values of ϕ_2 (represented by $\pm_2$) and to the freedom for $\mathbf{b}$ to rotate in a right- or left-handed direction (represented by $\pm$):

$$\begin{aligned}\mathbf{r} = &\begin{pmatrix} \cos^2\frac{\phi_2}{2}\sin\phi_1 \\ 0 \\ 0 \end{pmatrix}\frac{1}{4}(\sin 2u - \sin 2v) + \begin{pmatrix} 0 \\ \cos^2\frac{\phi_2}{2}\sin\phi_1 \\ 0 \end{pmatrix}\frac{1}{4}(-\cos 2u \pm \cos 2v) \\ &+ \begin{pmatrix} -\sin\phi_2\cos\phi_1\sin\theta_2 \\ -\sin\phi_2\cos\phi_1\cos\theta_2 \\ \sin\phi_2\sin\phi_1\sin\theta_2 \end{pmatrix}\frac{1}{2}(\sin u - (\pm_2)\sin v) + \begin{pmatrix} \sin\phi_2\cos\phi_1\cos\theta_2 \\ -\sin\phi_2\cos\phi_1\sin\theta_2 \\ \sin\phi_2\sin\phi_1\cos\theta_2 \end{pmatrix}\frac{1}{2}(-\cos u \pm (\pm_2)\cos v) \\ &+ \begin{pmatrix} -\sin^2\frac{\phi_2}{2}\sin\phi_1\cos 2\theta_2 \\ \sin^2\frac{\phi_2}{2}\sin\phi_1\sin 2\theta_2 \\ -\cos\phi_2\cos\phi_1 \end{pmatrix} t\,. \end{aligned} \tag{45}$$

2. *c.m. string*

There are two ways to set both the zero harmonics equal to zero.

$$\phi_2' = \phi_2 = \frac{\pi}{2}, \frac{3\pi}{2}, \quad \phi_1' = \phi_1 = 0, \pi\,. \tag{46}$$

This reduces the string to a two-parameter $N = 1$ string where $\theta_1 = \theta_2$, $\phi_1 = \frac{\pi}{2}$, $\theta_1' = \theta_2'$, $\phi_1' = \frac{\pi}{2}$.

The second set of angles

$$\phi_2' = \phi_2 = 0,\ \phi_1' = \phi_1 = \frac{\pi}{2}, \frac{3\pi}{2}, \tag{47}$$

give a string with only a second harmonic

$$\mathbf{r} = \pm\frac{1}{4}\begin{pmatrix} (\sin 2u + \cos\psi\sin 2v) \\ -\cos 2u \mp \cos 2v \\ \sin\psi\sin 2v \end{pmatrix}, \tag{48}$$

which is simply a z rotation of u on Eq. (43).

Other solutions may be obtained by satisfying one of the above constraints on one half of the string, and another set of constraints on the other half. For example, setting $\phi_2 = 0$, $\phi_1 = \frac{\pi}{2}$, $\phi_2' = \frac{\pi}{2}$, $\phi_1' = 0$, we obtain the string

$$\mathbf{r} = \frac{1}{4}\begin{pmatrix} -2(\cos\theta_2\cos u + \sin\theta_2\sin u) + \cos\psi\sin 2v \\ 2(\cos u\sin\theta_2 - \cos\theta_2\sin u) - \cos 2v \\ \sin\psi\sin 2v \end{pmatrix}. \tag{49}$$

D. $N = 3$

1. *Traveling string*

The general equations fixing the zero harmonics so that they cancel are highly nonlinear, and the solution set is very difficult to define, but the following four solutions are fairly easy to see:

$$\begin{aligned} &\phi_3' = \phi_3, && \theta_3' = \theta_3, && \phi_2' = \phi_2, && \theta_2' = \theta_2, && \phi_1' = \phi_1 + \pi, \\ &\phi_3' = \phi_3, && \theta_3' = \theta_3, && \phi_2' = -\phi_2, && \theta_2' = \theta_2 + \pi, && \phi_1' = \phi_1 + \pi, \\ &\phi_3' = -\phi_3, && \theta_3' = \theta_3 + \pi, && \phi_2' = \phi_2, && \theta_2' = \theta_2, && \phi_1' = \phi_1 + \pi, \\ &\phi_3' = -\phi_3, && \theta_3' = \theta_3 + \pi, && \phi_2' = -\phi_2, && \theta_2' = \theta_2 + \pi, && \phi_1' = \phi_1 + \pi. \end{aligned}$$

These all give the same two string equations [in the more concise form of $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\rho}}$ from (42)],

$$\begin{aligned}\mathbf{r} = &-\frac{1}{3}\cos^2\frac{\phi_3}{2}\cos^2\frac{\phi_2}{2}\sin\phi_1\sin 3t\,\hat{\boldsymbol{\rho}}(3\sigma) + \frac{1}{2}\cos^2\frac{\phi_3}{2}\sin\phi_2\cos\phi_1\sin 2t\,\hat{\boldsymbol{\phi}}(2\sigma - \theta_2) \\ &+\frac{1}{4}\sin\phi_3\sin\phi_2\sin\phi_1\sin 2t\,\hat{\boldsymbol{\rho}}(2\sigma - \theta_3 + \theta_2) - \frac{1}{2}\sin\phi_3\cos^2\frac{\phi_2}{2}\sin\phi_1\sin 2t\sin(2\sigma + \theta_3)\hat{\mathbf{z}} \\ &-\cos^2\frac{\phi_3}{2}\sin^2\frac{\phi_2}{2}\sin\phi_1\sin t\,\hat{\boldsymbol{\rho}}(\sigma - 2\theta_2) - \sin^2\frac{\phi_3}{2}\sin^2\frac{\phi_2}{2}\sin\phi_1\sin t\,\hat{\boldsymbol{\rho}}(\sigma - 2\theta_3 + 2\theta_2)\end{aligned}$$

$$
\begin{aligned}
&+\sin\phi_3\cos\phi_2\cos\phi_1\sin t\,\hat{\boldsymbol{\phi}}(\sigma-\theta_3)-\sin^2\frac{\phi_3}{2}\cos^2\frac{\phi_2}{2}\sin\phi_1\sin t\,\hat{\boldsymbol{\rho}}(-\sigma-2\theta_3)\\
&-\cos\phi_3\sin\phi_2\sin\phi_1\sin t\sin(\sigma+\theta_2)\hat{\mathbf{z}}+\sin\phi_3\sin\phi_2\cos\phi_1\sin t\cos(\sigma+\theta_3-\theta_2)\hat{\mathbf{z}}\\
&-\frac{1}{2}\sin\phi_3\sin\phi_2\sin\phi_1\,t\,\hat{\boldsymbol{\rho}}(-\theta_3-\theta_2)-\sin^2\frac{\phi_3}{2}\sin\phi_2\cos\phi_1\,t\,\hat{\boldsymbol{\rho}}(2\theta_3-\theta_2)\\
&-\left(\cos\phi_3\cos\phi_2\cos\phi_1+\sin\phi_3\sin^2\frac{\phi_2}{2}\sin\phi_1\sin(\theta_3-2\theta_2)\right)t\,\hat{\mathbf{z}}\,,
\end{aligned}\tag{50}
$$

or

$$
\begin{aligned}
\mathbf{r}=&-\frac{1}{3}\cos^2\frac{\phi_3}{2}\cos^2\frac{\phi_2}{2}\sin\phi_1\cos3\sigma\,\hat{\boldsymbol{\phi}}(-3t)-\frac{1}{2}\cos^2\frac{\phi_3}{2}\sin\phi_2\cos\phi_1\cos2\sigma\,\hat{\boldsymbol{\rho}}(-2t-\theta_2)\\
&+\frac{1}{4}\sin\phi_3\sin\phi_2\sin\phi_1\cos2\sigma\,\hat{\boldsymbol{\phi}}(-2t-\theta_3+\theta_2)-\frac{1}{2}\sin\phi_3\cos^2\frac{\phi_2}{2}\sin\phi_1\cos2\sigma\cos(2t-\theta_3)\hat{\mathbf{z}}\\
&-\cos^2\frac{\phi_3}{2}\sin^2\frac{\phi_2}{2}\sin\phi_1\cos\sigma\,\hat{\boldsymbol{\phi}}(-t-2\theta_2)-\sin^2\frac{\phi_3}{2}\sin^2\frac{\phi_2}{2}\sin\phi_1\cos\sigma\,\hat{\boldsymbol{\phi}}(-t-2\theta_3+2\theta_2)\\
&-\sin\phi_3\cos\phi_2\cos\phi_1\cos\sigma\,\hat{\boldsymbol{\rho}}(-t-\theta_3)+\sin^2\frac{\phi_3}{2}\cos^2\frac{\phi_2}{2}\sin\phi_1\cos\sigma\,\hat{\boldsymbol{\phi}}(t-2\theta_3)\\
&-\cos\phi_3\sin\phi_2\sin\phi_1\cos\sigma\cos(t-\theta_2)\hat{\mathbf{z}}+\sin\phi_3\sin\phi_2\cos\phi_1\cos\sigma\sin(t-\theta_3+\theta_2)\hat{\mathbf{z}}\\
&-\frac{1}{2}\sin\phi_3\sin\phi_2\sin\phi_1\,t\,\hat{\boldsymbol{\rho}}(-\theta_3-\theta_2)-\sin^2\frac{\phi_3}{2}\sin\phi_2\cos\phi_1\,t\,\hat{\boldsymbol{\rho}}(2\theta_3-\theta_2)\\
&-\left(\cos\phi_3\cos\phi_2\cos\phi_1+\sin\phi_3\sin^2\frac{\phi_2}{2}\sin\phi_1\sin(\theta_3-2\theta_2)\right)t\,\hat{\mathbf{z}}.
\end{aligned}\tag{51}
$$

2. c.m. string

There are many ways to set both zero harmonics equal to zero. Because any combination of these string halves is possible, we present only the equation for **a**.

There are three solutions which produce a string with first, second, and third harmonics:

$$\phi_3=0,\quad \phi_2=\frac{\pi}{2},\frac{3\pi}{2}\,,\tag{52}$$

$$
\begin{aligned}
\mathbf{a}=&\left(\frac{1}{6}\sin\phi_1\sin3u-(\pm_2)\frac{1}{2}\cos\phi_1\cos(2u-\theta_2)+\frac{1}{2}\sin\phi_1\sin(u-2\theta_2)\right)\hat{\mathbf{x}}\\
&+\left(-\frac{1}{6}\sin\phi_1\cos3u-(\pm_2)\frac{1}{2}\cos\phi_1\sin(2u-\theta_2)-\frac{1}{2}\sin\phi_1\cos(u-2\theta_2)\right)\hat{\mathbf{y}}\\
&+[-(\pm_2)\sin\phi_1\cos(u+\theta_2)]\,\hat{\mathbf{z}}\,,
\end{aligned}\tag{53}
$$

$$\phi_3=\frac{\pi}{2},\ \ \frac{3\pi}{2},\ \phi_2=0,\tag{54}$$

$$
\begin{aligned}
\mathbf{a}=&\left(\frac{1}{6}\sin\phi_1\sin3u-(\pm_3)\cos\phi_1\cos(u-\theta_3)+\frac{1}{2}\sin\phi_1\sin(u+2\theta_3)\right)\hat{\mathbf{x}}\\
&+\left(-\frac{1}{6}\sin\phi_1\cos3u-(\pm_3)\cos\phi_1\sin(u-\theta_3)+\frac{1}{2}\sin\phi_1\cos(u+2\theta_3)\right)\hat{\mathbf{y}}\\
&+\left(-(\pm_3)\frac{1}{2}\sin\phi_1\cos(2u+\theta_3)\right)\hat{\mathbf{z}}\,,
\end{aligned}\tag{55}
$$

$$\phi_2=0,\quad \phi_1=\frac{\pi}{2},\frac{3\pi}{2},\tag{56}$$

$$
\begin{aligned}
\mathbf{a}=&(\pm_1)\left(\frac{1}{3}\cos^2\frac{\phi_3}{2}\sin3u+\sin^2\frac{\phi_3}{2}\sin(u+2\theta_3)\right)\hat{\mathbf{x}}\\
&+(\pm_1)\left(-\frac{1}{3}\cos^2\frac{\phi_3}{2}\cos3u+\sin^2\frac{\phi_3}{2}\cos(u+2\theta_3)\right)\hat{\mathbf{y}}\\
&+(\pm_1)\left(-\frac{1}{2}\sin\phi_3\cos(2u+\theta_3)\right)\hat{\mathbf{z}}\,,
\end{aligned}\tag{57}
$$

The angles

$$\phi_3 = 0, \quad \phi_1 = \frac{\pi}{2}, \frac{3\pi}{2}, \tag{58}$$

give a string with only first and third harmonics

$$\begin{aligned}\mathbf{a} = (\pm_1)&\left(\frac{1}{3}\cos^2\frac{\phi_2}{2}\sin 3u + \sin^2\frac{\phi_2}{2}\sin(u-2\theta_2)\right)\hat{\mathbf{x}}\\ &+(\pm_1)\left(-\frac{1}{3}\cos^2\frac{\phi_2}{2}\cos 3u - \sin^2\frac{\phi_2}{2}\cos(u-2\theta_2)\right)\hat{\mathbf{y}}\\ &+(\pm_1)[-\sin\phi_2\cos(u+\theta_2)]\,\hat{\mathbf{z}}\,.\end{aligned} \tag{59}$$

The eight other solutions found reduce to strings with only a first harmonic. These solutions are

$$\begin{aligned}&\phi_3 = \pi, && \phi_1 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, &&\\ &\phi_3 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, && \phi_2 = \pi, && \phi_1 = 0, \pi,\\ &\phi_3 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, && \phi_2 = \pi, && \theta_3 = 2\theta_2,\\ &\phi_3 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, && \phi_2 = \pi, && \theta_3 = 2\theta_2 + \pi,\\ &\phi_2 = \pi, && \phi_1 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, && \theta_3 = 2\theta_2,\\ &\phi_2 = \pi, && \phi_1 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, && \theta_3 = 2\theta_2 + \pi,\\ &\phi_2 = \pi, && \theta_3 = 2\theta_2 + \tfrac{\pi}{2}, && \phi_3 = -\phi_1 + \tfrac{\pi}{2}, -\phi_1 + \tfrac{3\pi}{2},\\ &\phi_2 = \pi, && \theta_3 = 2\theta_2 + \tfrac{3\pi}{2}, && \phi_3 = \phi_1 + \tfrac{\pi}{2}, \phi_1 + \tfrac{3\pi}{2}.\end{aligned} \tag{60}$$

An example of an $N = 3$ string is given here,

$$\begin{aligned}\mathbf{r} = &\frac{1}{12}\sin\phi_1\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix}\sin 3u - \frac{1}{12}\sin\phi_1\begin{pmatrix}-\sin\alpha\cos\beta\\ \cos\alpha\cos\beta\\ \sin\beta\end{pmatrix}\cos 3u\\ &+\frac{1}{4}\cos\phi_1\begin{pmatrix}\sin\alpha\cos\beta\\ -\cos\alpha\cos\beta\\ -\sin\beta\end{pmatrix}\sin(2u-\theta_2) - \frac{1}{4}\cos\phi_1\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix}\cos(2u-\theta_2)\\ &+\frac{1}{4}\sin\phi_1\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix}\sin(u-2\theta_2) - \frac{1}{4}\sin\phi_1\begin{pmatrix}-\sin\alpha\cos\beta\\ \cos\alpha\sin\beta\\ \sin\beta\end{pmatrix}\cos(u-2\theta_2)\\ &-\frac{1}{2}\sin\phi_1\begin{pmatrix}\sin\alpha\sin\beta\\ -\cos\alpha\sin\beta\\ -\cos\beta\end{pmatrix}\cos(u+\theta_2) + \frac{1}{12}\begin{pmatrix}\sin\phi_1'\\ 0\\ 0\end{pmatrix}\sin 3v\\ &-\frac{1}{12}\begin{pmatrix}0\\ \sin\phi_1'\\ 0\end{pmatrix}\cos 3v + \frac{1}{4}\begin{pmatrix}0\\ -\cos\phi_1'\\ 0\end{pmatrix}\sin(2v-\theta_2')\\ &-\frac{1}{4}\begin{pmatrix}\cos\phi_1'\\ 0\\ 0\end{pmatrix}\cos(2v-\theta_2') + \frac{1}{4}\begin{pmatrix}\sin\phi_1'\\ 0\\ 0\end{pmatrix}\sin(v-\theta_2')\\ &-\frac{1}{4}\begin{pmatrix}0\\ \sin\phi_1'\\ 0\end{pmatrix}\cos(v-2\theta_2') - \frac{1}{2}\begin{pmatrix}0\\ 0\\ \sin\phi_1'\end{pmatrix}\cos(v+\theta_2').\end{aligned} \tag{61}$$

This string is displayed in Fig. 2 for a given choice of parameters.

V. PREVIOUS STRINGS

In this section we rewrite other existing parametrizations in terms of our rotation angles. (These rewritten versions have also been referenced earlier in [15].) The Turok string [9], shown in Fig. 3 for a particular choice of parameters,

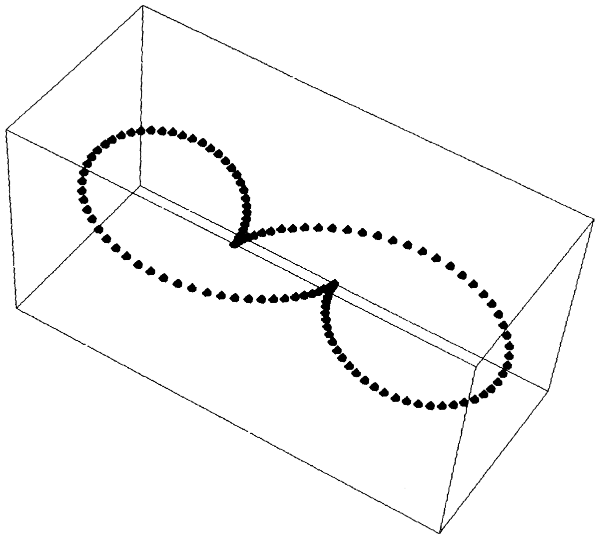

FIG. 2. An $N = 3$ c.m. string with $\alpha = 1.589$, $\beta = 1.712$, $\theta = 1.23$, $\theta' = 3.011$, $\phi = 0.785$, and $\phi' = 2.901$ at $t = 2$.

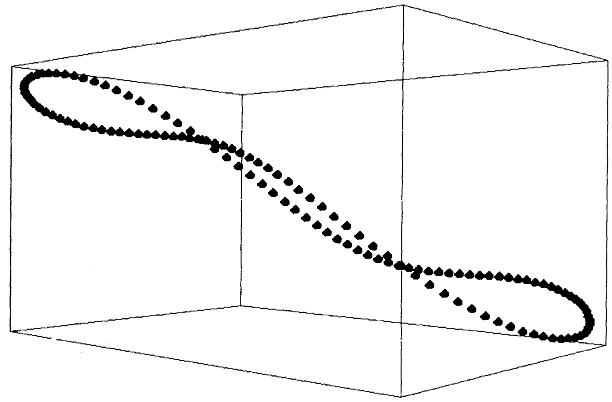

FIG. 3. The Turok string with $\alpha = 0.8$ and $\phi = \frac{\pi}{2}$ at $t = 0$.

is a two-parameter string which has a first and third harmonic in **a**, and a first in **b**. Defining Turok's parameter α to be $\alpha \equiv \sin^2 \frac{\eta}{2}$, the string equation is

$$\mathbf{r}_T = \frac{1}{2}\begin{pmatrix} \frac{1}{3}\sin^2\frac{\eta}{2}\sin 3u + \cos^2\frac{\eta}{2}\sin u \\ -\frac{1}{3}\sin^2\frac{\eta}{2}\cos 3u - \cos^2\frac{\eta}{2}\cos u \\ -\sin\eta\cos u \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sin v \\ -\cos\phi\cos v \\ -\sin\phi\cos v \end{pmatrix}. \tag{62}$$

The product of rotations which yields the 1-3 half of this string is

$$\mathbf{a}'_T = R_z(2u)R_x(\pi - \eta)R_z(u)\hat{\mathbf{x}}, \tag{63}$$

which corresponds to our complete product representation for $N = 3$ with $\phi_3 = 0$, $\phi_2 = \pi - \eta$, $\theta_2 = 0$, $\phi_1 = \frac{\pi}{2}$, $\theta_1 = -\frac{\pi}{2}$, using the identity $R_z(\frac{\pi}{2})R_x(\frac{\pi}{2})R_z(-\frac{\pi}{2})\hat{\mathbf{z}} = \hat{\mathbf{x}}$.

The second string half is given by

$$\mathbf{b}'_T = R_x(\phi)R_z(v)\hat{\mathbf{x}}. \tag{64}$$

The leftmost rotation is due to the fact that the Turok string is not in what we call standard form. We see that this reduces to the Kibble-Turok [8] formula for $\phi = 0$.

The Chen, DiCarlo, Hotes (CDH) string [12], shown in Fig. 4 for a given choice of parameters, is a 1-3/1 string with an added parameter in **a**, which contains the Turok string as a limit. We redefine the original CDH parameter $\eta_{\rm CDH} \equiv \frac{\pi}{2} - \eta$ to be consistent with our definition of α in the Turok string. The string equation becomes

$$\mathbf{r}_{\rm CDH} = \frac{1}{12}\begin{pmatrix} C\cos\theta \\ S \\ -C\sin\theta \end{pmatrix}\sin 3u - \frac{1}{12}\begin{pmatrix} -S\cos\theta \\ C \\ S\sin\theta \end{pmatrix}\cos 3u$$
$$+\frac{1}{4}\begin{pmatrix} 2 - C\cos\theta \\ -S \\ C\sin\theta \end{pmatrix}\sin u - \frac{1}{4}\begin{pmatrix} 3S\cos\theta \\ \cos\theta + \cos\eta \\ -\sin\eta(1 - 3\cos^2\theta) \end{pmatrix}\cos u + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\sin v - \frac{1}{2}\begin{pmatrix} 0 \\ \cos\phi \\ \sin\phi \end{pmatrix}\cos v, \tag{65}$$

where

$$C = \cos\theta - \cos\eta, \quad S = \sin\theta\sin\eta. \tag{66}$$

The product representation is

$$\mathbf{a}'_{\rm CDH} = R_y(\theta)R_z(\chi)R_z(2u)R_z(-\theta_2)R_x(\phi_2)R_z(\theta_2)R_z(u)\hat{\mathbf{x}}, \tag{67}$$

$$\mathbf{b}'_{\rm CDH} = R_x(\phi)R_z(v)\hat{\mathbf{x}}, \tag{68}$$

where

$$\cos\chi = C/(1-\cos\theta\cos\eta), \quad \sin\chi = S/(1-\cos\theta\cos\eta), \tag{69}$$

$$\cos\theta_2 = \cos\theta\sin\eta/d, \quad \sin\theta_2 = \sin\theta/d, \tag{70}$$

$$\cos\phi_2 = -\cos\theta\cos\eta, \quad \sin\phi_2 = d, \tag{71}$$

$$d = \pm(1-\cos^2\theta\cos^2\eta)^{\frac{1}{2}}. \tag{72}$$

DeLaney, Engle, and Scheick (DES) have derived a general five-parameter 1-3/1-3 string equation [13]

$$\begin{aligned}\mathbf{r}_{\rm DES} = {} & \frac{1}{6}\begin{pmatrix}\sin\phi\\ -\cos\phi\cos\gamma\\ \cos\phi\sin\gamma\end{pmatrix}\sin 3u - \frac{1}{6}\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\cos\gamma+\cos\theta\sin\gamma\\ -\sin\theta\sin\phi\sin\gamma+\cos\theta\cos\gamma\end{pmatrix}\cos 3u\\ & +\frac{1}{2}\begin{pmatrix}1-c\sin\phi\\ c\cos\phi\cos\gamma\\ -c\cos\phi\sin\gamma\end{pmatrix}\sin u - \frac{1}{2}\begin{pmatrix}x\\ y\cos\gamma+z\sin\gamma\\ -y\sin\gamma+z\cos\gamma\end{pmatrix}\cos u\\ & +\frac{1}{6}\begin{pmatrix}\sin\phi'\\ -\cos\phi'\\ 0\end{pmatrix}\sin 3v - \frac{1}{6}\begin{pmatrix}\sin\theta'\cos\phi'\\ \sin\theta'\sin\phi'\\ \cos\theta'\end{pmatrix}\cos 3v + \frac{1}{2}\begin{pmatrix}1-c'\sin\phi'\\ c'\cos\phi'\\ 0\end{pmatrix}\sin v - \frac{1}{2}\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}\cos v,\end{aligned} \tag{73}$$

with

$$\begin{aligned} x &= -3c\sin\theta\cos\phi,\\ y &= \sin\theta(1-3c\sin\phi),\\ z &= \cos\theta\sin\phi - c\sec\theta(1-3\sin^2\theta),\\ c &= \cos^2\theta(1+\sin\phi)/2,\end{aligned} \tag{74}$$

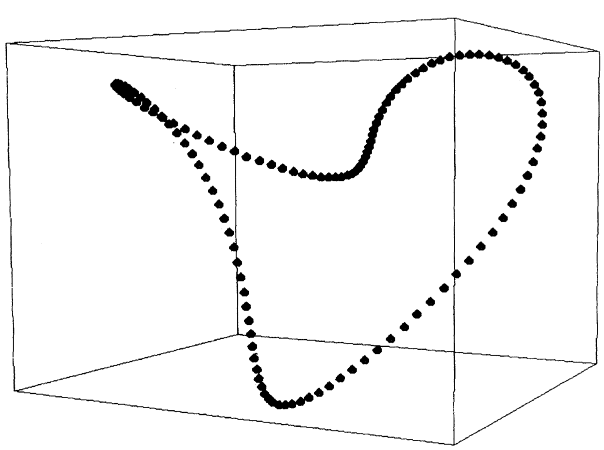

FIG. 4. The CDH string with $\theta = \frac{4\pi}{10}$, $\eta = 1.601\,60$, and $\phi = 2.644\,60$ at $t = 2.03$.

and with the same relationships between the primed variables.

In terms of the product representation,

$$\mathbf{a}'_{\rm DES} = R_x(-\gamma)R_z(\phi-\pi/2)R_x(\pi/2-\theta)R_z(2u)R_z(-\theta_2) \times R_x(\phi_2)R_z(\theta_2)R_z(u)\hat{\mathbf{x}}, \tag{75}$$

$$\mathbf{b}'_{\rm DES} = R_z(\phi'-\pi/2)R_x(\pi/2-\theta')R_z(2v)R_z(-\theta'_2) \times R_x(\phi'_2)R_z(\theta'_2)R_z(v)\hat{\mathbf{x}}, \tag{76}$$

with

$$\cos\theta_2 = \pm\sin\theta(1+\sin\phi)/f, \quad \sin\theta_2 = \mp\cos\phi/f, \tag{77}$$

$$\cos\phi_2 = \cos^2\theta\sin\phi - \sin^2\theta, \quad \sin\phi_2 = \pm f\cos\theta, \tag{78}$$

$$f = [\cos^2\phi + \sin^2\theta(1+\sin\phi)^2]^{\frac{1}{2}}, \tag{79}$$

and similarly for the primed angles. This string is displayed in Fig. 5 for a given choice of parameters.

These early parametrizations contained only odd harmonics. For this reason, they satisfy the symmetry relation $\mathbf{r}(\sigma+\pi,t) = -\mathbf{r}(\sigma,t)$. On the other hand, in Ref. [10] Burden introduced a simple class of strings with m and n harmonics in the left and right sectors, respectively,

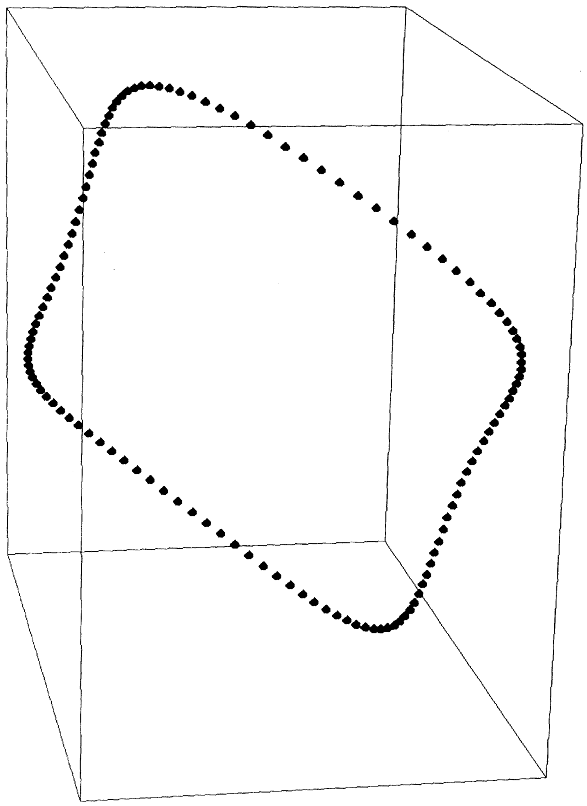

FIG. 5. The DES string with $\theta = \frac{4\pi}{3}$, $\theta' = 2.03$, $\phi = 0.644\,60$, $\phi' = 2.2345$, and $\gamma = \frac{7\pi}{5}$ at $t = 2.0$.

$$\mathbf{a}'(u) = \cos(mu)\hat{\mathbf{x}} - \sin(mu)\hat{\mathbf{z}},$$
$$\mathbf{b}'(v) = \cos(nv)(\cos\Psi\hat{\mathbf{x}} + \sin\Psi\hat{\mathbf{y}}) - \sin(nv)\hat{\mathbf{z}}. \quad (80)$$

For m and n relatively prime, this symmetry no longer holds, and gravitational radiation is no longer suppressed [10]. We can describe this class of strings as a subclass of our general parametrization in the following way,

$$\mathbf{a}'(u) = [R_y(u)]^m\,\hat{\mathbf{x}},$$
$$\mathbf{b}'(v) = R_z(\Psi)[R_y(v)]^n\,\hat{\mathbf{x}}. \quad (81)$$

Another string without the aforementioned symmetry is the Vachaspati-Vilenkin string [11], an example of which is shown in Fig. 6. This has first, second, and third harmonics in the left-going string half, and a first in the right-going half:

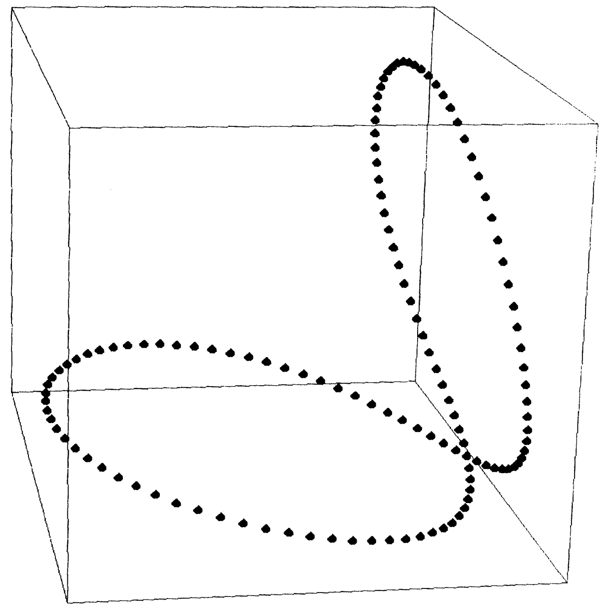

FIG. 6. The Vilenkin-Vachaspati string with $\alpha = 0.6$ and $\phi = 2.644\,60$ at $t = 0$.

$$\mathbf{r}_{\rm VV} = \frac{1}{2}\begin{pmatrix} -\frac{1}{3}\alpha\sin 3u + (1-\alpha)\sin u + \sin v \\ -\frac{1}{3}\alpha\cos 3u - (1-\alpha)\cos u - \cos\phi\cos v \\ \sqrt{\alpha(1-\alpha)}\,\sin 2u - \sin\phi\cos v \end{pmatrix}, \quad (82)$$

where $0 \le \alpha \le 1$, and $-\pi \le \phi \le \pi$. Defining ϕ_3 by the relation $\alpha \equiv \cos^2\frac{\phi_3}{2}$, the product representation gives

$$\mathbf{a}'_{\rm VV} = -R_x(\pi)R_z(u)R_z(-\pi/2)R_x(\phi_3)R_z(\pi/2)R_z(2u)\hat{\mathbf{x}}, \quad (83)$$

$$\mathbf{b}'_{\rm VV} = R_x(\phi)R_z(v)\hat{\mathbf{x}}. \quad (84)$$

Kibble and Garfinkle and Vachaspati (GV) have derived a formula for cuspless loops [6], shown in Fig. 7, again with arbitrary parameters:

$$\mathbf{r}_{\rm GV} = \frac{1}{2}\frac{1}{p^2+2}\frac{1}{2p^2+1}\begin{pmatrix} \frac{p^2}{4}\sin 4u + (p^2+1)^2\sin 2u \\ -\frac{2\sqrt{2}}{3}p\cos 3u - 2\sqrt{2}p(p^2+2)\cos u \\ -\frac{p^2}{4}\cos 4u - ((p^2+1)^2-2)\cos 2u \end{pmatrix} + \frac{1}{2}\frac{1}{p^2+2}\frac{1}{2p^2+1}\begin{pmatrix} \frac{p^2}{4}\sin 4v + (p^2+1)^2\sin 2v \\ \frac{p^2}{4}\cos 4v + ((p^2+1)^2-2)\cos 2v \\ \frac{2\sqrt{2}}{3}p\cos 3v + 2\sqrt{2}p(p^2+2)\cos v \end{pmatrix} \quad (85)$$

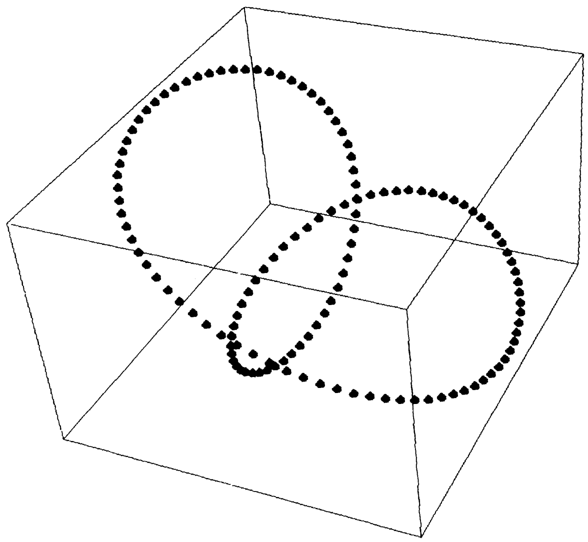

FIG. 7. The GV cuspless, kinkless string with $p = 0.2$ at $t = \frac{\pi}{2}$.

where p is a constant. In terms of the product representation, this is

$$\mathbf{a}' = R_x\left(\frac{\pi}{2}\right) R_z(u) R_x(\phi_4) R_z(2u) R_x(\phi_2) R_z(u)\hat{\mathbf{x}}, \quad (86)$$

$$\mathbf{b}' = -R_y(\pi) R_z(v) R_x(\phi_4) R_z(2v) R_x(\phi_2) R_z(v)\hat{\mathbf{x}}, \quad (87)$$

where ϕ_4 and ϕ_2 are related to p by the definition

$$p \equiv -\sqrt{2}\cot\frac{\phi_4}{2} \equiv \tan\frac{\phi_2}{2}/\sqrt{2}, \quad (88)$$

so that

$$\sin\phi_4 = -2\sqrt{2}p/(p^2+2), \quad \cos\phi_4 = (p^2-2)/(p^2+2), \quad (89)$$

$$\sin\phi_2 = 2\sqrt{2}p/(2p^2+1), \qquad \cos\phi_2 = -(2p^2-1)/(2p^2+1). \quad (90)$$

VI. KINK PARAMETRIZATIONS

While a cusp is a point on a string where $\mathbf{r}_\sigma = 0$, a kink is a discontinuity in $\mathbf{r}_\sigma$. A discontinuity of this type may occur after intercommutation (e.g., if a loop crosses itself, it splits and reconnects as two closed loops). The produced kink splits into right and left traveling kinks residing in $\mathbf{a}$ and $\mathbf{b}$, respectively. We should like to define these as right kinks and left kinks, for short. Indeed, a picture of a loop with both a left and right kink will show only one bend at the instant when the two pass through each other.

A. Paired kinks

Parametrization of loops with kinks has been investigated by Garfinkle and Vachaspati [6]. Using a greatest integer function, they were able to put symmetric sets of kinks (i.e., two symmetric pairs of left and right kinks) in an extension of Burden trajectories [10]. With four parameters, p, q, δ, and Ψ, define

$$\alpha = \pi(1-p)[2(\sigma - t)/L], \qquad \beta = \pi(1-q)[2(\sigma+t)/L] + \delta, \quad (91)$$

where $[x]$ is the greatest integer less than or equal to x. The Garfinkle-Vachaspati solution is given by

$$\mathbf{a}' = \sin(2\pi p(\sigma - t)/L + \alpha)\hat{\mathbf{x}} + \cos(2\pi p(\sigma - t)/L + \alpha)\hat{\mathbf{z}}, \quad (92)$$

$$\mathbf{b}' = \sin(2\pi q(\sigma + t)/L + \beta)[\cos\Psi\hat{\mathbf{x}} + \sin\Psi\hat{\mathbf{y}}] + \cos(2\pi q(\sigma+t)/L + \beta)\hat{\mathbf{z}}.$$

B. Symmetric kinks

The paired kinks can be generalized to a symmetric set. Define

$$f(u) = p\,u + \frac{2\pi}{n}(1-p)\left[u\Big/\frac{2\pi}{n}\right] \quad (93)$$

where $p \in (0,1)$ is real and $n > 1$ is an integer. $[x]$ is defined to be the greatest integer less than or equal to x. Then a string with n symmetric kinks ($n > 1$) is given by

$$\mathbf{a}' = (\cos(f(u)), \sin(f(u)), 0). \quad (94)$$

Due to the symmetric placement of the kinks, the integral of $\mathbf{a}'$ over $u = 0$ to 2π clearly vanishes.

For an even number of symmetric kinks ($n =$ even), this parametrization can be generalized in the manner discussed in Sec. III. For example, a large multiparameter set of symmetric kinked strings is given by

$$\mathbf{a}' = \left(\prod R_{x_i}(2m_i u + \beta_i)\right) R_{x_3}(f(u))\,\hat{\mathbf{x}}_1, \quad (95)$$

where the x_i, m_i, and β_i are, respectively, arbitrary axes, integers, and angles.

C. Single kinks

Parametrizations having a single kink in either $\mathbf{a}$ or $\mathbf{b}$, (left or right kink), or both, instead of a pair of symmetric kinks such as shown above, are difficult to express in closed form. To see this, consider the continuity constraint,

$$\int_0^{2\pi} \mathbf{r}_\sigma(\sigma) d\sigma = 0. \tag{96}$$

For symmetric kinks, this constraint becomes trivial as the first half of the integral cancels the second half for all components.

For the single-kink string we can satisfy this continuity constraint in the following way. As before, we split $\mathbf{r}$ into its left-going and right-going modes. Letting $\mathbf{b}$ be smooth, we derive the discontinuity solely from $\mathbf{a}'$. We ask that $\mathbf{a}'$ lie in the x-y plane and that the discontinuity occur at $u = 2n\pi$ where n is an integer. Letting

$$\mathbf{a}'(0) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \quad , \quad \mathbf{a}'(2\pi) = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{pmatrix}, \tag{97}$$

will give us a kink with a discontinuity angle varying with time, and depending on $\mathbf{b}'$ and an arbitrary constant α. The problem now lies in finding a parametrization of the path that $\mathbf{a}'$ takes along the unit circle such that its integral vanishes.

We can set

$$\mathbf{a}'(u) = \begin{pmatrix} \cos(\alpha + \frac{\pi-\alpha}{\pi} f(u)) \\ \sin(\alpha + \frac{\pi-\alpha}{\pi} f(u)) \\ 0 \end{pmatrix}, \tag{98}$$

asking that $f(0) = 0$ and $f(2\pi) = 2\pi$. To get the integral to vanish, one component can be taken care of by symmetry condition on f. For the other component, we need f to change slowly over a smaller part of the circle and then quickly over the rest. A simple example is

$$f(u) = u - \delta \sin u\,. \tag{99}$$

δ is positive and constant. The integration constraint results in a nontrivial relation between α and δ,

$$\tan\alpha = \frac{J_{\frac{\pi-\alpha}{\pi}}(\delta\frac{\pi-\alpha}{\pi}) - \cos\alpha\, J_{\frac{\pi-\alpha}{\pi}}(-\delta\frac{\pi-\alpha}{\pi}) - \sin\alpha\, E_{\frac{\pi-\alpha}{\pi}}(-\delta\frac{\pi-\alpha}{\pi})}{E_{\frac{\pi-\alpha}{\pi}}(\delta\frac{\pi-\alpha}{\pi}) + \sin\alpha\, J_{\frac{\pi-\alpha}{\pi}}(-\delta\frac{\pi-\alpha}{\pi}) - \cos\alpha\, E_{\frac{\pi-\alpha}{\pi}}(-\delta\frac{\pi-\alpha}{\pi})}, \tag{100}$$

where the Anger function and Weber function, J and E respectively, are given by

$$J_\nu(z) = \tfrac{1}{\pi}\int_0^\pi \cos(\nu s - z\sin s) ds,$$
$$E_\nu(z) = \tfrac{1}{\pi}\int_0^\pi \sin(\nu s - z\sin s) ds. \tag{101}$$

This transcendental relation is well defined, and has as a sample numerical solution, $\delta = 0.7365$ for $\alpha = \frac{\pi}{4}$. This solution generalizes easily to an infinite parameter set of string solutions, where $f(u)$ is

$$f(u) = u + \sum_{n=0}^{\infty} \delta_n \sin(nu), \tag{102}$$

subject to a single integral condition as a generalization of Eq. (97).

Other modulating functions can also be considered, such as polynomial functions

$$f(u) = u + \sum_{n=\text{odd}} \alpha_n \left[\frac{(u-\pi)}{\pi}\right]^n, \tag{103}$$

subject to two conditions. The first being the continuity condition, which is transcendental in the arguments α_n, and the condition that $f(0) = 0$, which means that the sum of the coefficients α_n vanishes.

VII. KINKS AFTER INTERCOMMUTATION

In the previous section we have discussed representations of strings with single kinks traveling in one direction around the string loop, where only $\mathbf{a}'$ or $\mathbf{b}'$ but not both, contains a discontinuity. We have also mentioned, however, that this description is not adequate for describing a string kinked as a result of intercommutation. We will first consider a generic case of intercommutation. This is followed by an explicit analytic example.

A. A construction algorithm

At the point of intercommutation (say, $\sigma = t = 0$), locally the two legs of the newly formed kink describe a plane in three-space. The directions of these legs are given by $\mathbf{r}_\sigma$ to the left and right of the kink, and we define these vectors as $\mathbf{r}_\sigma(-\epsilon, 0)$ and $\mathbf{r}_\sigma(+\epsilon, 0)$, respectively. Because the two legs of the kink were originally parts of string segments crossing through each other, each leg moves transverse to itself and the plane, and opposite, generally, to each other. That is, the initial conditions require that $\mathbf{r}_t$ has a nonzero component transverse to this plane that changes in sign when crossing through the kink. We will show that this initial condition is contradictory to the conditions of a string containing only a single left and right kink.

Consider the initial time-derivatives of $\mathbf{r}$ to the left and right of the kink, $\mathbf{r}_t(-\epsilon, 0)$ and $\mathbf{r}_t(+\epsilon, 0)$, respectively. Suppose only $\mathbf{a}'$ is discontinuous, as in the examples in the previous section. From Eq. (1) we have,

$$\begin{aligned} \mathbf{r}_\sigma(-\epsilon,0) &= \tfrac{1}{2}[\mathbf{a}'(-\epsilon)+\mathbf{b}'(0)],\\ \mathbf{r}_\sigma(+\epsilon,0) &= \tfrac{1}{2}[\mathbf{a}'(+\epsilon)+\mathbf{b}'(0)],\\ \mathbf{r}_t(-\epsilon,0) &= \tfrac{1}{2}[-\mathbf{a}'(-\epsilon)+\mathbf{b}'(0)],\\ \mathbf{r}_t(+\epsilon,0) &= \tfrac{1}{2}[-\mathbf{a}'(+\epsilon)+\mathbf{b}'(0)]. \end{aligned} \tag{104}$$

Noticing that the right-hand sides contain only three independent vectors, we can write one in terms of the others,

$$\mathbf{r}_t(+\epsilon,0) = \mathbf{r}_t(-\epsilon,0) + \mathbf{r}_\sigma(+\epsilon,0) - \mathbf{r}_\sigma(-\epsilon,0). \tag{105}$$

And clearly since the last two vectors on the right-hand side of Eq. (105) lie in the above-mentioned plane, the component of $\mathbf{r}_t$ transverse to this plane must be equal on both sides of the kink, in contradiction with the initial motion. We see therefore that discontinuities must occur in both $\mathbf{a}'$ and $\mathbf{b}'$ to correctly describe a kink after intercommutation.

Under these circumstances, the kink will split into left- and right-moving parts (left and right kinks) after intercommutation. Given a string parametrization of a string that self-intersects, we should like to show how one constructs the parametrization for the daughter loops. Consider a low-harmonic string that at some point in time crosses itself. Let this crossing occur at $t=\tau$, for $\sigma=\sigma_0,\sigma_1$. To describe the resulting motion, we concentrate on one of the daughter loops. Note that at the point of intercommutation, this loop is described by its left and right moving parts: $\mathbf{a}(u), u\in[\sigma_0-\tau,\sigma_1-\tau]$ and $\mathbf{b}(v), v\in[\sigma_0+\tau,\sigma_1+\tau]$, respectively. We can define a new string with $\mathbf{A}$, $\mathbf{B}$, of invariant length $\Delta\sigma=\sigma_1-\sigma_0$, in terms of these functions $\mathbf{a},\mathbf{b}$. We ask that the left- and right-moving parts of this new string, $\mathbf{A}$, $\mathbf{B}$, take the same values as $\mathbf{a}$ and $\mathbf{b}$ over the intervals given above, respectively, and demand that their derivatives $\mathbf{A}'$, $\mathbf{B}'$ be periodic with period $\Delta\sigma$.

$\mathbf{A}$, $\mathbf{B}$ will not generally be periodic, with the daughter usually acquiring some center of mass velocity. To wit, let

$$\mathbf{k} = \mathbf{a}(\sigma_1-\tau)-\mathbf{a}(\sigma_0-\tau) = -(\mathbf{b}(\sigma_1+\tau)-\mathbf{b}(\sigma_0+\tau)). \tag{106}$$

$\mathbf{k}$ determines the resulting c.m. velocity of the piece. Define $\mathbf{A}$, $\mathbf{B}$ in the following way. For $s\in[0,\Delta\sigma]$, set

$$\begin{aligned} \mathbf{A}(s) &= \mathbf{a}(\sigma_0-\tau+s),\\ \mathbf{B}(s) &= \mathbf{b}(\sigma_0+\tau+s). \end{aligned} \tag{107}$$

For $s=n\Delta\sigma+s', s'\in[0,\Delta\sigma], n\in\mathbb{Z}$ we let

$$\begin{aligned} \mathbf{A}(s) &= \mathbf{a}(\sigma_0-\tau+s)+n\mathbf{k},\\ \mathbf{B}(s) &= \mathbf{b}(\sigma_0+\tau+s)-n\mathbf{k}. \end{aligned} \tag{108}$$

The functions $\mathbf{A}$ and $\mathbf{B}$ in Eq. (108) describe a closed string of period $\Delta\sigma$ containing a single kink at the time of intercommutation that subsequently splits into left- and right-moving kinks. The closed-loop trajectory

$$\mathbf{r}(\sigma,t) = \frac{1}{2}(\mathbf{A}(\sigma-t)+\mathbf{B}(\sigma+t)) \tag{109}$$

satisfies the gauge conditions discussed in Sec. II. The loop moves with c.m. velocity $-\mathbf{k}/\Delta\sigma$, corresponding to the shift relation

$$\mathbf{r}(\sigma,t+\Delta\sigma) = \mathbf{r}(\sigma,t)-\mathbf{k}. \tag{110}$$

We thus have a procedure for determining the equation of motion for a cosmic string after intercommutation. Given the harmonic parameterization for a string, one first needs to calculate the points of self-intersection (see Refs. [12,13,26] for typical calculation). Then the above procedure can be used to find an analytical expression for the resulting string motion.

B. Example

The construction of a loop with a single kink has been discussed in the previous section. It was explained how a kink could be described in terms of a phase modification of a low harmonic form. Here we wish to look at the more realistic situation where a daughter loop is produced upon the self-intersection of a closed low-harmonic string.

In Ref. [12], the range of parameters have been carefully determined for which self-intersections occur in the Turok string of Eq. (62). We shall follow the evolution of one of these strings whose parameters lie in this range, before and after self-intersection. The harmonic forms can be adapted according to the equations of the above subsection in order to describe the daughter loops.

Using the techniques of Ref. [12], we find, for our example, that a Turok-string self-intersection occurs for $\alpha=0.5, \phi=9\pi/20$ at $\tau=4.887\,07, \sigma_0=-0.404\,027, \sigma_1=0.333\,862$. One can add π to the σ values to describe the other self-intersection occurring simultaneously (due to the symmetry of this class of loops). This example will subsequently split into three subloops. Using the aforementioned procedure, we have calculated the resulting disintegration of the string. Figure 8 displays the parent

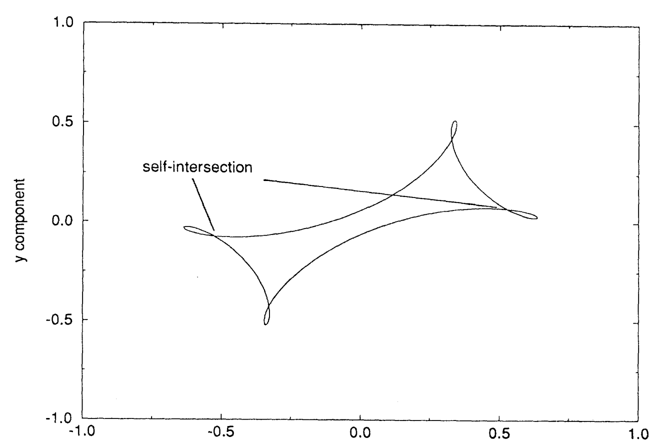

FIG. 8. Self-intersection of a Turok string.

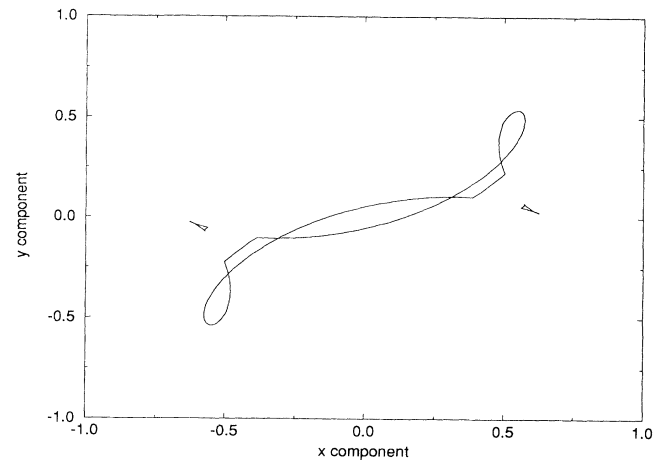

FIG. 9. The Turok string shortly after intercommutation.

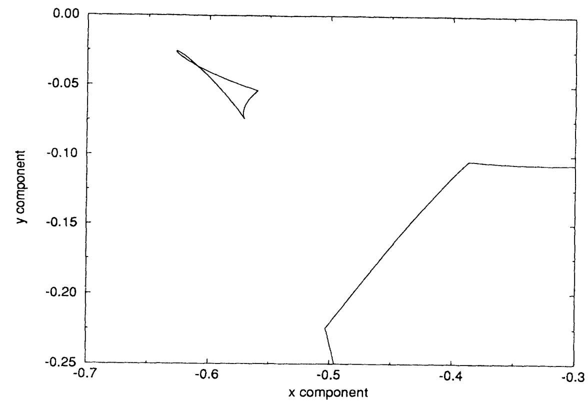

FIG. 10. A close-up of the intercommutation region.

string at the point of self-intersection. Figure 9 shows the split, and Fig. 10 is a close-up of the point of intercommutation after the separation.

The evolution of the example is faithful to the gauge conditions chosen (and described in Sec. II). All strings segments have the appropriate transverse motion and unit energy density, in view of the fact that both $\mathbf{A}'$ and $\mathbf{B}'$ are unit vectors. The two outside daughter loops have indeed acquired c.m. motion (with one kink in each) but the middle daughter (with two symmetric kinks) having none. Of course, the number of left and right kinks is twice the number of kinks. Correspondingly the middle loop spins after the split, conserving angular momentum. The outer loops are more circular and have smaller periods; they are observed to shrink rapidly down to rather small size, as seen in Fig. 9. Finally, we note that as the kink separates into the left and right kinks, the string segment between the two kinks is curved. In general, there will not be a straight line between the left and right kinks.

VIII. CONCLUSIONS

The solutions to the classical relativistic string model for a fixed number of low harmonics are useful in a number of different cosmic string applications. Starting with the study of a simple first plus third harmonic string [8], the issue of self-intersections has been addressed for increasingly more general parametrizations [9,21,12,22–26,13].

Harmonic parametrizations have been called upon in the calculation of string angular momentum [9,13], models of cosmic strings in an expanding universe [27,28] and also of the string gravitational radiation with and without kinks [29–33]. Gravitational particle production of cosmic strings has been studied using harmonic forms [34]. It has been observed that string trajectories containing only odd harmonics do not emit gravitational radiation [10,35]. It is hoped that the more general trajectories presented in this paper will aid in giving a more realistic description of the gravitational radiation of cosmic strings. The electromagnetic self-interaction of a string has been calculated using harmonic parametrizations [36–38]. The probability that harmonically parametrized string loops will collapse to black holes has also been addressed [39] and the interaction between harmonic strings and domain walls has been studied [40]. Solutions incorporating these parametrizations have the advantage that analytic precision is not sacrificed for increasingly complex structure.

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