Retardation corrections to the quarkonium spectrum and the form of the confinement kernel

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We reconsider the problem of the retardation corrections to the quarkonium spectrum, due to some errors occurring in an earlier paper. We find that such corrections are too large ($\simeq 1$ GeV) and rule out the significance of the usual confining part $I_{\rm conf}$ of the BS kernel as obtained by a simple "covariantization" of the instantaneous form implied by the linear potential. This suggests to keep $I_{\rm conf}$ instantaneous so that the only retardation correction comes from the perturbative part of the kernel. Under this assumption we calculate the correction to the hyperfine splitting of heavy quarkonium and find it to be of the order of few tens MeV and in the right direction.

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I. INTRODUCTION

In a preceding paper we evaluated the retardation corrections to some S states of the charmonium and bottomonium spectrum for an appropriate Bethe-Salpeter (BS) kernel [1]. The kernel was assumed to be the sum of a short-range perturbative part, an intermediate part, and a confinement one:

$$I=I_{\rm pert}+I_{\rm inter}+I_{\rm conf}$$
$$=(2\pi)^3\left\{\frac{4}{3}\frac{\alpha_s}{2\pi^2}\left[-\frac{\gamma_0^{(1)}\gamma_0^{(2)}}{\mathbf{Q}^2}-\frac{1}{t}[\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}-(\boldsymbol{\gamma}^{(1)}\cdot\hat{\mathbf{Q}})(\boldsymbol{\gamma}^{(2)}\cdot\hat{\mathbf{Q}})]\right]+\frac{C}{2\pi^2}\delta^3(\mathbf{Q})\left[1+O\left[\frac{1}{m^2}\right]\right]-\frac{\sigma}{\pi^2}\left[\frac{1}{t^2}+O\left[\frac{1}{m^2}\right]\right]\right\}. \tag{1.1}$$

Here the $O(1/m^2)$ terms stand for certain well-defined expressions for which we refer to [1,5], center-mass-variables have been assumed and it has been set $\mathbf{Q}=\mathbf{q}'-\mathbf{q}$ and $t=(p_1'-p_1)^2=(q_0'-q_0)^2-(\mathbf{q}'-\mathbf{q})^2$. Because of the lack of a consistent method for deriving the long-range part of the kernel from first principles, the choice (1.1) was motivated by a comparison (via the so-called instantaneous approximation) with the semirelativistic $q\bar{q}$ potential $V_{\rm WL}$ as obtained in the Wilson loop formalism [2–4]. We recall that $V_{\rm WL}$ can be written as the sum of a static part $V_{\rm stat}=-\frac{4}{3}(\alpha_s/r)+C+\sigma r$ and two relativistic corrections: a spin-dependent $V_{\rm SD}$ and a spin-independent one $V_{\rm VD}$.

The $O(1/m^2)$ terms in (1.1) are essential in order to reproduce the correct $V_{\rm VD}$ but they are immaterial for the present discussion since they were kept in their instantaneous form. Apart from such terms, the choice (1.1) for $I_{\rm conf}$ is that currently used in the literature [6], as the simplest covariant generalization of the instantaneous form

$$I_{\rm conf}^{(0)}=-(2\pi)^3\frac{\sigma}{\pi^2}\left[\frac{1}{\mathbf{Q}^4}+O\left[\frac{1}{m^2}\right]\right] \tag{1.2}$$

which in turn is directly tied to the linear part of the potential $V_{\rm stat}$:

$$\langle\mathbf{q}'|\sigma r|\mathbf{q}\rangle=-\frac{\sigma}{\pi^2}\frac{1}{\mathbf{Q}^4}\ . \tag{1.3}$$

The retardation corrections to the quarkonium levels were defined as the differences between the values obtained by solving the bound-state problem for the BS kernel I defined by (1.1) and for the potential $V_{\rm WL}$. In Ref. [1] they were written as

$$W_a^{\rm ret}=\langle\Phi_a|V_{\rm ret}^{(1)}|\Phi_a\rangle\ , \tag{1.4}$$

Φ_a being the Schrödinger eigenfunctions and $V_{\rm ret}^{(1)}$ denoting an additional contribution expressed in terms of the quantities

$$I_1(q^0;\mathbf{q}',\mathbf{q})=I(0,\mathbf{q}';q^0,\mathbf{q})-I_0(\mathbf{q}',\mathbf{q})\ . \tag{1.5}$$

The kernel (1.1) was supposed to be regularized according to the prescription

$$\left[\frac{1}{t^2}\right]_{\rm reg}=\frac{1}{2}\left[\frac{\partial^2}{\partial\varepsilon^2}-\varepsilon\ln\frac{\varepsilon}{\mu}\frac{\partial^3}{\partial\varepsilon^3}\right]\frac{1}{t-\varepsilon^2+i0}$$
$$=D_\varepsilon^\mu\frac{1}{t-\varepsilon^2+i0} \tag{1.6}$$

μ being a constant with the dimension of a mass and the limit $\varepsilon\to 0$ being understood at the end of all manipulations.

The results obtained in [1] seemed to indicate that the

corrections were small. Unfortunately they contain an error of sign and miss a normalization factor $(2\pi)^{3/2}$, which amounts to a change by an order of magnitude. This circumstance and new calculations performed for the P states force us to modify the conclusions of Ref. [1]. Indeed, the new corrections range from 0.5 to 2 GeV and the replacement of the instantaneous kernel (1.2) by the covariant one turns out to be a major change. In such a situation, since (1.3) is the basis for the choice of $I_{\rm conf}$, the instantaneous kernel (1.2) has to be considered more fundamental than the covariant form occurring in (1.1), in spite of the more aesthetic appearance of the last one. Therefore, it is more consistent to assume directly $I_{\rm conf}=I^{(0)}_{\rm conf}$. As a matter of fact such an assumption is also in line with the models based on the flux-tube idea which suggests that the confinement comes from a longitudinal field [7].

Naturally if $I_{\rm conf}=I^{(0)}_{\rm conf}$, the only retardation corrections to the levels come from the subleading part of the perturbative kernel and concern the spin dependence of the spectrum, specifically the hyperfine and the fine structure. Actually for the first few S states the hyperfine retardation corrections turn out to be of the order of few tens MeV and tend to reduce the separation as evaluated by the pure δ-potential term $(32\pi\alpha_s/9m^2)\delta(\mathbf{r})$ which usually turns out too large.

In Sec. II first we correct the final equations given in Ref. [1] and generalize them to any angular momentum; then we report the new numerical results having assumed $I_{\rm conf}$ as in (1.1). In Sec. III on the contrary, we assume $I_{\rm conf}=I^{(0)}_{\rm conf}$, outline the calculations of the correction to the hyperfine separation for few $c\bar{c}$ and $b\bar{b}$ states and compare the results with the data when available. No attempt is made of new fits. In Sec. IV we draw our conclusions.

II. LEADING RETARDATION CORRECTIONS FOR THE COVARIANT CONFINEMENT KERNEL

If one assumes (1.1), at the leading order it is

$$I_1(q_0;\mathbf{q}',\mathbf{q})=-(2\pi)^3\frac{\sigma}{\pi^2}D^\mu_\varepsilon\left[\frac{1}{q_0^2-\mathbf{Q}^2-\varepsilon^2}+\frac{1}{\mathbf{Q}^2+\varepsilon^2}\right] \tag{2.1}$$

and then one finds [1]

$$W^{\rm ret}_{nl}=-8\sqrt{2}\frac{\sigma}{\pi^{3/2}}D^\mu_\varepsilon\int_0^\infty dk\,k^2\int_0^\infty dp\,p\int_0^\infty dr\,r\tilde{\phi}_{ns}(k)\left[\frac{k^2}{2m}-\frac{W^{(0)}_{ns}}{2}\right]$$
$$\times j_l(kr)\sin(pr)\phi_{ns}(r)\mathcal{P}\frac{1}{k^2/2m-W^{(0)}_{nS}/2+\sqrt{p^2+\varepsilon^2}}\frac{1}{p^2+\varepsilon^2}\,, \tag{2.2}$$

which corrects Eq. (3.7) of Ref. [1] and generalizes it to any angular momentum ($\mathcal{P}$ stands for the principal value prescription). If, furthermore, one replaces the configurational and momentum "exact" eigenfunctions $\phi(r)$ and $\tilde{\phi}(k)$ with the spherical oscillator eigenfunctions (with λ_{nl} variationally determined), one ends up with

$$W^{\rm ret}_{nl}=2\sigma C_{nl}\int_0^\infty dk\,k\left[\frac{k^2}{2m}-\frac{W^{(0)}_{nl}}{2}\right]e^{-\lambda^2_{nl}k^2}\int_0^\infty dp\,p\,e^{-\lambda^2_{nl}p^2/2}g_{nl}(p,k)$$
$$\times D^\mu_\varepsilon\mathcal{P}\frac{1}{k^2/2m-W^{(0)}_{nl}/2+\sqrt{p^2+\varepsilon^2}}\frac{1}{p^2+\varepsilon^2}=2\sigma C_{nl}D^\mu_\varepsilon I_{nl}(\varepsilon)\,, \tag{2.3}$$

TABLE I. Leading retardation corrections (in GeV) for the covariant kernel for the two sets of parameters reported in Table III. In parentheses are reported the values of $W^{\rm ret}_{nl}$ for $\xi=0$ obtained by direct calculation of (2.3) where possible (negative unperturbed eigenvalues).

State	C_{nl}	$c\bar{c}(A)$	$b\bar{b}(A)$	$c\bar{c}(B)$	$b\bar{b}(B)$
$1S$	$-\frac{16\lambda}{\pi^{3/2}}$	$-0.98\pm0.094+0.425\xi$	$-0.80\pm0.13+0.27\xi$	$-1.26\pm0.11+0.30\xi$	$-1.2\pm0.17+0.17\xi$
		(-1.24)	(-0.70)	(-1.23)	(-0.88)
$2S$	$-\frac{32\lambda}{3\pi^{3/2}}$	$-0.52\pm0.13+2.03\xi$	$-1.07\pm0.09+0.52\xi$	$-4.96\pm1.29-0.76\xi$	$-1.32\pm0.09+0.36\xi$
		(-0.63)	(-1.42)		(-1.31)
$3S$	$-\frac{15\lambda}{\pi^{3/2}}$	$-2.54\pm0.25-0.08\xi$	$-0.90+1.18\xi$	$-0.35\pm0.13+0.39\xi$	$2.50\pm1.48+2.16\xi$
					(4.02 ± 0.63)
$1P$	$-\frac{64}{3}\frac{\lambda}{\pi^{3/2}}$	$-2.52\pm0.12+0.88\xi$	$-1.92\pm0.28+0.47\xi$	$-2.8\pm0.39+1.34\xi$	$-2.50\pm0.20+0.63\xi$
		(-2.36)	(-1.98)	(-2.72)	(-2.46)
$2P$	$-\frac{8}{15}\frac{\lambda}{\pi^{3/2}}$		$-2.46\pm0.17+0.94\xi$		$-2.60\pm0.25+1.43\xi$
			(-2.37)		(-2.38)

TABLE II. Retardation corrections (in MeV) to the hyperfine separation in the case of $I_{\rm conf}=I^{(0)}_{\rm conf}$ for the two sets of parameters reported in Table III.

	A			B			
	$\frac{32\pi\alpha_s}{9m^2}\delta(\mathbf{r})$	$W^{SS}_{\rm ret}$	Tot.	$\frac{32\pi\alpha_s}{9m^2}\delta(\mathbf{r})$	$W^{SS}_{\rm ret}$	Tot.	Expt.
$1S\ c\bar{c}$	182	−44	136	126	−33	93	116
$2S\ c\bar{c}$	102	−15	87	83	−16	67	93
$1S\ b\bar{b}$	265	−29	236	145	−23	122	
$2S\ b\bar{b}$	89	−9	80	61	−7	54	

in which the corrections to Eq. (3.10) of Ref. [1] have been incorporated in the expression of the coefficient C_{nl}. The new coefficients C_{nl} are given in Table I; the functions g_{nl} for the $1S$, $2S$, and $3S$ states are identical to those reported in Ref. [1], while the ones for the $1P$ and $2P$ states are shown in the Appendix.

From (2.3) the calculation goes on as in Ref. [1]. Written $I(\varepsilon)$ in the form

$$I(\varepsilon)=A_0+A_1\varepsilon+A_2\varepsilon^2+A_3\varepsilon^3 + \cdots +(B_2\varepsilon^2+B_4\varepsilon^4+\cdots)\ln\frac{\varepsilon}{\mu}\ , \tag{2.4}$$

one has, for a fixed value μ_0 of μ,

$$W^{\rm ret}_{nl}(\mu_0)=\sigma C_{nl}[2A^{nl}_2(\mu_0)+3B^{nl}_2]\ , \tag{2.5}$$

and, for an arbitrary value,

$$W^{\rm ret}_{nl}(\mu)=W^{\rm ret}_{nl}(\mu_0)+\sigma C_{nl}B^{nl}_2\ln\frac{\mu}{\mu_0}\ . \tag{2.6}$$

The analytical expressions of the coefficients A_1, A_3, and B_2 for the S states are again the same as in Ref. [1]; those for the P states are reported in the Appendix; $A_2(\mu_0)$ is finally obtained by interpolation of the numerical curve $I=I(\varepsilon)$.

The new results are reported as functions of $\xi=\ln(\mu/\mu_0)$ (with $\mu_0=0.05$ GeV), for two choices of parameters, the first used in [1] (coming from a fit in which the quark pair creation effect was included [8]) and the second new (from a fit on the bare spectrum [8]).

III. RETARDATION CORRECTIONS TO THE HYPERFINE SEPARATIONS FOR INSTANTANEOUS CONFINEMENT KERNEL

Assuming $I_{\rm conf}=I^{(0)}_{\rm conf}$ with $I^{(0)}_{\rm conf}$ given by Eq. (1.2) we have simply

$$I_1(\omega,\mathbf{q}',\mathbf{q})=-(2\pi)^3\frac{4}{3}\alpha_s\frac{1}{2\pi^2}\left[\left[\frac{1}{q_0^2-(\mathbf{q}'-\mathbf{q})^2+\varepsilon^2}+\frac{1}{(\mathbf{q}'-\mathbf{q})^2+\varepsilon^2}\right][\gamma^{(1)}\cdot\gamma^{(2)}-(\gamma^{(1)}\cdot\hat{\mathbf{Q}})(\gamma^{(2)}\cdot\hat{\mathbf{Q}})]\right]\ . \tag{3.1}$$

In the $l=0$ case, keeping again only the leading contribution, we obtain the retardation correction to the hyperfine splitting in the form

$$W^{\rm hyperf}_{n0}=\frac{8\alpha_s}{9m^2}C_{n0}\int_0^\infty dk\,k\left[\frac{k^2}{2m}-\frac{W^{(0)}_{n0}}{2}\right]e^{-\lambda_n^2k^2}\int_0^\infty dp\,p\,e^{-\lambda_n^2p^2/2}\mathcal{P}\frac{g_{n0}(k,p)}{-W^{(0)}_{n0}/2+k^2/2m+p}\ , \tag{3.2}$$

where the coefficients C_{n0} and the functions g_{n0} are the same as before. The evaluation of the integral in (3.2) can be done directly; the results obtained are reported in Table II together with the leading splitting as given by the potential theory and with the data.

IV. CONCLUSIONS

In conclusion we stress the following.

(a) We have recalculated the leading retardation correction to the spectrum of the heavy quarkonium for the kernel (1.1) for two different choices of the parameters (Table III). The corrections turn out to be too large, in the range 0.5–2 GeV, and thus the covariant generalization represents a major change with respect to the instantaneous kernel. Notice that for the choice A the correction obtained for the S states are not too far from the value of the parameter C of the potential [with the exception of $c\bar{c}(3S)$]. This could seem to support a conjecture by Gromes on the origin of the constant C [9] (which

TABLE III. Choices A and B of the parameters are taken from Ref. [8]. They are obtained by a best fit of $c\bar{c}$ and $b\bar{b}$ entire spectrum including (A) or not including (B) the quark pair creation.

	$\frac{4}{3}\alpha_s$	σ	C	m_c	m_b
A	0.587	0.146	−0.707	1.913	5.268
B	0.484	0.178	−0.781	1.788	5.183

then should be subtracted from the calculated value to avoid double counting), in which case the covariant kernel would not be ruled out. However, the circumstance no longer occurs for choice B or for the P states [and the $c\bar{c}(3S)$ state] even with choice A and so (1.1) is really inconsistent.

(b) Since only the instantaneous kernel has a theoretical basis, the above results suggest that one has to assume $I_{\rm conf}=I^{(0)}_{\rm conf}$, with $I^{(0)}_{\rm conf}$ as given by Eq. (1.2). After all this is consistent with the indication of a purely longitudinal confining field coming, e.g., from the tube flux model.

(c) Under the assumption $I_{\rm conf}=I^{(0)}_{\rm conf}$, the only retardation corrections to the potential results come from the perturbative kernel (1.1) and regard the spin dependence of the energy levels, both the hyperfine and the fine structure of the spectrum.

(d) We have evaluated the corrections to the hyperfine separations for the $c\bar{c}$ and $b\bar{b}$ systems and have found that they improve significantly the agreement with the data at least for one of the two choices of the parameters. This is in line with the second-order potential calculations by Gupta, Radford, and Repko [10] who have implicitly introduced perturbative retardation corrections through the contribution of the box diagram.

APPENDIX

The explicit expressions for the functions g_{1P} and g_{2P} in Eq. (2.4) are

$$
\begin{aligned}
&g_{1P}=(1+k^2\lambda_{1P}^2)\sinh(kp\lambda_{1P}^2)-kp\lambda_{1P}^2\cosh(kp\lambda_{1P}^2)\ ,\\
&g_{2P}=(10-4k^2\lambda_{2P}^2)\{\sinh(kp\lambda_{2P}^2)[6+2k^2\lambda_{2P}^2+4p^2\lambda_{2P}^2+4k^4\lambda_{2P}^4+12k^2p^2\lambda_{2P}^4]\\
&\qquad\qquad+\cosh(kp\lambda_{2P}^2)[-6kp\lambda_{2P}^2-4kp^3\lambda_{2P}^4-12k^3p\lambda_{2P}^4]\}\ ,\\
&A_1^{1P}=-\frac{3\pi^{3/2}}{16\lambda},\quad A_1^{2P}=-\frac{15\pi^{3/2}}{2\lambda}\ ,\\
&B_2^{1P}=-m\lambda\frac{\sqrt{\pi}}{4}\{1-2z^2+2\sqrt{\pi}z^3e^{z^2}[1-\Phi(z)]\}\ ,\\
&B_2^{2P}=-2m\sqrt{\pi}\lambda\{\sqrt{\pi}[1-\Phi(z)]e^{z^2}(25z^3+20z^5+4z^7)+5-18z^2-18z^4-4z^6\}\ ,\\
&A_3^{1P}=-\frac{5}{64}\pi^{3/2}\lambda\ ,\quad A_3^{2P}=-\frac{45}{8}\pi^{3/2}\lambda\ .
\end{aligned}
\tag{A1}
$$

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