Viability of lattice perturbation theory

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In this paper we show that the apparent failure of QCD lattice perturbation theory to account for Monte Carlo measurements of perturbative quantities results from choosing the bare lattice coupling constant as the expansion parameter. Using instead "renormalized" coupling constants defined in terms of physical quantities, such as the heavy-quark potential, greatly enhances the predictive power of lattice perturbation theory. The quality of these predictions is further enhanced by a method for automatically determining the coupling-constant scale most appropriate to a particular quantity. We present a mean-field analysis that explains the large renormalizations relating lattice quantities, such as the coupling constant, to their continuum analogues. This suggests a new prescription for designing lattice operators that are more continuumlike than conventional operators. Finally, we provide evidence that the scaling of physical quantities can be asymptotic or perturbative already at (quenched) β's as low as 5.7, provided the evolution from scale to scale is analyzed using renormalized perturbation theory. This result indicates that reliable simulations of QCD are possible at these same low β's.

I. INTRODUCTION

In principle, nonperturbative lattice simulations allow the calculation of any quantity in QCD, without recourse to perturbation theory. In practice, however, perturbation theory is important to lattice QCD in several ways. Firstly, it provides the essential connection between lattice simulations, which are most effective for low-energy phenomena, and the high-energy arena of perturbative QCD phenomenology. This is accomplished through such constructs as the operator-product expansion. Secondly, perturbation theory can account for effects on low-energy phenomena due to the physics at distance scales shorter than the lattice spacing. Provided the lattice spacing a is small enough, systematic errors of order a and higher can be removed from the theory by perturbatively correcting the action and operators that define the lattice theory. This approach provides a cost-effective alternative to simply reducing the lattice spacing when systematic errors must be removed. Finally, lattice simulations and perturbation theory must agree for short-distance quantities, where both approaches should be reliable, if we are to have confidence in simulation results for nonperturbative quantities.

It is disturbing therefore that Monte Carlo estimates for most short-distance quantities seem to agree poorly with perturbative calculations. An example is the vacuum expectation value of the lattice gluon operator U_μ in Landau gauge. This is the lattice analogue of the expectation value $\langle A_\mu^2 \rangle$ of the square of the bare gauge field A_μ. Since $\langle A_\mu^2 \rangle$ is quadratically divergent, the loop integral in first-order perturbation theory is dominated by momenta of order the cutoff, and perturbation theory

should be effective for cutoffs of order a couple of GeV or larger. However, the perturbative result, when expressed in terms of the bare coupling constant $\alpha_{\text{lat}} \equiv g_{\text{lat}}^2/4\pi$ of the lattice theory, is

$$\langle 1 - \tfrac{1}{3}\text{Tr}U_\mu \rangle_{\text{PT}} = 0.97\,\alpha_{\text{lat}} = .078 \tag{1}$$

at $\beta \equiv 6/g_{\text{lat}}^2 = 6$.[1] This is almost a factor of 2 smaller than the nonperturbative result

$$\langle 1 - \tfrac{1}{3}\text{Tr}U_\mu \rangle_{\text{MC}} = 0.139, \tag{2}$$

obtained from Monte Carlo simulations [1]. The coupling constant is quite small here ($\alpha_{\text{lat}} = 0.08$), and the loop momenta typically large ($q \approx \pi/a \approx 6$ GeV). Perturbation theory ought to work; instead it seems to fail completely. Discouraging results such as this have arisen in a wide range of lattice calculations, leading to considerable pessimism about the viability of lattice perturbation theory at moderate β's.

In this paper we show that although these facts are true they are misleading. We find that the key problem with previous calculations of this sort is in the choice of the expansion parameter for the perturbation series: α_{lat} is generally a very poor choice. There is no compelling reason in a field theory for using the bare coupling constant as the expansion parameter in weak-coupling perturbation theory. Standard practice is to express perturbation

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[1]β is the parameter used to specify the bare coupling constant in the standard lattice action for QCD.

series in terms of some renormalized coupling constant, one usually defined in terms of a physical quantity. Indeed the renormalized coupling is usually a running coupling "constant" whose value in a particular expansion depends upon the length scales relevant in that process; there is no single expansion parameter for all series. The perturbative quantities important in lattice QCD generally involve lengths of order the lattice spacing a, and so one might expect little renormalization of the coupling from its bare value. However this argument, the usual rationale for using α_{lat}, ignores the possibility of a large scale-independent renormalization of the bare coupling. We find that just such a renormalization does occur in lattice QCD, making expansions in α_{lat} useless except at very large β's.

Faced with large renormalizations, we must replace α_{lat} by a renormalized coupling. It is straightforward to reexpress lattice perturbation expansions in terms of any of the expansion parameters that have proven effective in continuum perturbation theory, for example, $\alpha_{\overline{\text{MS}}}(q)$ with some physically motivated momentum q, where $\overline{\text{MS}}$ denotes the modified minimal subtraction scheme. When this is done we find that lattice perturbation theory becomes far more reliable. In fact, perturbation theory becomes about as effective for lattice quantities as it is for continuum quantities at comparable momenta.

The large renormalization of α_{lat} is due to the structure of the link operators from which the theory is built. The nonlinear relation between the link operator and the gauge field leads to large renormalizations of lattice operators relative to their continuum analogues, and these in turn result in large shifts of the coupling constants in the action. In this paper we present a simple nonperturbative procedure for removing the bulk of these large "tadpole" renormalizations from gluon and quark operators. This procedure elucidates the problems with α_{lat}. More importantly, perturbative expansions of the renormalization constants that relate quark currents and other composite operators on the lattice to their continuum counterparts become far more convergent once the tadpole contributions are removed.

In Sec. II we discuss the symptoms that result from a poor choice of expansion parameter in a perturbation series. We show how these symptoms afflict lattice expansions expressed in terms of α_{lat}, and we suggest a new, physically motivated procedure for renormalizing lattice perturbation theory.[2] In Sec. III, we compare predictions from our renormalized perturbation theory with nonperturbative results obtained from Monte Carlo simulations. We examine quark masses $\langle \text{Tr} U_\mu \rangle$ and a variety of Wilson loops and Creutz ratios. We find impressive agreement for all quantities, with no tuning of the theory, even at β's as low as 5.7. In Sec. IV we discuss the origins of the large renormalizations that arise when comparing lattice quantities with their continuum analogues. We develop a new prescription for building lattice operators that are

much closer in behavior to their continuum counterparts; in particular the large renormalizations disappear.[3] The success of renormalized perturbation theory at low β's suggests that the evolution of the coupling constant with lattice spacing is also perturbative and scaling asymptotic at these β's. This important issue is discussed in Sec. V. Finally, in Sec. VI, we summarize our conclusions, stressing their implications concerning the reliability of simulations on relatively coarse (and therefore much less costly) lattices. The data for the plots throughout the paper are tabulated in the Appendix.

II. RENORMALIZED LATTICE PERTURBATION THEORY

A. A poor expansion parameter

If an expansion parameter α_{good} produces well-behaved perturbation series for a variety of quantities, using an alternative expansion parameter $\alpha_{\text{bad}} \equiv \alpha_{\text{good}}(1 - 10\,000\,\alpha_{\text{good}})$ will lead to second-order corrections that are uniformly large, each roughly equal to $10\,000\,\alpha_{\text{bad}}$ times the first-order contribution. Series expressed in terms of α_{bad}, although formally correct, are misleading if truncated and compared with data. The signal for a poor choice of expansion parameter is the presence in a variety of calculations of large second-order coefficients that are all roughly equal relative to first order.

A large coefficient appears in the first second-order calculation done for the lattice theory: the calculation of the gluonic three-point function used to relate the Λ parameter of the bare lattice coupling α_{lat} to the Λ's of various continuum coupling constants [5,6]. The coupling constant $\alpha(q)_{\overline{\text{MOM}}}$, defined in terms of this three-point function at momentum q, has the expansion

$$\alpha(q)_{\overline{\text{MOM}}} = \alpha_{\text{lat}} \left\{ 1 + \alpha_{\text{lat}}[\beta_0 \ln(\pi/aq)^2 + 5.419] \right\}, \quad (3)$$

where $\beta_0 = 11/4\pi$ and $\overline{\text{MOM}}$ denotes the momentum-space subtraction scheme. Naively, one expects that $\alpha(q = \pi/a)_{\overline{\text{MOM}}} \approx \alpha_{\text{lat}}$, since π/a is roughly the largest momentum on the lattice. However, the constant 5.419 spoils the equality; it results in very large ratios between continuum and lattice Λ's.

Since continuum quantities are usually well behaved when expanded in terms of $\alpha(q)_{\overline{\text{MOM}}}$, it is immediately obvious that most other continuum quantities have a similar constant term when expressed in terms of α_{lat}. For example, the heavy quark potential $V(q)$ at momentum transfer q has the expansion [7]

$$V(q) = -\frac{C_f 4\pi \alpha_{\text{lat}}}{q^2} \left\{ 1 + \alpha_{\text{lat}} \left[\beta_0 \ln \left(\frac{\pi}{aq} \right)^2 + 4.70 \right] \right\}, \quad (4)$$

where $C_f = 4/3$ is the quark's color (Casimir) charge. Similar results hold for the $e^+ e^-$ hadronic cross section,

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[2] For a discussion of these issues in the context of dimensionally regularized QCD perturbation theory, see [2]. A preliminary version of our lattice analysis is in [3].

[3] A preliminary version of this analysis is published in [4].

derivatives of moments for deep inelastic ep scattering, etc.

A crucial point is that a similar constant term appears in the expansions for all short-distance lattice quantities that have been studied. For example, the corrections to the heavy-quark potential as a function of distance have the form [8]

$$V(R) = -\frac{C_f \alpha_{\text{lat}}}{R} \left\{ 1 + \alpha_{\text{lat}} \left[\beta_0 \ln \left(\frac{\pi R}{a} \right)^2 \right. \right.$$
$$\left. \left. + C(R/a) \right] \right\}, \qquad (5)$$

where $C(R/a)$ for various values of R is given in the table

R/a	2	4	6	∞
$C(R/a)$	5.5	5.5	5.6	5.711

[The constant for $R = \infty$ can be obtained by Fourier transforming the equation for $V(q)$ above.] Note that the constants $C(R/a)$ at finite R vary little from the one at $R = \infty$. This is expected since these corrections are dominated by quadratically UV divergent tadpole loops that are insensitive to the external momenta.

As we show later, similar terms are present in Wilson loops and Creutz ratios. Thus the pattern of second-order coefficients for lattice quantities strongly suggests that α_{lat} is a poor choice of expansion parameter.

B. A better expansion parameter

To define an improved (renormalized) expansion parameter, we must both choose a definition of the running coupling $\alpha_s(q)$ ("fix the scheme") and specify how the scale q of the coupling is to be chosen ("set the scale"). It is natural and convenient in perturbation theory to tie the scale of the coupling to that of the loop momenta circulating in the Feynman diagrams. Thus, we want to define $\alpha_s(q)$ so that it approximates the coupling strength of a gluon with momentum q.[4] It is also important that $\alpha_s(q)$ be defined in terms of a *physical* quantity, so as to avoid confusions, such as that between the minimal subtraction (MS) and $\overline{\text{MS}}$ schemes, that are artifacts of arbitrary definitions.

1. Fixing the scheme

Of the many physical quantities one might use to define an $\alpha_s(q)$, the heavy-quark potential $V(q)$ is among the most attractive [2]. Typically there is an integral over the momentum of the leading-order gluon, but the gluon in $V(q)$ has only momentum q. Thus it is particu-

larly easy to tie the coupling constant's argument to the gluon's momentum for this quantity: we define $\alpha_V(q)$, the coupling strength of a gluon with momentum q, such that

$$V(q) = -\frac{C_f 4\pi \alpha_V(q)}{q^2} \qquad (6)$$

with no higher-order corrections. We can easily relate α_V to the bare lattice coupling constant α_{lat} since $V(q)$ has been computed in terms of α_{lat} [Eq. (4)]:

$$\alpha_{\text{lat}} = \alpha_V(q) \left\{ 1 - \alpha_V \left[\beta_0 \ln(\pi/aq)^2 + 4.702 \right] \right\} + O(\alpha_V^3) \qquad (7)$$

for SU(3) color with no light-quark vacuum polarization. With this expression, any one-loop or two-loop lattice expansion can be reexpressed as a series in α_V. The q dependence of α_V is given by the usual formula

$$\alpha_V^{-1}(q) = \beta_0 \ln(q/\Lambda_V)^2 + \beta_1/\beta_0 \ln\ln(q/\Lambda_V)^2 + O(\alpha_V(q)), \qquad (8)$$

where $\beta_0 = 11/4\pi$ (as before), $\beta_1 = 102/16\pi^2$, and

$$\Lambda_V = 46.08 \, \Lambda_{\text{lat}} \qquad (9)$$

is the scale parameter for this scheme. The scale parameter Λ_{lat} for α_{lat} is defined implicitly by

$$\alpha_{\text{lat}}^{-1} = \beta_0 \ln(1/a\Lambda_{\text{lat}})^2 + \beta_1/\beta_0 \ln\ln(1/a\Lambda_{\text{lat}})^2 + \cdots. \qquad (10)$$

The coupling $\alpha_{\overline{\text{MS}}}(q)$ is numerically fairly close to $\alpha_V(q)$, and thus is another useful alternative to α_{lat}. In this case,

$$\alpha_{\text{lat}} = \alpha_{\overline{\text{MS}}}(q) \left\{ 1 - \alpha_{\overline{\text{MS}}} \left[\beta_0 \ln(\pi/aq)^2 + 3.880 \right] \right\} + O(\alpha_{\overline{\text{MS}}}^3), \qquad (11)$$

and the scale parameter is

$$\Lambda_{\overline{\text{MS}}} = 28.81 \, \Lambda_{\text{lat}}. \qquad (12)$$

Note that the relation between α_V (or $\alpha_{\overline{\text{MS}}}$) and α_{lat} can be elegantly reexpressed in terms of the inverse couplings:

$$\alpha_{\text{lat}}^{-1} = \alpha_V^{-1}(q) + \beta_0 \ln(\pi/aq)^2 + 4.702 + O(\alpha_V), \qquad (13)$$

$$\alpha_{\text{lat}}^{-1} = \alpha_{\overline{\text{MS}}}^{-1}(q) + \beta_0 \ln(\pi/aq)^2 + 3.880 + O(\alpha_{\overline{\text{MS}}}). \qquad (14)$$

As we discuss in Sec. IV B, it turns out that these expressions are also more accurate.

2. Setting the scale

The coupling constant α_V is defined so that $\alpha_V(q^*)$ is the appropriate expansion parameter for a process in which the typical gluon momentum is q^*. For many processes it is possible to guess q^* fairly accurately. For example, power-law UV divergent quantities such as $\langle \text{Tr} U \rangle$ are controlled by the lattice modes with the highest momenta, and so one expects $q^* \approx \pi/a$. (The largest lattice

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[4]It is natural in a gauge theory to associate the scale of the coupling with the gluon's momentum since every g in the theory is associated with a particular A_μ by gauge invariance. This association allows us to set the scale in a gauge invariant way.

momentum is actually $2\pi/a$, but there are few modes with such high momenta; thus the effective cutoff is more like π/a.) Although such guesses are often sufficient, there is a simple automatic procedure that takes the guessing out of q^*. This procedure has proven invaluable in our systematic study of the reliability of perturbation theory.

Consider a one-loop perturbative contribution in our scheme:

$$I = \alpha_V(q^*) \int d^4q \, f(q), \qquad (15)$$

where q is the gluon's momentum. The natural definition of q^* would be

$$\alpha_V(q^*) \int d^4q \, f(q) \equiv \int d^4q \, \alpha_V(q) f(q) \qquad (16)$$

except that the second integral is singular. The singularity is due to the pole in the coupling constant at $q = \Lambda_V$. This pole is an artifact of the all orders summation of perturbative logarithms that is implicit in the formula for $\alpha_V(q)$ [Eq. (8)]. The singularity does not arise in any finite order of perturbation theory, as may be seen by replacing the running coupling constant $\alpha_V(q)$ in Eq. (16) by its expansion in terms of the coupling constant renormalized at some fixed scale μ:

$$\alpha_V(q) = \alpha_V(\mu)\{1 + \beta_0 \ln(q/\mu)^2\alpha_V(\mu)$$
$$+ [\beta_0 \ln(q/\mu)^2\alpha_V(\mu)]^2 + \cdots\}. \qquad (17)$$

None of these terms separately results in a singularity, but the sum of all terms diverges.

In fact it is incorrect to sum to all orders since the QCD perturbation series is an asymptotic series. The proper procedure is to retain only those terms consistent with the accuracy of the rest of the calculation. For our purposes we should retain only the first two terms in Eq. (17):

$$\alpha_V(q^*) \int d^4q \, f(q) \equiv \alpha_V(\mu) \int d^4q \, f(q)$$
$$+ \beta_0 \, \alpha_V(\mu)^2 \int d^4q \, f(q) \, \ln(q/\mu)^2$$
$$+ \cdots. \qquad (18)$$

Expanding $\alpha_V(q^*)$ in terms of $\alpha_V(\mu)$ in this equation we obtain a simple definition for q^* (independent of μ):

$$\ln(q^{*\,2}) \equiv \frac{\int d^4q \, f(q) \, \ln(q^2)}{\int d^4q \, f(q)}. \qquad (19)$$

3. Summary

To summarize, our general procedure for analyzing a perturbation series in lattice QCD involves replacing α_{lat}

by $\alpha_V(q^*)$ using Eq. (7). The scale q^* is determined by probing the first-order calculation with a factor $\ln q^2$, as in Eq. (19). In calculations that extend through two-loop order we assume that the one-loop q^*, determined this way, is also appropriate for the two-loop contribution.

A special feature of expansions in $\alpha_V(q^*)$ for continuum quantities is that they are unaffected through second order by quark vacuum-polarization insertions in the gluon propagator. All such contributions are automatically absorbed into $\alpha_V(q^*)$, by virtue of its definition. Consequently perturbative expansions are identical (in second order as well as first order) for quenched and unquenched versions of QCD when they are expressed in terms of $\alpha_V(q^*)$. This result is only approximately true for lattice quantities.[5] Since α_V is defined in the continuum, finite lattice-spacing corrections due to quark vacuum polarization can become important for quantities with the largest q^*'s. In practice, however, such effects appear to be small. The only other relevant difference between quenched and unquenched QCD is in the evolution of $\alpha_V(q)$: $\beta_0 \rightarrow (11 - \frac{2}{3}n_f)/4\pi$ and $\beta_1 \rightarrow (102 - \frac{38}{3}n_f)/16\pi^2$ in Eq. (8) for n_f light-quark flavors. The n_f independence of α_V expansions leads to an alternative procedure for determining q^* that is analyzed extensively, for continuum QCD, in [2].

III. TESTING RENORMALIZED PERTURBATION THEORY

Our procedure for defining a renormalized coupling constant with a proper scale (Sec. II B) follows solely from known results in lattice perturbation theory, without regard to Monte Carlo data. Only now are we ready to consider the extent to which our renormalized perturbation theory agrees with Monte Carlo simulations of short-distance quantities.

Having converted all of our perturbative expansions from α_{lat} to α_V, we need some way of determining values of α_V that are appropriate to particular simulations. The most straightforward procedure is to measure α_V in the simulations.[6] This can be done, for example, by measuring the heavy-quark potential, or, more simply, by measuring the trace of the plaquette operator U_{plaq} (the 1×1 Wilson loop). The improved perturbative expansion for the logarithm of $\text{Tr}U_{\text{plaq}}$ is

$$-\ln\langle\tfrac{1}{3}\text{Tr}U_{\text{plaq}}\rangle = \frac{4\pi}{3} \alpha_V(3.41/a) \, \{1 - (1.19 + 0.017\,n_f)$$
$$\times \alpha_V + O(\alpha_V^2)\}. \qquad (20)$$

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[5]We thank Urs Heller for alerting us to an inconsistency concerning this point in an earlier draft of this paper.

[6]As we discuss in Sec. IV, α_V can also be computed directly from α_{lat} without using Monte Carlo data. However this procedure is less accurate than measuring α_V.

TABLE I. Monte Carlo data for logarithm of the plaquette, together with the coupling constant values used in this study.

β	$-\ln W_{11}$	α_{lat}	$\alpha_{\overline{\text{MS}}}(1/a)$	$\alpha_V(1/a)$	$\alpha_V(\pi/a)$
5.7	0.5995	0.0838	0.2579	0.3552	0.1885
6	0.5214	0.0796	0.1981	0.2467	0.1557
6.1	0.5025	0.0783	0.1860	0.2275	0.1483
6.2	0.4884	0.0770	0.1774	0.2144	0.1430
6.3	0.4740	0.0758	0.1690	0.2020	0.1377
6.4	0.4610	0.0746	0.1617	0.1913	0.1329
9	0.2795	0.0531	0.0815	0.0878	0.0739
12	0.1954	0.0398	0.0532	0.0558	0.0499
18	0.1227	0.0265	0.0317	0.0326	0.0305

Given data for this quantity, one can easily solve for $\alpha_V(3.41/a)$. The coupling $\alpha_V(q)$ for other q's can then be obtained using standard two-loop evolution [Eq. (8)]. We have extracted $\alpha_V(3.41/a)$ in this way from data for quenched QCD at several β's. The results, evolved down to $q = 1/a$ and to $q = \pi/a$, are given in Table I. We also give values for α_{lat} and for $\alpha_{\overline{\text{MS}}}(1/a)$, the latter being obtained from the measured α_V using the relation $\Lambda_{\overline{\text{MS}}} = 0.6252\,\Lambda_V$.

Note that the coupling-constant scale of $3.41/a$ used in this expansion is fixed by our scale-fixing procedure [Eq. (19)]. Also we use the logarithm of the Wilson loop here, rather than the loop itself, since its perturbative expansion is better behaved (see below).

Our choice of $-\ln\langle\frac{1}{3}\text{Tr}U_{\text{plaq}}\rangle$ for determining α_V is for convenience; we have not attempted to optimize this choice. One could use any other short-distance quantity whose α_V expansion is known through second order. Other alternatives might be Creutz ratios of small loops, whose perturbative expansions might be more convergent, or a combination of Wilson loops chosen so that potential nonperturbative area-law contributions cancel.

As discussed above, the formula used in measuring α_V [Eq. (20)] is almost identical for quenched and unquenched QCD. This should be true for all of the α_V expansions that follow. Thus the same techniques and tests we use here can be applied to the unquenched case. We have not yet done this, but we expect similar results.

A. $\langle A_\mu^2 \rangle$

The lattice equivalent of $\langle A_\mu^2 \rangle$ is $\langle 1 - \frac{1}{3}\text{Tr}U_\mu\rangle$, which is given in perturbation theory by $0.97\,\alpha_s$. (We use α_s to represent a generic choice for the coupling.) The one-loop contribution comes from a quadratically divergent tadpole graph, and we therefore expect that it is dominated by momenta of order the lattice cutoff π/a. Using the procedure of Sec. II B we find $q^* = 2.80/a$. In Fig. 1 we compare perturbative results for $\langle 1 - \frac{1}{3}\text{Tr}U_\mu\rangle$ with Monte Carlo data [1] at several values of β. We present results from perturbation expansions in α_{lat}, in our favorite coupling constant $\alpha_V(q^*)$, and in $\alpha_{\overline{\text{MS}}}(q^*)$. The data agree with perturbation theory to within 10–15% for all $\beta \geq 5.7$ when α_V or $\alpha_{\overline{\text{MS}}}$ is used. Uncalculated terms of order α_V^2 or higher in the perturbation theory could easily account for the remaining differences; the
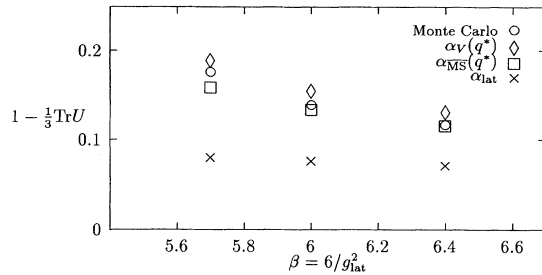

FIG. 1. The expectation value of the trace of a link in Landau gauge, calculated by Monte Carlo simulations (circles), and in first-order perturbation theory using for the expansion parameter $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{\text{MS}}}(q^*)$ (boxes), and α_{lat} (crosses).

differences between the α_V and $\alpha_{\overline{\text{MS}}}$ predictions give an indication of how important such terms might be. Of course part of the difference between perturbation theory and Monte Carlo might be nonperturbative, particularly at the lowest β. The data disagree with the α_{lat} expansion by almost a factor of 2.

B. Mass renormalization for Wilson quarks

A famous example of the "failure" of lattice perturbation theory is the calculation of the additive mass renormalization for Wilson quarks. The bare mass in Wilson's formulation of the lattice quark action is renormalized by an additive power-law divergent term. The critical quark mass, for which this term is canceled (leaving the quark massless), is given in perturbation theory by $m_c a \equiv 1/2\kappa_c - 4 = -5.457\alpha_s$ [9]. (Here, κ is the "hopping parameter" used to parametrize the quark mass in lattice gauge theory.) The linear divergence in this result suggests that the important momenta here are of order π/a. We find $q^* = 2.58/a$ using our procedure [Eq. (19)]. In Fig. 2 we compare perturbative results for m_c with Monte Carlo data [10] at several values of β. Again we see that the data agree with our renormalized perturbation theory to within 10–15% for all β's, but disagree
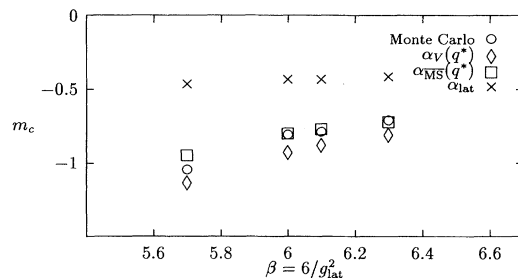

FIG. 2. The critical quark mass m_c for Wilson quarks, calculated by Monte Carlo simulations (circles), and in first-order perturbation theory using for the expansion parameter $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{\text{MS}}}(q^*)$ (boxes), and α_{lat} (crosses). Statistical errors in the Monte Carlo results are negligible.

with perturbation theory using α_{lat} by almost a factor of 2.

C. Wilson loops and Creutz ratios

Aside from the heavy-quark potential and the coupling constant, Wilson loops are the only lattice quantities for which two-loop perturbation theory has been calculated. Consequently they provide the most stringent tests of perturbation theory. Large Wilson loops have badly behaved perturbative expansions for a trivial reason: they contain a self-energy contribution proportional to the length of the loop. For large loops, contributions to this self-energy approximately exponentiate, so we expect that the logarithm of a Wilson loop is better behaved in perturbation theory than the loop itself.[7] Taking Creutz ratios [11] of Wilson loops further improves perturbation theory by reducing the effects of both the divergent contributions associated with the perimeter of the loop and those coming from the corners of the loop.

For these reasons we concentrate in this study on the logarithms of small Wilson loops and on Creutz ratios χ_{mn} defined by

$$\chi_{mn} \equiv -\ln\left(\frac{W_{mn}W_{m-1\,n-1}}{W_{m\,n-1}W_{m-1\,n}}\right), \qquad (21)$$

where W_{mn} is one third the expectation value of the trace of the $m \times n$ planar Wilson loop:

$$W_{mn} \equiv \tfrac{1}{3}\langle \text{Tr}\,U_{m\times n}\rangle. \qquad (22)$$

We compare perturbative predictions with new data generated on a 16^4 lattice at β's ranging from 5.7 to 18 [12]. We use one-loop and two-loop perturbation-theory coefficients computed for a 16^4 lattice [8], and include the leading-order contribution from the zero mode [13]. Thus our perturbation theory is accurate up to uncalculated terms of order α_V^3, and of order α_V^2/V, due to the zero mode, where $V = 16^4$ is the volume of the lattice. The finite-volume errors become significant for larger loops and so we limit ourselves to 5×5 loops and smaller.

In Fig. 3 we show results for χ_{22}, calculated through first order in α_s, and also through second order. The pattern at first order is similar to that in our previous examples: expansions in $\alpha_V(q^*)$ and $\alpha_{\overline{\text{MS}}}(q^*)$ give reliable results at all β's; the expansion in α_{lat} is off by almost a
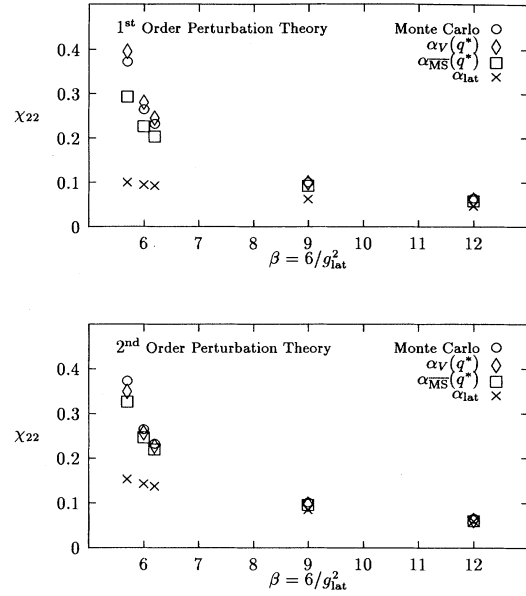

FIG. 3. Results for Creutz ratio χ_{22} at different couplings β from Monte Carlo simulations (circles), and from perturbation theory [using $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{\text{MS}}}(q^*)$ (boxes), and α_{lat} (crosses)]. The first plot shows perturbation theory through one-loop order, and the second through two-loop order. Statistical errors in the Monte Carlo results are negligible.

factor of 4 at $\beta = 5.7$, and still by almost 30% at $\beta = 12$. The second-order corrections significantly improve agreement between the data and the α_V and $\alpha_{\overline{\text{MS}}}$ expansions, with errors ranging from a few percent at $\beta = 5.7$ to a few tenths of a percent at $\beta = 12$. The remaining discrepancy could easily be accounted for by uncalculated corrections of order α_s^3, although again nonperturbative effects may play a role at the lowest β's. The second-order expansion with α_{lat} gives results that are at least an order of magnitude worse than those from the other two expansions (at all β's). By comparison with the others, the convergence of this expansion is very sluggish—an unambiguous symptom of a bad expansion parameter.[8]

In Fig. 4 we show two-loop results with each of the coupling constants for a variety of different Creutz ratios at $\beta = 6.2$. The α_V and $\alpha_{\overline{\text{MS}}}$ expansions are again far superior for all of the ratios.

We expect smaller momentum scales for Creutz ratios than for the loops themselves since many of the divergent contributions to loop expectation values cancel in the

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[7]Our data confirm that perturbation theory works better for logarithms of the W_{mn} than for the W_{mn} themselves, the expansions for the latter failing completely for even modestly large loops. Curiously the pathologies in the W_{mn} expansions seem to cancel the pathologies in α_{lat} when m and n are small, making the α_{lat} expansion more accurate than the α_V expansion for these loops. Neither expansion is as accurate as expanding $-\ln W_{mn}$ in powers of α_V and then exponentiating. This last procedure gives good results (when β is large) for all loops out to 8×8, the largest we checked.

[8]Note that some of our results have been anticipated in the literature. The fact that perturbative results for Creutz ratios are better behaved when expanded in terms of $\alpha_{\overline{\text{MS}}}$ than when expanded in terms of α_{lat} was pointed out in [14]. The fact that perturbative results for Creutz ratios are better behaved when expanded in terms of an α_s defined from any given ratio than when expanded in terms of α_{lat} was pointed out in [15].
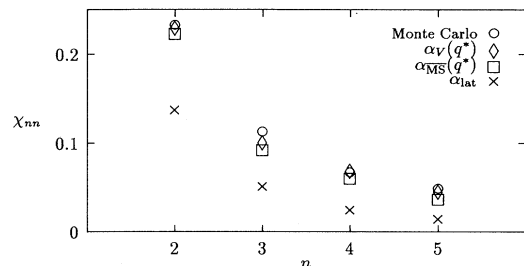
FIG. 4. Results from perturbation theory [with $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{MS}}(q^*)$ (boxes), and α_{lat} (crosses)] and Monte Carlo simulations (circles) for diagonal Creutz ratios $\chi_{n,n}$ at $\beta = 6.2$. Statistical errors in the Monte Carlo results are negligible.

ratios. Our scale-setting procedure indicates that q^* is $1.09/a$ for χ_{22}, and smaller for ratios involving larger loops.

The importance of choosing a proper q^* is illustrated in Fig. 5, where the two-loop prediction for χ_{22} has been reexpressed in terms of $\alpha_V(q)$ and plotted versus q. Taking $q = \pi/a$, for example, rather than $q = q^* \equiv 1.09/a$ results in a 10% error rather than a 1% error. This situation should be contrasted with that for $-\ln W_{22}$ (Fig. 6). This quantity is significantly more ultraviolet than χ_{22}, having $q^* = 2.65/a$. Here our perturbative estimate degrades significantly if we use, say, $q = 1/a$ rather than q^* to set the scale of our expansion parameter. These examples illustrate the importance of our scale-setting procedure when high precision is required. Small departures from q^* are unimportant, at least for reasonably convergent series; but deviations by factors of 2 or more can affect the reliability of a perturbative estimate.

IV. MEAN-FIELD THEORY

We have shown that perturbation theory works well when a proper coupling constant is used, but it is still important to understand the origins of the large mismatch between the lattice coupling and the continuum couplings. This mismatch is one of many examples where a large renormalization is required to relate a lattice quantity to its continuum analogue. In this section we
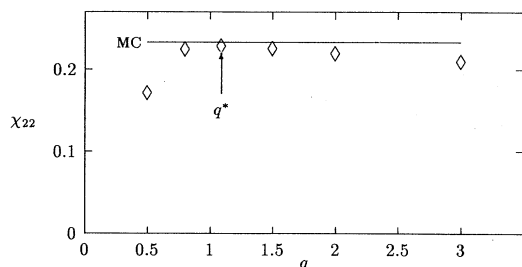

FIG. 5. Results for the Creutz ratio χ_{22} from Monte Carlo simulations (line) and from the perturbation expansion in $\alpha_V(q)$ (diamonds) vs the scale q.
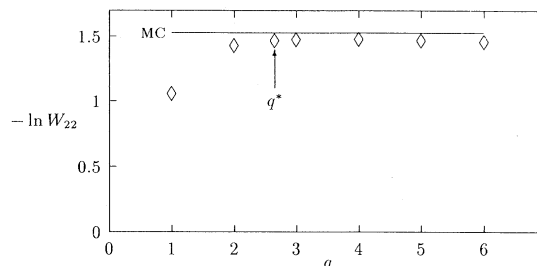

FIG. 6. Results for $-\ln W_{22}$ from Monte Carlo simulations (line) and from the perturbation expansion in $\alpha_V(q)$ (diamonds) vs the scale q.

explore the connection between operators on the lattice and in the continuum.

A. Tadpole improvement

We usually design lattice operators by mapping them onto analogous operators in the continuum theory. For gauge fields, this mapping is based upon the expansion

$$U_\mu(x) \equiv e^{iagA_\mu(x)} \rightarrow 1 + iagA_\mu(x). \qquad (23)$$

This expansion seems plausible when the lattice spacing a is small, but it is misleading since further corrections do *not* vanish as powers of a in the quantum theory. Higher-order terms in the expansion of U_μ contain additional factors of gaA_μ, and the A_μ's, if contracted with each other, generate ultraviolet divergences that precisely cancel the additional powers of a. Consequently these terms are suppressed only by powers of g^2 (not a), and turn out to be uncomfortably large. These are the QCD tadpole contributions.

The tadpoles spoil our intuition about the connection between lattice operators and the continuum, and so we should not be surprised if the lattice theory is not quite what we expected (because of large renormalizations). In order to regain this intuition we must refine the naive formula that connects the lattice operator to the continuum operator [Eq. (23)]. Consider the vacuum expectation values of these operators. In the continuum, the expectation value of $1 + iagA_\mu(x)$ is 1. In the lattice theory, tadpole corrections renormalize the link operator so that its vacuum expectation value (in, say, Landau gauge) is considerably smaller than 1 (see Fig. 1). This suggests that the appropriate connection with continuum fields is more like

$$U_\mu(x) \rightarrow u_0 \left[1 + iagA_\mu(x)\right], \qquad (24)$$

where u_0, a number less than one, represents the mean value of the link. Gauge invariance requires that parameter u_0 enter as an overall constant.

This formula follows simply from a renormalization-group argument. The tadpole contributions come mainly from the gauge-field modes with the highest momenta. Consequently the tadpoles can be removed by splitting

the gauge field into ultraviolet (UV) and infrared (IR) parts (in a smooth gauge), and integrating out the UV parts. Averaging over the UV modes, the link operator is replaced by its IR part:

$$U_\mu \to u_0 \, e^{iagA_\mu^{\mathrm{IR}}} \approx u_0 \, (1 + iagA_\mu^{\mathrm{IR}}), \qquad (25)$$

where now the Taylor expansion of the exponential is quite convergent. Parameter u_0 contains the averaged UV contribution. It enters only as an overall constant since the link operator functions as a gauge connection both before and after averaging.

The mean-field parameter u_0 depends upon the parameters of the theory. It can be measured easily in a simulation. Simply measuring the link expectation value gives zero since the link operators are gauge dependent. However relations such as Eq. (24) only make sense in smooth gauges, such as the Landau gauge. Thus one might define u_0 to be the expectation value of the link operator in the Landau gauge. A simpler, gauge-invariant definition uses the measured value of the plaquette:

$$u_0 \equiv \langle \tfrac{1}{3} \mathrm{Tr} U_{\mathrm{plaq}} \rangle^{1/4}. \qquad (26)$$

Several other definitions are possible based upon such things as the expectation value of long Wilson lines or the critical hopping parameter for Wilson quarks. All give similar results: at $\beta = 6$, for example, u_0 is 0.86 from the Landau-gauge link and 0.88 from the plaquette.

Our improved relation, Eq. (24), between lattice and continuum gauge-field operators suggests that all links U_μ that appear in lattice operators should be replaced by U_μ/u_0, where u_0 is measured in the simulation. The operators U_μ/u_0 are much closer in their behavior to their continuum analogues; large tadpole renormalizations are largely canceled out by the u_0 (and the cancellation is nonperturbative since u_0 is measured rather than calculated). This is the key ingredient in our tadpole-improvement procedure for lattice operators. Several illustrations follow in succeeding sections.

B. α_V from α_{lat}

Our new prescription for building continuumlike operators suggests that

$$\tilde{S}_{\mathrm{gluon}} = \sum \frac{1}{\tilde{g}^2 u_0^4} \mathrm{Tr}(U_{\mathrm{plaq}} + \mathrm{H.c.}) \qquad (27)$$

is a better gluon action for lattice QCD. In particular, perturbation theory in $\tilde{g}^2$ is much more like continuum perturbation theory (i.e., no tadpoles). Of course this tadpole-improved action becomes the normal gluon action if we identify

$$\begin{aligned}\tilde{g}^2 &= g_{\mathrm{lat}}^2/u_0^4 \\ &= g_{\mathrm{lat}}^2/\langle \tfrac{1}{3}\mathrm{Tr}(U_{\mathrm{plaq}})\rangle.\end{aligned} \qquad (28)$$

This relationship is very important; it tells us that the correct expansion parameter for the *usual* theory is $\tilde{g}^2$
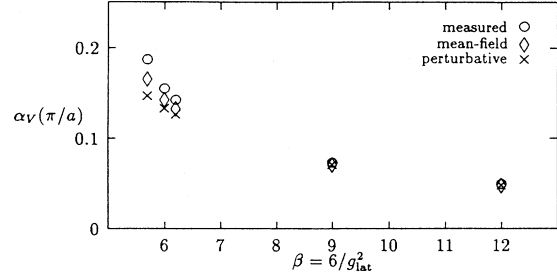

FIG. 7. Values of $\alpha_V(\pi/a)$ as determined by measuring $-\ln\langle\tfrac{1}{3}\mathrm{Tr}U_{\mathrm{plaq}}\rangle$ (circles), by using a nonperturbative mean-field formula to relate it to the bare coupling (diamonds), and by using perturbation theory to relate it to the bare coupling (crosses).

rather than g_{lat}^2. The difference is significant; for example, $\tilde{g}^2 \approx 1.7 g_{\mathrm{lat}}^2$ at $\beta = 6$ (using the measured value of the plaquette to relate the couplings). It is a big mistake to expand in powers of α_{lat} rather than $\tilde{\alpha}_{\mathrm{lat}} \equiv \tilde{g}^2/4\pi$.[9]

If our mean-field analysis is correct, $\tilde{\alpha}_{\mathrm{lat}}$ should be roughly equal to $\alpha_V(\pi/a)$. This is confirmed by perturbation theory which implies that

$$\alpha_V(\pi/a) = \frac{\alpha_{\mathrm{lat}}}{\langle \tfrac{1}{3}\mathrm{Tr}U_{\mathrm{plaq}}\rangle} \, [1 + 0.513\,\alpha_V + O(\alpha_V^2)]; \qquad (29)$$

the difference between the two coupling constants is only a few percent at $\beta = 6$. This formula provides a nonperturbative relationship between the bare lattice coupling α_{lat} and α_V when measured values for the plaquette are used.

Note that since the renormalization is multiplicative, its main effect is to rescale the argument of the running coupling constant. This suggests that we define α_V by

$$\alpha_V(46.08/a) = \alpha_{\mathrm{lat}} \, [1 + O(\alpha_V^2)] \qquad (30)$$

(since $\Lambda_V = 46.08\Lambda_{\mathrm{lat}}$), and then use two-loop evolution to determine $\alpha_V(q)$ for any other scale q. This provides a purely perturbative relation between α_V and α_{lat}. Relating the couplings by rescaling the arguments is roughly equivalent to using the formulas that relate the inverse couplings [Eq. (13)].

In Fig. 7 we compare the measured values of $\alpha_V(\pi/a)$ from Sec. III with values obtained from the mean-field formula, Eq. (29), and from the perturbative formula,

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[9]Our analysis of the gauge-field action was anticipated by Parisi [16] who gives a simple analysis for the compact Abelian theory. To see what effect the (UV-divergent) tadpoles have on infrared modes, we can split the gauge field in UV and IR components, and average over the UV part. Then the Abelian gauge action becomes

$$\langle g^{-2}\cos(gF_{\mu\nu}^{\mathrm{UV}} + gF_{\mu\nu}^{\mathrm{IR}})\rangle^{\mathrm{UV}} = g^{-2}\langle\cos(gF_{\mu\nu}^{\mathrm{UV}})\rangle \, \cos(gF_{\mu\nu}^{\mathrm{IR}})$$

and the effective coupling for the IR modes is g^2 divided by the UV part of the plaquette expectation value.

Eq. (30). Large coupling-constant renormalizations are automatically incorporated when α_V is measured, and so the validity of our mean-field analysis is tested by the extent to which the mean-field values agree with the measured values. All three methods produce results consistent up to corrections of order α_V^3. The first two methods of determining α_V are preferable to the perturbative formula, particularly at low β's, since they incorporate some higher-order and nonperturbative effects.

C. Field strengths

Our prescription for defining tadpole-improved lattice operators is crucial in other, related contexts. One example is in defining operators to represent the chromo-electric and magnetic fields. These are needed for the operators that remove $O(a, a^2)$ errors from quark actions. The standard cloverleaf definitions $\mathbf{E}_{cl}$ and $\mathbf{B}_{cl}$ involve products of four link operators, just like the plaquette [17]. Thus the tadpole-improved operators

$$\tilde{\mathbf{E}}_{cl} = \mathbf{E}_{cl}/\langle \tfrac{1}{3}\mathrm{Tr}(U_{\mathrm{plaq}})\rangle, \tag{31}$$

$$\tilde{\mathbf{B}}_{cl} = \mathbf{B}_{cl}/\langle \tfrac{1}{3}\mathrm{Tr}(U_{\mathrm{plaq}})\rangle, \tag{32}$$

are almost twice as large at $\beta = 6$. The plaquette factors account for the bulk of the very large renormalizations found in perturbation theory for operators containing cloverleaf fields. Such operators play an important role in all formulations of heavy-quark dynamics [18,19]; omitting the tadpole renormalization leads to severe underestimates of their effects.

D. Improved Wilson fermions

Our tadpole-improvement scheme provides valuable insights into the pattern of large renormalizations in lattice QCD, and it is generally trivial to implement. As another example consider the tadpole-improved action for Wilson quarks:

$$S_q = \sum_x \bar{\psi}\psi + \tilde{\kappa} \sum_{x,\mu} \bar{\psi}\left((1+\gamma_\mu)\frac{U_\mu}{u_0}\right)\psi + \cdots. \tag{33}$$

Again, this action is identical to the usual one if we relate the modified parameters, here the hopping parameter $\tilde{\kappa}$, to the usual ones by rescaling with u_0:

$$\tilde{\kappa} = \kappa\, u_0. \tag{34}$$

The modified hopping parameter should be more continuumlike; for example, the tree-level value that gives massless quarks, $\tilde{\kappa}_c = 1/8$, should be roughly correct for interacting quarks as well, at least at high β's. Thus an approximate nonperturbative formula for the critical value of the usual hopping parameter is

$$\kappa_c \approx 1/8u_0, \tag{35}$$

or, in terms of the critical bare mass,

TABLE II. Mass renormalization for Wilson fermions at different couplings β as computed using ordinary α_V perturbation theory, tadpole-improved perturbation theory, and Monte Carlo simulation. Also listed is the first-order perturbative correction to the mean-field estimate of m_c, $\delta m_c a = -1.268\alpha_V(1.03/a)$.

β	$\delta m_c a$	Perturbative	Tadpole improved	Measured
5.7	-0.44	-1.11	-1.00	-1.04(2)
6.0	-0.31	-0.91	-0.80	-0.80(2)
6.1	-0.29	-0.86	-0.76	-0.78(2)
6.3	-0.22	-0.79	-0.70	-0.70(2)

$$m_c a \equiv 1/2\kappa_c - 4 \approx 4u_0 - 4. \tag{36}$$

This formula accounts for about 75% of the renormalization of the hopping parameter when β is large, as is evident if we compare the perturbative expansions for the two sides (defining $u_0 = \langle \tfrac{1}{3}\mathrm{Tr}U_{\mathrm{plaq}}\rangle^{1/4}$). By combining these perturbative expansions, we obtain a tadpole-improved perturbation theory for the critical bare mass m_c:

$$m_c a = -4\left(1 - \langle \tfrac{1}{3}\mathrm{Tr}U_{\mathrm{plaq}}\rangle^{1/4}\right) - 1.268\,\alpha_V(1.03/a)$$
$$+ O(\alpha_V^2). \tag{37}$$

This formula has the same perturbative expansion as that used in Sec. III B, but, by rearranging the expansion and using the measured value of the plaquette, we make it far more accurate since large tadpole renormalizations are summed to all orders. Higher-order perturbative corrections should be smaller for the improved formula, as should nonperturbative effects. This seems to be the case as we show in Table II, where the two predictions are compared with Monte Carlo data. Tadpole-improved one-loop perturbation theory predicts m_c (and κ_c) about as accurately as it can be measured.

In the continuum limit, the tadpole-improved lagrangian for massless quarks becomes

$$2\tilde{\kappa}\, \bar{\psi}\gamma_\mu \partial^\mu \psi + O(a). \tag{38}$$

This indicates that $\sqrt{2\tilde{\kappa}}\,\psi$ is the lattice quantity that corresponds to the continuum quark field. Since $\tilde{\kappa} \approx 1/8$ when the quarks are massless (or, equivalently, $u_0 \approx 1/8\kappa_c$), a tadpole-improved operator for massless quarks on the lattice is[10]

$$\tilde{\psi} \equiv \psi/2. \tag{39}$$

This lattice operator has roughly the same normalization as the continuum field; in particular, there are no

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[10]This result is only valid for $\kappa \approx \kappa_c$. Away from κ_c, $\tilde{\psi}$ becomes [20,4]

$$\tilde{\psi} \equiv (1 - 3\kappa/4\kappa_c)^{1/2}\,\psi.$$

Note that the κ dependence here is quite different from that of the commonly used (but incorrect) $\sqrt{2\kappa}\psi$.

TABLE III. Renormalization constants relating continuum ($\overline{\text{MS}}$ at $\mu = 1/a$) and lattice operators for massless quarks. Results are for the ratio of continuum to lattice matrix elements, both without and with tadpole factors removed.

Operator		Perturbative	Tadpole improved
$\bar{\psi}\psi$	[21]	$(1 - 1.39\,\alpha_V)\,2\kappa_c$	$(1 - 0.03\,\alpha_V)\,/4$
$\bar{\psi}\gamma^5\psi$	[21]	$(1 - 2.40\,\alpha_V)\,2\kappa_c$	$(1 - 1.03\,\alpha_V)\,/4$
$\bar{\psi}\gamma^\mu\psi$	[21]	$(1 - 2.19\,\alpha_V)\,2\kappa_c$	$(1 - 0.82\,\alpha_V)\,/4$
$\bar{\psi}\gamma^\mu\gamma^5\psi$	[21]	$(1 - 1.68\,\alpha_V)\,2\kappa_c$	$(1 - 0.31\,\alpha_V)\,/4$
$\bar{\psi}\sigma^{\mu\nu}\psi$	[21]	$(1 - 1.80\,\alpha_V)\,2\kappa_c$	$(1 - 0.44\,\alpha_V)\,/4$
$C_1^L[\psi\psi\psi]$	[22]	$(1 - 2.78\,\alpha_V)\,(2\kappa_c)^{1.5}$	$(1 - 0.73\,\alpha_V)\,/8$
$d_1^L[\psi\psi\psi]$	[22]	$(1 - 2.89\,\alpha_V)\,(2\kappa_c)^{1.5}$	$(1 - 0.84\,\alpha_V)\,/8$
$d_4^L[\psi\psi\psi]$	[22]	$(1 - 2.76\,\alpha_V)\,(2\kappa_c)^{1.5}$	$(1 - 0.71\,\alpha_V)\,/8$
$\bar{\psi}\psi O_+\psi\psi$	[23]	$(1 - 3.81\,\alpha_V)\,(2\kappa_c)^2$	$(1 - 1.08\,\alpha_V)\,/16$
$\bar{\psi}\psi O_-\psi\psi$	[23]	$(1 - 3.66\,\alpha_V)\,(2\kappa_c)^2$	$(1 - 0.93\,\alpha_V)\,/16$
$\bar{\psi}\psi O_1\psi\psi$	[23]	$(1 - 3.78\,\alpha_V)\,(2\kappa_c)^2$	$(1 - 1.05\,\alpha_V)\,/16$
$\bar{\psi}\psi O_2\psi\psi$	[23]	$(1 - 3.55\,\alpha_V)\,(2\kappa_c)^2$	$(1 - 0.83\,\alpha_V)\,/16$

large tadpole contributions to the renormalization constant relating them. This is important in designing new lattice operators involving quark fields. For example, if one wants matrix elements of the continuum current $\bar{\psi}\gamma^\mu\gamma^5\psi$, then one should simulate with the lattice operator

$$\bar{\tilde{\psi}}\gamma^\mu\gamma^5\tilde{\psi} = \tfrac{1}{4}\,\bar{\psi}\gamma^\mu\gamma^5\psi. \tag{40}$$

Our procedure differs somewhat from common practice. Frequently the quark renormalization factor is taken to be $\sqrt{2\kappa_c}$ for massless quarks, rather than the factor $\sqrt{2\tilde{\kappa}_c} = 1/2$ that we use. Then an operator O containing n quarks is renormalized with a factor of the form $(2\kappa_c)^{n/2}\,Z_O$, where κ_c is measured and Z_O is computed in perturbation theory. With our procedure, such an operator is renormalized with a (simpler) factor $\tilde{Z}_O/2^n$, where $\tilde{Z}_O = (8\kappa_c)^{n/2}\,Z_O$ is again determined perturbatively. This is simply a rearrangement of perturbation theory, but by incorporating the κ_c factors into $\tilde{Z}_O$ we cancel most of the tadpole contribution. Thus perturbation theory should be significantly more convergent for $\tilde{Z}_O$ than it is for Z_O. We have verified this for a variety of two-, three-, and four-quark operators. Our results are presented in Table III.[11] There we present the renormalization factors for each operator (continuum divided by lattice) as computed in the standard fashion, and with our tadpole improvements. Perturbation theory is much more convergent with our procedure.

The continuum operators used in this comparison were defined using the $\overline{\text{MS}}$ scheme. Our choice of normalization scale, $\mu = 1/a$, was somewhat empirical; a more systematic determination of the appropriate scale is possible using a variation of the techniques discussed in Sec. II. Also, there is another obvious nonperturbative procedure for normalizing the operator for lattice quarks. The

quark field's normalization should be roughly the square root of the normalization of either the vector or axial-vector current since these currents are conserved (or partially conserved). Inspection of our table indicates that using the average normalization of the two currents to define the quark normalization gives even better results than those shown there.

The results here are all for massless or nearly massless quarks. Tadpole-improved operators and actions for heavy quarks are discussed in [20,4], for Wilson quarks, and in [19], for nonrelativistic quarks (NRQCD).

V. IMPLICATIONS FOR SCALING

An important test of any lattice analysis is that physical results are independent of the lattice spacing a, or, equivalently, that they are independent of the bare coupling α_{lat} used in the action (each value of α_{lat} corresponds to a particular value of a). To apply this test we need to know how variations in α_{lat} are related to variations in a; that is, we need to know the β function, $d\alpha_{\text{lat}}/d\ln a$. This is because simulation results are all in units of a, while a itself is specified only indirectly in a simulation, through the value of α_{lat}. (The lattice spacing sets the scale for a simulation, and so the variation of the coupling with varying lattice spacing is referred to as the "scaling" of the coupling constant.)

There are two common procedures by which the β function is determined. One uses simulation results for a long-distance quantity such as the string tension or a glueball mass. Simulations of a glueball mass M, for example, produce values of $M a$ for each value of α_{lat} used in the simulation's action. From these one can compute $d\alpha_{\text{lat}}/d\ln M a$, which is the same as $d\alpha_{\text{lat}}/d\ln a$ insofar as mass M is a independent. Unfortunately, any quantity such as M is only approximately a independent because of finite-lattice-spacing errors in the action. This additional a dependence results in systematic errors in the determination of the β function that are difficult to quantify without extensive simulation.

A second approach for obtaining the β function is to compute it analytically, using weak-coupling perturbation theory. This becomes feasible when a is small since the β function is predominantly determined by the short-distance behavior of the theory, and this behavior is perturbative for QCD. A perturbative calculation of the β function avoids ambiguities due to finite-lattice-spacing errors in the action since these are readily analyzed in perturbation theory. The main issue in this approach is whether or not the lattice spacing is small enough to justify using perturbation theory.

Our results in Sec. III show that perturbation theory successfully predicts a range of short-distance quantities, and so it is likely that it also correctly predicts the β function, a short-distance quantity like any other. Thus our results provide indirect evidence in support of perturbative scaling of the coupling. Our results also test the scaling properties of the coupling constant directly. This is because at each β we measure the coupling α_V at

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[11]These results are for Wilson parameter $r = 1$; very similar results arise for $r = \tfrac{1}{2}$.

$q = 3.41/a$, using data for the very ultraviolet-divergent plaquette, and then we perturbatively evolve α_V down to scales ranging from $0.4/a$ to $2.8/a$ to compute estimates for a variety of less ultraviolet quantities. (Note that α_V at $\beta = 6$ increases by more than 50% when evolving from $q = 3.41/a$ down to $q = 1.09/a$, the scale for Creutz ratio χ_{22}.) The success of our many perturbative estimates is compelling evidence that coupling-constant evolution is mostly perturbative for all β's down to 5.7, and possibly even for lower ones. Of course, this discussion only applies to the α_V and $\alpha_{\overline{\text{MS}}}$ definitions of the coupling; α_{lat} is poorly behaved, but also largely irrelevant given our new perturbative techniques.

The q dependence of $\alpha_V(q)$ is readily extracted from our data. The results for three values of β are shown in Fig. 8. To obtain these plots, we fit second-order expansions in $\alpha_V(q^*)$ to Monte Carlo data for the six smallest Creutz ratios, and for the logarithms of the six smallest Wilson loops. The value of $\alpha_V(q^*)$ obtained from the fit for each quantity is plotted versus the q^* for that quantity. The q^*'s for the 12 quantites used here range from $0.43/a$ (for χ_{44}) to $3.41/a$ (for $-\ln W_{11}$)—about a factor of 8. For comparison, we have included the (two-loop) perturbative prediction for $\alpha_V(q)$ (solid line), arbitrarily normalizing α_V so that the curve passes through the data point for $-\ln W_{22}$. The data are quite consistent with perturbative scaling, even at $\beta = 5.7$.[12] Note that statistical errors in the Monte Carlo data are negligible here; the fluctuations visible in the plots are due to uncalculated third-order terms in perturbation theory, which differ from quantity to quantity, and, at the lowest β, to nonperturbative effects. (The onset of the long-distance area law in the logarithms of the Wilson loops is apparent in the plot for $\beta = 5.7$, although the effect is not all that large even for the 3×3 loop.)

Our conclusion, that the coupling constant scales perturbatively even at $\beta = 5.7$, contradicts standard lore. This lore derives in part from studies of scale-invariant ratios of Λ_{lat}, defined perturbatively from α_{lat}, with physical quantities such as the deconfining temperature T_c or the string tension σ that are determined nonperturbatively in simulations. Such ratios, which should become independent of β with the onset of perturbative scaling, show considerable variation with β for β's less than 6.2. This is illustrated by the upper plots in Fig. 9. These show the $1P$-$1S$ mass splitting ΔM divided by Λ_{lat} for the ψ [18] and Υ [25] meson families, as well as $\sqrt{\sigma}/\Lambda_{\text{lat}}$ [26], for a range of β's. Variations of order 30–40 % are readily apparent between $\beta = 5.7$ to $\beta = 6.1$. However, the situation changes completely if we replace the Λ_{lat}'s
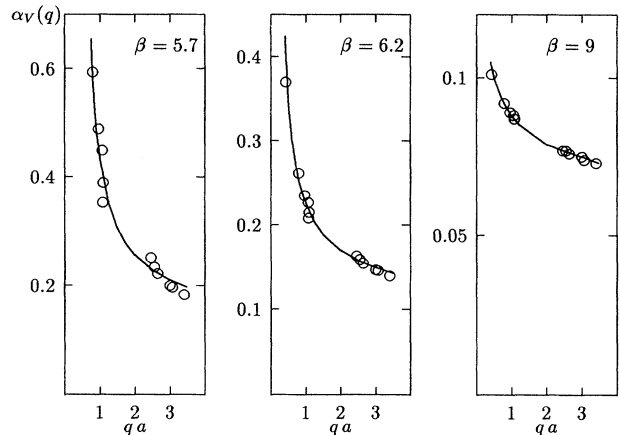

FIG. 8. Values of $\alpha_V(q)$ for a range of q's as determined from lattice QCD measurements at various β's. The data points (circles) are measured values (with negligible statistical errors) obtained by fitting second-order perturbation theory to Monte Carlo simulation data for various short-distance quantities. The solid line shows the variation in $\alpha_V(q)$ expected from two-loop perturbation theory.

in these ratios by Λ_V's determined (perturbatively) from the α_V's we used in Sec. III. When compared with Λ_V, the ratios are constant to within a few percent.

The β independence of these ratios is not exact. There are certainly finite-lattice-spacing errors in each of the measured quantities we use here. These errors have been analyzed carefully for the Υ data [4]; they lead to variations of order 10% in the ratio with Λ_V over the range shown. Errors for the other two quantities are probably smaller since Υ's are the smallest mesons.[13]

There are, of course, physical quantities that are more sensitive to finite-lattice-spacing errors than those discussed here, and scale-invariant ratios involving these will show considerably more variation than we find for the quarkonium and string tension data. This does not indicate a failure of perturbative scaling for the coupling constant; rather, it indicates that $O(a, a^2)$ corrections are needed in the action for accurate simulations of these physical quantities.

These results suggest that the two methods for determining the β function, indirectly, using nonperturbative simulations of long-distance quantities, and directly, using perturbation theory, agree even at low β's, provided the action is sufficiently accurate that the long-distance quantities are insensitive to finite-lattice-spacing errors. We feel that the perturbative determination is probably more reliable, at least until it is common practice to correct actions for $O(a, a^2)$ errors. Standard lore about

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[12]Similar results are reported for the SU(2) lattice theory in [24], although the finite-scaling technique used there is quite different from our procedure. That study probes different scales by examining a single quantity on a series of lattices with different lattice spacings. Our study probes different scales on single lattices by examining a variety of quantities, some more ultraviolet than others. In both cases the evolution of the coupling constant is tracked over a large range of scales.

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[13]Note that the $1P$-$1S$ splitting in quarkonium mesons is one of few hadronic measurements that is suitable for studying scaling. This is because the splitting is almost completely insensitive to the heavy-quark mass, and so depends only upon the coupling constant. This is also why the ψ results shown are nearly indistinguishable from the Υ results.
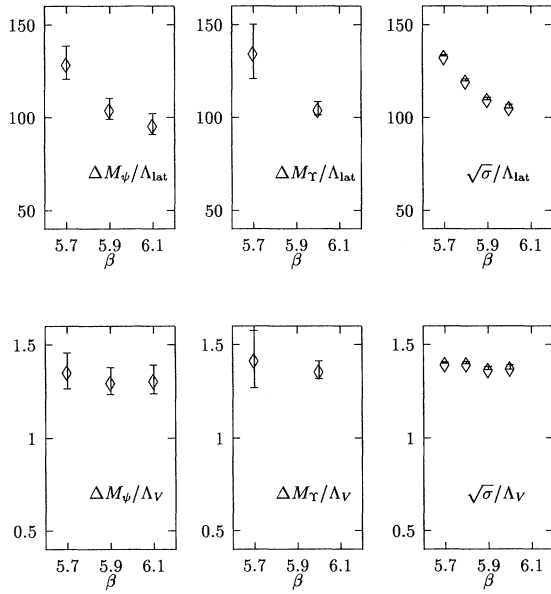
FIG. 9. Ratios of the square root of the string tension σ and the $1P$-$1S$ mass splittings ΔM for ψ's and Υ's with Λ_{lat} (top row) and with Λ_V (bottom row).

scaling is invalid partly because it is based on simulation results from uncorrected actions, and partly because it relies upon perturbative expansions in the bare coupling α_{lat}. The coupling α_{lat} does not scale perturbatively at low β's (at least through two-loop order), but renormalized couplings such as $\alpha_V(\pi/a)$ are perturbative. This is why the ratios with Λ_V scale so well. (Of course, ratios of physical quantities should be β independent as well, and they are; see Fig. 10.)

It has been apparent for some time that the deconfining temperature scales better when analyzed in terms of a modified coupling constant similar to ours [27]. Now it is clear that the modified coupling constant is just a continuum coupling constant such as α_V or $\alpha_{\overline{\text{MS}}}$. Furthermore it is clear that the failure of perturbative scaling for the coupling constant was intimately related to the lack of convergence of perturbation theory for short-distance quantities such as κ_c or χ_{22}. Both problems are resolved by replacing α_{lat} with α_V.
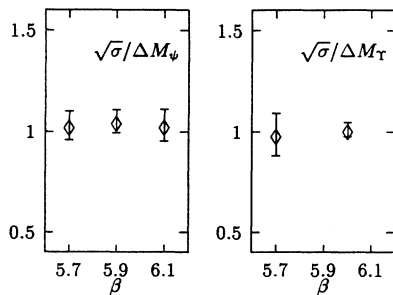

FIG. 10. Ratios of the square root of the string tension σ with the $1P$-$1S$ mass splittings ΔM for ψ's and Υ's.

VI. SUMMARY

The use of lattice perturbation theory in conjunction with simulations has been hampered by two problems: expansions in powers of the bare lattice coupling α_{lat} consistently underestimate perturbative effects, sometimes by factors of 2 or 4; expansions for many quantities (and particularly renormalization constants for lattice operators) have large coefficients due to tadpole diagrams and consequently converge poorly, if at all.

We have addressed both problems in this paper. We have shown here that lattice perturbation theory works well when a proper coupling constant is used; and it can be made about as convergent as the continuum theory by systematically removing tadpole contributions.

The first problem is remedied by replacing α_{lat} with a renormalized coupling constant such as $\alpha_V(q^*)$, where scale q^* is customized (in a predetermined way) to the quantity under study. The coupling constant α_V is defined in terms of a physical quantity, the heavy-quark potential, and it can either be measured (Sec. III) or it can be determined from the bare lattice coupling α_{lat} using formulas from mean-field theory (Sec. IV B). Perturbation theory, when expressed in terms of $\alpha_V(q^*)$, is remarkably effective even at $\beta = 5.7$.

The second problem, large tadpole-induced renormalizations, is remedied by simple redefinitions of the basic operators used to define the lattice theory. Every U_μ in a naive lattice operator is replaced with U_μ/u_0, where u_0 is a measured constant representing the mean value of the link (Sec. IV); and every renormalized low-mass (Wilson) quark field $\sqrt{2\kappa_c}\psi$ is replaced by $\psi/2$. The new operators obtained this way are rescaled versions of the naive operators. Their normalizations are very close to those of their continuum analogues; renormalization constants for composite operators built from these tadpole-improved operators have perturbative expansions that are far more convergent. Tadpole improvement is essential for operators, such as the cloverleaf operators for $F_{\mu\nu}$, that involve many links; without it normalizations are wrong by as much as a factor of 2, and perturbation theory becomes useless.

Our examples suggest that lowest-order perturbation theory in α_V gives results for short-distance quantities that are typically correct to within 10–20 % at $\beta = 6$. Expansions in α_{lat} can be off by factors 2 or 4 at the same β. Adding higher-order corrections usually reduces errors by factors of 2–5 for α_V expansions, and by very little for α_{lat} expansions. In many situations, α_V expansions can be made still more accurate through tadpole improvement, where powers of the mean-field parameter u_0 are factored out of the expansion leaving behind a more convergent series. Our tadpole-improved one-loop formula [Eq. (37)] for the critical value of the hopping parameter in Wilson's quark action, for example, is about as accurate as the best numerical determinations of this quantity. Finally, our procedure for determining the proper scale q^* for the coupling consistently leads to excellent expansions, although expansions in $\alpha_V(\pi/a)$ for quantities defined over one or two lattice spacings usually give

errors that are within a factor of 2–3 of those obtained with $\alpha_V(q^*)$.

The fact that perturbation theory seems to be working at $\beta = 5.7$ implies that asymptotic or perturbative scaling should also work. We verified this here, for a number of physical quantities, by comparing their dependence on β with that of the scale parameter Λ_V for the renormalized coupling α_V; scale invariant ratios of these quantities with Λ_V (as opposed to Λ_{lat}) showed little variation all the way down to $\beta = 5.7$. These results suggest that the lattice spacings used in current simulations are small enough for reliable studies of QCD. Indeed, if anything, they are unnecessarily small. It is probably much more cost effective to simulate QCD at $\beta = 5.7$, while removing the $O(a, a^2)$ errors that are important by correcting the action. Previous efforts at improving lattice actions have not been too successful; but these relied upon the use of expansions in α_{lat}, and naive lattice operators. The perturbative quantities we examine in this paper are very similar in character to the new coupling constants that appear in corrected actions and operators. Thus our success in computing these quantities (to within a few percent in most cases) suggests that the use of α_V perturbation theory and tadpole-improved operators to correct the action will be much more successful. The po-

tential savings in computer resources make it imperative that this possibility be thoroughly investigated.

ACKNOWLEDGMENTS

APPENDIX

This appendix presents data for some of the figures in Tables IV–XII.

TABLE IV. $\langle 1 - \frac{1}{3}\text{Tr}U_\mu \rangle$ (Landau gauge) the expectation value of the trace of a link in the Landau gauge calculated in first-order perturbation theory and by Monte Carlo simulation for various β's. Statistical errors in all data presented are of order one in the last digit quoted or smaller.

| β | Perturbation theory | | | MC |
	α_{lat}	$\alpha_{\overline{\text{MS}}}(q^*)$	$\alpha_V(q^*)$	data
5.7	0.081	0.161	0.191	0.176
6	0.077	0.136	0.157	0.139
6.4	0.072	0.118	0.133	0.117

TABLE V. $a\,m_c$—the critical quark mass m_c for Wilson quarks with r=1.0, calculated in first-order perturbation theory and by Monte Carlo simulation for various β's. Statistical errors in the simulation data are of order 2 in the last digit quoted.

| β | Perturbation theory | | | MC |
	α_{lat}	$\alpha_{\overline{\text{MS}}}(q^*)$	$\alpha_V(q^*)$	data
5.7	-0.46	-0.93	-1.12	-1.04
6	-0.43	-0.78	-0.91	-0.80
6.1	-0.43	-0.75	-0.86	-0.78
6.3	-0.41	-0.70	-0.79	-0.70

TABLE VI. χ_{22}—the expectation value of the Creutz ratio χ_{22} calculated in first-order and second-order perturbation theories and by Monte Carlo simulation for various β's. Statistical errors in the Monte Carlo results are negligible.

β	First order			Second order			MC data
	α_{lat}	$\alpha_{\overline{\text{MS}}}$	α_V	α_{lat}	$\alpha_{\overline{\text{MS}}}$	α_V	
5.7	0.10160	0.29790	0.40100	0.15429	0.33181	0.35347	0.37343
6	0.09652	0.23170	0.28560	0.14407	0.25225	0.26150	0.26558
6.2	0.09341	0.20840	0.24990	0.13794	0.22497	0.23146	0.23317
9	0.06435	0.09749	0.10490	0.08548	0.10112	0.10167	0.10173
12	0.04826	0.06399	0.06703	0.06015	0.06556	0.06570	0.06574
18	0.03217	0.03826	0.03930	0.03746	0.03882	0.03885	0.03885

TABLE VII. χ_{nn}—diagonal Creutz ratios χ_{nn} as computed in second-order perturbation theory and by Monte Carlo simulation at $\beta = 6.2$. Statistical errors in the Monte Carlo results are negligible.

n	Perturbation theory			MC data
	α_{lat}	$\alpha_{\overline{\text{MS}}}(q^*)$	$\alpha_V(q^*)$	
2	0.13794	0.22497	0.23146	0.23317
3	0.05207	0.09467	0.10312	0.11348
4	0.02525	0.06283	0.07121	0.06793
5	0.01512	0.03881	0.04780	0.04949

TABLE VIII. Perturbative predictions for Creutz ratio χ_{22} using expansion parameter $\alpha_V(q)$ with various q's at $\beta = 6.2$. Monte Carlo simulation gives $\chi_{22} = 0.233$.

$a\,q$	0.5	0.8	$a\,q^* = 1.09$	1.5	2.0	3.0
χ_{22}	0.174	0.227	0.231	0.228	0.222	0.212

TABLE IX. Perturbative predictions for $-\ln W_{22}$ using expansion parameter $\alpha_V(q)$ with various q's at $\beta = 6.2$. Monte Carlo simulation gives $W_{22} = 1.527$.

$a\,q$	1	2	$a\,q^* = 2.65$	3	4	6
$-\ln W_{22}$	1.074	1.446	1.485	1.492	1.494	1.474

TABLE X. $\alpha_V(\pi/a)$ as measured (from $-\ln W_{11}$), as computed from the bare lattice coupling using nonperturbative mean-field theory, and as computed in perturbation theory. Statistical errors in the Monte Carlo results are negligible.

β	Measured	Mean field	Perturbative
5.7	0.188	0.168	0.148
6	0.156	0.145	0.134
6.2	0.143	0.135	0.127
9	0.074	0.073	0.072
12	0.050	0.050	0.049
18	0.031	0.030	0.030

TABLE XI. Scale invariant ratios for the $1P$-$1S$ mass differences ΔM for ψ's and Υ's and the square root of the string tension σ.

β	$\Delta M_\psi/\Lambda_{\text{lat}}$	$\Delta M_\Upsilon/\Lambda_{\text{lat}}$	$\sqrt{\sigma}/\Lambda_{\text{lat}}$	$\Delta M_\psi/\Lambda_V$	$\Delta M_\Upsilon/\Lambda_V$	$\sqrt{\sigma}/\Lambda_V$
5.7	130(9)	136(15)	134(1)	1.36(9)	1.42(15)	1.40(1)
5.8			120(1)			1.40(1)
5.9	105(6)		110(1)	1.31(7)		1.37(1)
6.0		105(4)	105(1)		1.36(5)	1.38(1)
6.1	96(6)			1.31(7)		

TABLE XII. $\alpha_V(q^*)$ as determined by fitting second-order perturbation theory to Monte Carlo simulation results for logarithms and Creutz ratios of small Wilson loops. Statistical errors in the Monte Carlo results are negligible.

	q^*	Fitted $\alpha_V(q^*)$ $\beta = 5.7$	$\beta = 6.2$	$\beta = 9.0$
$-\ln W_{11}$	3.41	0.183	0.140	0.073
$-\ln W_{12}$	3.07	0.196	0.146	0.074
$-\ln W_{13}$	3.01	0.200	0.147	0.075
$-\ln W_{22}$	2.65	0.222	0.155	0.076
$-\ln W_{23}$	2.56	0.234	0.159	0.077
$-\ln W_{33}$	2.46	0.251	0.163	0.077
$\chi_{2.2}$	1.09	0.353	0.208	0.087
$\chi_{2.3}$	1.10	0.390	0.215	0.087
$\chi_{2.4}$	0.97	0.489	0.235	0.089
$\chi_{3.3}$	1.07	0.449	0.227	0.088
$\chi_{3.4}$	0.80	0.594	0.262	0.092
$\chi_{4.4}$	0.43	1.642	0.370	0.101

[1] We thank Steve Sharpe for providing us with $\text{Tr}U_\mu$ data from the staggered-fermion collaboration.

[2] S. J. Brodsky, G. P. Lepage, and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983).

[3] G. P. Lepage and P. B. Mackenzie, in *Lattice '90*, Proceedings of the International Symposium, Tallahassee, Florida, 1990, edited by U. M. Heller et al. [Nucl. Phys. B (Proc. Suppl.) **20**, 173 (1991)].

[4] G. P. Lepage, in *Lattice '91*, Proceedings of the International Symposium, Tsukuba, Japan, 1991, edited by M. Fukugita et al. [Nucl. Phys. B (Proc. Suppl.) **26**, 45 (1992)].

[5] A. Hasenfratz and P. Hasenfratz, Phys. Lett. **93B**, 165 (1980).

[6] R. Dashen and D. Gross, Phys. Rev. D **23**, 2340 (1981).

[7] A. Billoire, Phys. Lett. **104B**, 472 (1981); Eve Kovacs, Phys. Rev. D **25**, 871 (1981).

[8] U. Heller and F. Karsch, Nucl. Phys. **B251**, 254 (1985).

[9] R. Groot, J. Hoek, and J. Smit, Nucl. Phys. **B237**, 111 (1984), and references therein.

[10] The data for κ_c are from P. Bacilieri et al., Phys. Lett. B **214**, 115 (1988) ($\beta = 5.7$); M. Guagnelli (private communication) ($\beta = 6.0$); C. Bernard (private communication) ($\beta = 6.1, 6.3$).

[11] We use the notation for Creutz ratios in M. Creutz, *Quarks, Gluons, and Lattices* (Cambridge University Press, Cambridge, England, 1983).

[12] G. Hockney, G. P. Lepage, and P. B. Mackenzie (unpublished).

[13] A. Coste et al., Nucl. Phys. **B262**, 67 (1985).

[14] R. Kirschner et al., Nucl. Phys. **B210** [FS6], 567 (1982); see also F. Gutbrod, Z. Phys. C **37**, 143 (1987).

[15] G. Curci and R. Petronzio, Phys. Lett. **132B**, 133 (1983).

[16] G. Parisi, in *High Energy Physics–1980*, Proceedings of the XXth International Conference, Madison, Wisconsin, edited by L. Durand and L. G. Pondrom (American Institute of Physics, New York, 1981); see also F. Green and S. Samuel, Nucl. Phys. **B194**, 107 (1982); S. Samuel, O. Martin, and K. Moriarty, Phys. Lett. **153B**, 87 (1985); Yu. M. Makeenko and M. I. Polikarpov, Nucl. Phys. **B205** [FS5], 386 (1982).

[17] See, for example, B. Sheikholeslami and R. Wohlert, Nucl. Phys. **B259**, 572 (1985).

[18] A. X. El-Khadra et al., Phys. Rev. Lett. **69**, 729 (1992).

[19] G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea, and K. Hornbostel, Phys. Rev. D. **46**, 4052 (1992).

[20] P. B. Mackenzie, in *Lattice '92*, Proceedings of the International Symposium, Amsterdam, The Netherlands, 1992, edited by J. Smit and P. van Baal [Nucl. Phys. B (Proc. Suppl.) **30**, 35 (1993)]; A. S. Kronfeld, *ibid.*, p. 445; A. X. El-Khadra et al., Fermilab Report No. PUB-93/195-T, 1993 (unpublished).

[21] G. Martinelli and Y.-C. Zhang, Phys. Lett. **123B**, 433 (1983).

[22] D. G. Richards, C. T. Sachrajda, and C. J. Scott, Nucl. Phys. **B286**, 683 (1987).

[23] C. Bernard, A. Soni, and T. Draper, Phys. Rev. D **36**, 3224 (1987); G. Martinelli, Phys. Lett. **141B**, 395 (1984). The results of the first reference, adjusted to dimensional reduction form, have been used.

[24] M. Lüscher, R. Sommer, U. Wolff, and P. Weisz, CERN Report No. CERN-TH 6566/92 (unpublished).

[25] B. A. Thacker and G. P. Lepage, Phys. Rev. D **43**, 196 (1991); C. T. H. Davies and B. A. Thacker, in *Lattice '91*, [4], pp. 375 and 378.

[26] K. D. Born et al., in *Lattice '90* [3], p. 394; G. S. Bali and K. Schilling, Phys. Rev. D **46**, 2636 (1992). String tension data at $\beta = 6.1$ were obtained by interpolating between the values at $\beta = 6.0$ and 6.2.

[27] F. Karsch and R. Petronzio, Phys. Lett. **139B**, 403 (1984); J. Fingberg, U. Heller, and F. Karsch, Beilefeld Report No. BI-TP 92-26 (unpublished).