Rare decay $B \to K^*\gamma$: A more precise calculation

Patrick J. O'Donnell
Department of Physics, University of Toronto, 60 St. George Street, Toronto, Canada M5S 1A7

Humphrey K. K. Tung
Institute of Physics, Academia Sinica, Taipei, Taiwan 11529, Republic of China
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Efforts to predict the rare exclusive decay $B \to K^*\gamma$ from the well-known inclusive decay $b \to s\gamma$ are frustrated by the effect of the large recoil momentum. We show how to reduce the large uncertainty in calculating this decay by relating $B \to K^*\gamma$ to the semileptonic process $B \to \rho e\bar{\nu}$ using the heavy-quark symmetry in B decays and SU(3) flavor symmetry. A direct measurement of the q^2 spectrum for the semileptonic decay can provide accurate information for the exclusive rare decay.

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The inclusive rare decay $B \to X_s\gamma$ is now well understood in the context of the standard model [1] and the experimental upper bound [2] of 8.4×10^{-4} for the branching ratio is already playing an important role [3] in constraining the parameters of models other than the standard model. On the other hand, the most likely experimental observation to be made will be the exclusive decay $B \to K^*\gamma$. The recent limit from the CLEO Collaboration [4] of the branching ratio for this mode is 0.92×10^{-4}. It is this exclusive rare decay $B \to K^*\gamma$, however, which is the least well known theoretically due to the large recoil momentum of the K^* meson [5]. A recent paper [6] points out that heavy-quark symmetry together with SU(3) flavor symmetry could relate the rare decay $B \to K^*\gamma$ to a measurement of the semileptonic decay $B \to \rho e\bar{\nu}$. However, the relation is only valid at a single point in the Dalitz plot, a point where the semileptonic decay vanishes, so that there would still be a large uncertainty in such a measurement.

In this paper we obtain a similar relation that relates the exclusive rare decay $B \to K^*\gamma$ to the spectrum in q^2 for the semileptonic decay $B \to \rho e\bar{\nu}$. The q^2 spectrum for $B \to \rho e\bar{\nu}$ does not vanish at $q^2 = 0$ and so a direct measurement of the spectrum at this point can provide accurate information for $B \to K^*\gamma$. Of necessity this new result requires an extension of the heavy-quark symmetries to a consideration of the K^* and ρ systems. We show that this is not the same as demanding K^* or ρ to be a heavy-quark system in the conventional sense. Our result dramatically reduces the uncertainty from the earlier calculations.

First we discuss the application of the heavy-quark symmetry. Usually, it is the hadronic systems with a b or a c quark that have these symmetries. Here we derive the relations for matrix elements of either B or B^* with an unspecified vector meson V. We show how to extend the heavy-quark symmetry relations to the case when the meson V is K^* or ρ, and we estimate the possible errors using a set of quark-model calculations, both for nonrelativistic and for relativistic cases. Then we use the results to give a reliable relation between the decay $B \to K^*\gamma$ and the q^2 spectrum for $B \to \rho e\bar{\nu}$.

I. HEAVY-QUARK SYMMETRY RELATIONS

We first recapitulate the derivation of the heavy-quark symmetry relations for B decay. The hadronic matrix elements relevant to the decay $B(b\bar{q}) \to V(Q\bar{q})$ are given by

$$\langle V(k,\epsilon)|\bar{Q}\gamma_\mu b|B(p_B)\rangle = 2T_1(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}p_B^\lambda k^\sigma \ , \tag{1}$$

$$\langle V(k,\epsilon)|\bar{Q}\gamma_\mu\gamma_5 b|B(p_B)\rangle = -2(m_B^2-m_V^2)T_2(q^2)\epsilon_\mu^* - 2T_3(q^2)(\epsilon^*\cdot q)(p_B+k)_\mu - 2T_4(q^2)(\epsilon^*\cdot q)(p_B-k)_\mu \ , \tag{2}$$

$$\begin{aligned}\langle V(k,\epsilon)|\bar{Q}i\sigma_{\mu\nu}q^\nu b_R|B(p_B)\rangle = {} & f_1(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}p_B^\lambda k^\sigma + \left[(m_B^2-m_V^2)\epsilon_\mu^* - (\epsilon^*\cdot q)(p_B+k)_\mu\right]f_2(q^2) \\ & +(\epsilon^*\cdot q)\left[(p_B-k)_\mu - \frac{q^2}{m_B^2-m_V^2}(p_B+k)_\mu\right]f_3(q^2) \ ,\end{aligned} \tag{3}$$

where $q = p_B - k$. We show below that the hadronic form factors $f_{1,2,3}(q^2)$ and $T_{1,2,3,4}(q^2)$ for the decay $B \to V$ can all be related using just the spin symmetry and static limit of the heavy b quark.

In the heavy b limit, the spin of the b quark is decoupled from all other light fields in B [7]. We can therefore construct the spin operator S_b^Z for the b quark such that

$$S_b^Z|B(b\bar{q})\rangle = \frac{1}{2}|B_l^*(b\bar{q})\rangle \ ,$$

$$S_b^Z|B_l^*(b\bar{q})\rangle = \frac{1}{2}|B(b\bar{q})\rangle \ ,$$

where B_l^* stands for a longitudinal vector B^* meson. In $|B\rangle$ and $|B_l^*\rangle$, the spatial momentum of the b quark is in the z direction for the b spinor to be an eigenstate of S_b^Z. Using the relation $\langle V|\bar{Q}\Gamma b|B\rangle = -2\langle V|[S_b^Z,\bar{Q}\Gamma b]|B_l^*\rangle$ for Γ any product of γ matrices, we have the following identities between the $B\to V$ and $B_l^*\to V$ matrix elements:

$$\langle V|A_0|B\rangle = -\langle V|V_3|B_l^*\rangle \ , \tag{4}$$

$$\langle V|A_3|B\rangle = -\langle V|V_0|B_l^*\rangle \ , \tag{5}$$

$$\langle V|V_\pm|B\rangle = \mp\langle V|V_\pm|B_l^*\rangle \ , \tag{6}$$

$$\langle V|V_0|B\rangle = -\langle V|A_3|B_l^*\rangle \ , \tag{7}$$

$$\langle V|V_3|B\rangle = -\langle V|A_0|B_l^*\rangle \ , \tag{8}$$

$$\langle V|A_\pm|B\rangle = \mp\langle V|A_\pm|B_l^*\rangle \ , \tag{9}$$

where $V_\mu = \bar{Q}\gamma_\mu b$ and $A_\mu = \bar{Q}\gamma_\mu\gamma_5 b$. The covariant expansions of the vector and axial-vector matrix elements for the decay $B^*(b\bar{q})\to V(Q\bar{q})$ are defined to be

$$\langle V(k,\epsilon)|\bar{Q}\gamma_\mu b|B^*(p_B,\zeta)\rangle = \left[\,(\zeta\cdot\epsilon^*)\mathcal{A}_1(q^2) + (\zeta\cdot q)(\epsilon^*\cdot q)\mathcal{A}_2(q^2)\,\right](p_B+k)_\mu$$
$$+\left[\,(\zeta\cdot\epsilon^*)\mathcal{B}_1(q^2) + (\zeta\cdot q)(\epsilon^*\cdot q)\mathcal{B}_2(q^2)\,\right](p_B-k)_\mu + \mathcal{C}(q^2)(\epsilon^*\cdot q)\zeta_\mu + \mathcal{D}(q^2)(\zeta\cdot q)\epsilon^*_\mu \ , \tag{10}$$

$$\langle V(k,\epsilon)|\bar{Q}\gamma_\mu\gamma_5 b|B^*(p_B,\zeta)\rangle = \mathcal{E}(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}\zeta^\lambda(p_B+k)^\sigma + \mathcal{F}(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}\zeta^\lambda(p_B-k)^\sigma, \tag{11}$$

where ζ and ϵ^* are the polarization vectors of B^* and V, respectively. Using the matrix identities in Eqs. (4)–(9), we can relate $T_{1,2,3,4}$ to the $B^*\to V$ form factors:

$$\begin{aligned}
2m_BT_1 &= (\mathcal{A}_1-\mathcal{B}_1) \ ,\\
2(m_B^2-m_V^2)T_2 &= m_B(\mathcal{A}_1+\mathcal{B}_1) + E_V(\mathcal{A}_1-\mathcal{B}_1) \ ,\\
2m_B(T_3-T_4) &= -(\mathcal{A}_1-\mathcal{B}_1) \ ,\\
2m_B(T_3+T_4) &= -(\mathcal{A}_1+\mathcal{B}_1) - \mathcal{C} \ ,\\
\mathcal{D} &= (\mathcal{A}_1-\mathcal{B}_1) \ ,\\
\mathcal{E} &= \mathcal{A}_1 \ ,\\
\mathcal{F} &= \mathcal{B}_1 \ ,\\
\mathcal{A}_2 &= \mathcal{B}_2 = 0 \ .
\end{aligned} \tag{12}$$

Since the spatial momentum of the b quark is defined in the z direction, the above relations are worked out in the B rest frame. We choose the longitudinal polarization vector for B_l^* to be $\zeta_l^\mu = (0;0,0,1)$ and define the momentum of V to be $k^\mu = (E_V;k^1,k^2,k^3)$, where $E_V = (m_B^2+m_V^2-q^2)/(2m_B)$. The resulting form-factor relations in Eq. (12) are consistent with those in Ref. [7] using the spin symmetry of a heavy Q, except for the relation $T_3+T_4=0$, which is missing here.

We can relate the form factors $f_{1,2,3}$ to the form factors $T_{1,2,3,4}$ using the static limit of the b quark. In the B rest frame, the static b-quark spinor satisfies the equation of motion $\gamma_0 b = b$. We then have the relations between the γ_μ and $\sigma_{\mu\nu}$ matrix elements [8]:

$$\langle V|\bar{Q}\gamma_i b|B\rangle = \langle V|\bar{Q}i\sigma_{0i}b|B\rangle \ , \tag{13}$$

$$\langle V|\bar{Q}\gamma_i\gamma_5 b|B\rangle = -\langle V|\bar{Q}i\sigma_{0i}\gamma_5 b|B\rangle \ . \tag{14}$$

This gives the form-factor relations

$$\begin{aligned}
f_1 &= -(m_B-E_V)T_1 - \frac{m_B^2-m_V^2}{m_B}T_2 \ ,\\
f_2 &= -\frac{1}{2}\left[(m_B-E_V) - (m_B+E_V)\frac{q^2}{m_B^2-m_V^2}\right]T_1 - \frac{1}{2m_B}\left(m_B^2-m_V^2+q^2\right)T_2 \ ,\\
f_3 &= -\frac{1}{2}(m_B+E_V)T_1 + \frac{1}{2m_B}(m_B^2-m_V^2)(T_1+T_2+T_3-T_4) \ .
\end{aligned} \tag{15}$$

Using the spin-symmetry relations in Eq. (12), we can also write $f_{1,2,3}$ in terms of the $B^*\to V$ form factors as

$$\begin{aligned}
f_1 &= -\mathcal{A}_1 \ ,\\
f_2 &= -\frac{1}{2}\mathcal{A}_1 - \frac{1}{2}\left(\frac{q^2}{m_B^2-m_V^2}\right)\mathcal{B}_1 \ ,\\
f_3 &= \frac{1}{2}\mathcal{B}_1 \ .
\end{aligned} \tag{16}$$

Thus, using only the spin symmetry and static limit of the heavy b quark, we can relate the $B\to V$ hadronic form factors as

$$
\begin{aligned}
2(m_B^2 - m_V^2)T_2 &= [\,(m_B + m_V)^2 - q^2\,]\,T_1 + \alpha \ , \\
2T_3 &= -T_1 + \frac{\beta}{2m_B} \ , \\
2T_4 &= T_1 + \frac{\beta}{2m_B} \ , \\
f_1 &= -(m_B + m_V)T_1 - \frac{\alpha}{2m_B} \ , \\
2f_2 &= -\,[\,(m_B + m_V)^2 - q^2\,]\,\frac{T_1}{m_B + m_V} - \frac{\alpha}{2m_B}\left(1 + \frac{q^2}{m_B^2 - m_V^2}\right) \ , \\
2f_3 &= -(m_B - m_V)T_1 + \frac{\alpha}{2m_B} \ ,
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
\alpha &= m_B(\mathcal{A}_1 + \mathcal{B}_1) - m_V(\mathcal{A}_1 - \mathcal{B}_1) \ , \\
\beta &= -\mathcal{C} - (\mathcal{A}_1 + \mathcal{B}_1) \ .
\end{aligned}
$$

If we ignore the α and β terms in Eq. (17), the resulting form-factor relations are exactly the heavy-quark symmetry relations obtained using the large-mass limits of both b and Q quarks [5, 7, 8]. The function $-\sqrt{4m_Bm_V}T_1$ in this limit resembles the role of the Isgur-Wise function, with absolute normalization at $q^2 = t_m$ given in the quark model as [5]

$$
-\sqrt{4m_Bm_V}T_1(t_m) = \left(\frac{2\beta_B\beta_V}{\beta_B^2 + \beta_V^2}\right)^{3/2} \approx 1 \ ,
$$

where β_B and β_V are variational parameters of the momentum wave functions for B and V, respectively. We shall show below that the α and β terms in Eq. (17) can be regarded as small corrections to the heavy-quark relations coming from the weak binding, or $\Lambda_{\rm QCD}$ effects. Thus, the symmetry relations are dominated by the T_1 terms.

To show that the right-hand side (RHS) of Eq. (17) is dominated by the T_1 terms, we use the equation of motion for the b quark to get the matrix relation $\langle V|\bar{Q}\not{p}_B b|B^*\rangle = m_B\langle V|\bar{Q}b|B^*\rangle$ and work in the V rest frame. If we assume the static limit of Q so that the equation of motion $\bar{Q} = \bar{Q}\gamma_0$ is satisfied we can relate the matrix elements for $\bar{Q}b$ and $\bar{Q}\gamma_0 b$ as

$$
\langle V|\bar{Q}\not{p}_B b|B^*\rangle \approx m_B\langle V|\bar{Q}\gamma_0 b|B^*\rangle \ . \tag{18}
$$

The additional form-factor relations that follow from Eq. (18) are given by

$$
-\mathcal{C} = (\mathcal{A}_1 + \mathcal{B}_1) = \frac{m_V}{m_B}(\mathcal{A}_1 - \mathcal{B}_1) \ , \tag{19}
$$

corresponding to $\alpha = \beta = 0$ in Eq. (17). The correction to the static Q assumption is proportional to $\mathbf{p}/m_Q$, where $\mathbf{p}$ is the spatial momentum of Q in the V rest frame. Since

$$
m_V = \sqrt{\mathbf{p}^2 + m_Q^2} + \sqrt{\mathbf{p}^2 + m_q^2} + \text{binding energy} \ ,
$$

and $m_V \approx m_Q + m_q$ for a heavy enough vector meson, in the weak binding limit of V it is easy to show that $\mathbf{p}/m_Q \ll 1$. The corrections to Eq. (19) arising from the static Q assumption are therefore small, and consequently we have the suppression of $\alpha, \beta = O(\mathbf{p}/m_Q)$ in this limit.

We can use the quark model to show explicitly the suppression of α and β in the weak binding limit. In the quark-model calculations of α, β, and T_1, we have, in the B rest frame,

$$
m_B\beta = -\alpha = \sqrt{4m_BE_V}\left(h_1 - \frac{h_2}{E_V - m_V}\right) \ , \tag{20}
$$

$$
T_1 = -\sqrt{\frac{E_V}{m_B}}\frac{h_2}{(E_V + m_V)(E_V - m_V)} \ , \tag{21}
$$

where

$$
\begin{aligned}
h_1 &= \int d\mathbf{p}\,\phi_V\phi_B\sqrt{\frac{E_Q + m_Q}{2E_Q}}\sqrt{\frac{E_b + m_b}{2E_b}}\left(1 + \frac{(\mathbf{k} + \mathbf{p})\cdot\mathbf{p}}{(E_Q + m_Q)(E_b + m_b)}\right) \ , \\
h_2 &= \int d\mathbf{p}\,\phi_V\phi_B\sqrt{\frac{E_Q + m_Q}{2E_Q}}\sqrt{\frac{E_b + m_b}{2E_b}}\left(\frac{\mathbf{k}\cdot\mathbf{p}}{E_b + m_b} + \frac{\mathbf{k}\cdot(\mathbf{k} + \mathbf{p})}{E_Q + m_Q}\right) \ .
\end{aligned}
\tag{22}
$$

The term $\mathbf{k}$ is the recoil momentum of V in the B rest frame. The energies of the b and Q quarks in Eq. (22) are given by $E_b = \sqrt{\mathbf{p}^2 + m_b^2}$ and $E_Q = \sqrt{(\mathbf{k}+\mathbf{p})^2 + m_Q^2}$. The terms $\phi_B(\mathbf{p})$ and $\phi_V(\mathbf{p} + r\mathbf{k})$, where $r = m_q/(m_Q + m_q)$, are the momentum wave functions of B and V, respectively. In the weak binding limit of V and with $m_V \approx m_Q + m_q$, we have [5]

$$\frac{\mathbf{k}\cdot\mathbf{p}}{E_b + m_b} + \frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{p})}{E_Q + m_Q} \approx (E_V - m_V)\left(1 + \frac{(\mathbf{k}+\mathbf{p})\cdot\mathbf{p}}{(E_Q + m_Q)(E_b + m_b)}\right) . \quad (23)$$

This gives in Eq. (22) the relation

$$(E_V - m_V)h_1 \approx h_2 \ , \quad (24)$$

which is insensitive to the problem of how the overlap occurs between ϕ_B and ϕ_V. The suppression of α and β in this limit is then obvious from Eq. (20).

In the quark model, the relation between $-\alpha$ (and $m_B\beta$) and T_1 can also be written

$$m_B\beta = -\alpha = [\,(m_B + m_V)^2 - q^2\,]\, T_1\, \varepsilon \ , \quad (25)$$

where $\varepsilon = 1 - (E_V - m_V)h_1/h_2$. The correction to Eq. (23) is given by

$$\delta = [\,(m_b - m_Q) - (E_b - E_Q) - (E_V - m_V)\,]\left(1 + \frac{(\mathbf{k}+\mathbf{p})\cdot\mathbf{p}}{(E_Q + m_Q)(E_b + m_b)}\right) - 2(m_b - m_Q)\frac{(\mathbf{k}+\mathbf{p})\cdot\mathbf{p}}{(E_Q + m_Q)(E_b + m_b)} \ , \quad (26)$$

which corresponds to the binding effects in V. From Eq. (26), it is then easy to show that $\varepsilon \approx 0$ in the weak binding limit of V and with $m_V \approx m_Q + m_q$.

Using the Gaussian wave functions for ϕ_B and ϕ_V [5,9], we obtain throughout the whole kinematic region a rather stable ratio for $(E_V - m_V)h_1/h_2$, which is between 0.95 and 1.04 for $B \to K^*$, and between 0.89 and 0.98 for $B \to \rho$. These two ranges include the large uncertainty in the quark model associated with different recoil dependencies in the wave function overlap [5]. In all cases the value for ε is very small with $|\varepsilon| < 0.05$ for $B \to K^*$ and $|\varepsilon| < 0.11$ for $B \to \rho$ throughout the full kinematic range.

It is then clear from Eq. (25) that the symmetry relations in Eq. (17) are dominated by the T_1 terms as the α and β terms are suppressed by ε. [Near $q^2 = t_m$ some relations are further suppressed by $\varepsilon(m_V/m_B)$.] Thus, the $B \to V$ hadronic form factors satisfy the heavy-quark symmetry relations even if the quark Q is much lighter than the b quark. The breakdown of the relations is a measure of the weak binding approximation and is a small correction.

Since the effect is small and stable across the full kinematic range we can use the model to investigate, with some confidence, the $1/m_Q$ behavior of ε. This behavior can most easily be checked near the zero recoil part, in which the recoil effect becomes insignificant. In Fig. 1, we show the m_Q dependence of $\varepsilon(t_m)$ with $m_V - m_Q$ fixed at $m_\rho - m_u$. For m_Q greater than about 0.9 GeV, ε falls off like a power law of $1/m_Q$. As shown in the figure, the curve can be approximated, above 1.25 GeV, by a Taylor expansion of $\varepsilon(t_m)$ with respect to $\langle p^2\rangle/m_Q^2$ using Eq. (26). The leading terms in the expansion are given by

$$\varepsilon(t_m) \approx \left(\frac{m_V - m_Q}{m_Q}\right) - \frac{1}{3}\left(\frac{m_q}{m_Q + m_q}\right)\frac{\langle p^2\rangle}{\beta_V^2}\frac{m_V}{m_Q}\left(1 + \frac{m_Q}{m_b}\right) - \frac{3}{8}\frac{\langle p^2\rangle}{m_Q^2}\frac{m_V}{m_Q}\left(1 + \frac{7}{9}\frac{m_Q}{m_b}\right) \ , \quad (27)$$

where

$$\langle p^2\rangle = \frac{\int d\mathbf{p}\phi_V(\mathbf{p})\phi_B(\mathbf{p})\,p^2}{\int d\mathbf{p}\phi_V(\mathbf{p})\phi_B(\mathbf{p})} = \frac{3\,\beta_B^2\beta_V^2}{\beta_B^2 + \beta_V^2} \ .$$

The expansion parameter $\langle p^2\rangle$ is a stable function of m_Q with $\sqrt{\langle p^2\rangle} = 428\,\mathrm{MeV}$ for m_u and $\sqrt{\langle p^2\rangle} = 502\,\mathrm{MeV}$ for m_b. At about $m_Q = 0.9\,\mathrm{GeV}$, ε has a maximum and ceases to follow the $1/m_Q$ power law. It is this turnover that stops the correction ε from becoming very large for smaller m_Q and keeps the correction to the symmetry relation in Eq. (17) small for s and u quarks.

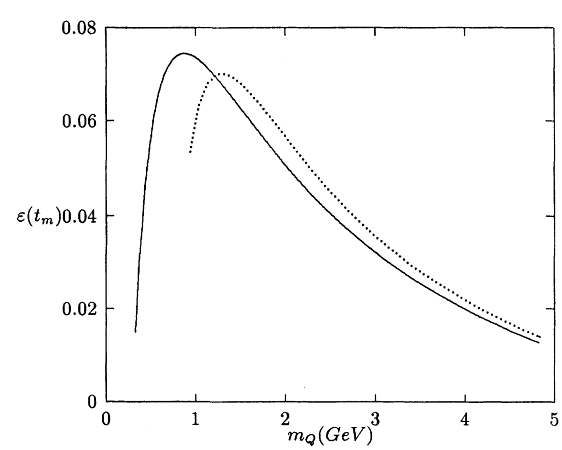

FIG. 1. The m_Q dependence of $\varepsilon(t_m)$ with $m_V - m_Q$ fixed at $m_\rho - m_u$. The solid line is the exact numerical result of $\varepsilon(t_m)$ using Eq. (26). The dotted line is a Taylor expansion of $\varepsilon(t_m)$ with respect to $\langle p^2\rangle/m_Q^2$.

II. THE DECAYS $B \to K^*\gamma$ AND $B \to \rho e\bar{\nu}$

The branching ratio for the exclusive $B \to K^*\gamma$ to the inclusive $b \to s\gamma$ processes can be written in terms of f_1 and f_2 at $q^2 = 0$, as [10, 11]

$$R(B \to K^*\gamma) = \frac{\Gamma(B \to K^*\gamma)}{\Gamma(b \to s\gamma)} \cong \frac{m_b^3(m_B^2 - m_{K^*}^2)^3}{m_B^3(m_b^2 - m_s^2)^3}\frac{1}{2}\left[\,|f_1(0)|^2 + 4|f_2(0)|^2\,\right] \ . \tag{28}$$

Using Eq. (17), we can write $f_2(0) = (1/2)f_1(0)$ at $q^2 = 0$. Although there is now only one form factor to calculate in Eq. (28), this is still a controversial model-dependent calculation. There is an uncertainty of about a factor of 10 depending on the way the large recoil of K^* is handled. In the nonrelativistic quark model, the exclusive to inclusive ratio $R(B \to K^*\gamma)$ is calculated to be within the range of 4.5% [10] to 25% [5, 11]. In the relativistic quark model [12] of Bauer, Stech, and Wirbel, we have instead $R = 12\%$. In the QCD sum-rule calculations, even higher values of $R = 28 \pm 11\,\%$ [13] and $R = 40\%$ [14] were obtained.

In an attempt to remove this uncertainty, Burdman and Donoghue [6] have discussed a method of relating $B \to K^*\gamma$ to the semileptonic process $B \to \rho e\bar{\nu}$ using the static b-quark limit and SU(3) flavor symmetry. Their main result is that the ratio

$$\Gamma(B \to K^*\gamma)\left(\lim_{q^2 \to 0,\mathrm{curve}} \frac{1}{q^2}\frac{d\Gamma(B \to \rho e\bar{\nu})}{dE_\rho dE_e}\right)^{-1} = \frac{4\pi^2}{G_F^2}\frac{|\eta|^2}{|V_{ub}|^2}\frac{(m_B^2 - m_{K^*}^2)^3}{m_B^4} \ , \tag{29}$$

is independent of hadronic form factors. Here, η represents the QCD corrections [1] to the decay $b \to s\gamma$, and the word "curve" denotes the region in the Dalitz plot for which $q^2 = 4E_e(m_B - E_\rho - E_e)$. The only uncertainty on the right-hand side is that of $|V_{ub}|$ for which [15] $|V_{ub}|/|V_{cb}| = 0.10 \pm 0.03$.

Their method proposes to overcome the uncertainty in the calculation at large recoil ($q^2 = 0$) of the $B \to K^*$ form factors by making a direct measurement of the semileptonic decay $B \to \rho e\bar{\nu}$. Notice that we use only the $q^2 = 0$ point on the "curve" to compare with the photonic decay in Eq. (29). The problem with this is that the semileptonic decay vanishes at the $q^2 = 0$ point on the "curve," which is why this kinematic factor is divided out in Eq. (29). This means that experimentally there should be no events at that point and very few in the neighborhood, making it a very difficult measurement.

We shall avoid this by considering instead the q^2 spectrum for the semileptonic decay $B \to \rho e\bar{\nu}$. The advantage here is that the q^2 spectrum does not vanish at $q^2 = 0$ since we integrate over the events from different electron energies across the Dalitz plot. The disadvantage is that in taking the ratio we do not have the simple cancellation of form factors, which made the previous relationship so appealing. However, we can relate the ratio to the knowledge of ε, which we have demonstrated to be a small number anywhere in the Dalitz plot.

The differential width for $B \to \rho e\bar{\nu}$ is given by

$$\frac{d\Gamma(B \to \rho e\bar{\nu})}{dq^2} = \frac{G_F^2}{12\pi^3}|V_{ub}|^2|\mathbf{k}|^3\Lambda_T \ , \tag{30}$$

where

$$\Lambda_T = T_1^2 q^2 + T_2^2\frac{(m_B^2 - m_\rho^2)^2}{2m_\rho^2}\left(1 + \frac{3q^2 m_\rho^2}{m_B^2|\mathbf{k}|^2}\right) + T_3^2\frac{2m_B^2|\mathbf{k}|^2}{m_\rho^2} + T_2T_3\frac{m_B^2 - m_\rho^2}{m_\rho^2}(m_B^2 - m_\rho^2 - q^2) \ .$$

At $q^2 = 0$, the differential width for $B \to \rho e\bar{\nu}$ reduces to

$$\left.\frac{d\Gamma(B \to \rho e\bar{\nu})}{dq^2}\right|_{q^2=0} = \frac{G_F^2}{192\pi^3}|V_{ub}|^2\frac{(m_B^2 - m_\rho^2)^5}{m_B^3 m_\rho^2}|T_2(0) + T_3(0)|^2 \ . \tag{31}$$

If we use the symmetry relations in Eq. (17) and SU(3) flavor symmetry in which $T_1^{B\to K^*} = T_1^{B\to\rho}$, we can express the ratio between $R(B \to K^*\gamma)$ and $d\Gamma(B \to \rho e\bar{\nu})/dq^2$ at $q^2 = 0$ as

$$\begin{aligned} R(B \to K^*\gamma)\left(\left.\frac{d\Gamma(B \to \rho e\bar{\nu})}{dq^2}\right|_{q^2=0}\right)^{-1} &= \frac{192\pi^3}{G_F^2}\frac{1}{|V_{ub}|^2}\frac{(m_B^2 - m_{K^*}^2)^5}{(m_B^2 - m_\rho^2)^5}\frac{(m_B - m_\rho)^2}{(m_B - m_{K^*})^2}\frac{m_b^3}{(m_b^2 - m_s^2)^3}|\mathcal{I}|^2 \\ &= 1.9 \times 10^{16}\,\mathrm{GeV}\left(\frac{0.1}{|V_{ub}/V_{cb}|}\right)^2 |\mathcal{I}|^2 \ . \end{aligned} \tag{32}$$

In the limit $\varepsilon \approx 0$, that is $\alpha \approx 0$ and $\beta \approx 0$ in Eq. (17), we have $\mathcal{I} = 1$ in Eq. (32). We estimate the correction to $\mathcal{I}$ using the quark-model results for α and β in Eq. (25) as

$$\mathcal{I} = \left(1 - \frac{m_B + m_{K^*}}{2m_B}\varepsilon^{B\to K^*}(0)\right)\Big/\left(1 - \frac{(m_B + m_\rho)(m_B^2 + m_\rho^2)}{4m_B^2 m_\rho}\varepsilon^{B\to\rho}(0)\right) \ . \tag{33}$$

Although ε is a small number, the value of $\varepsilon(0)$ is sensitive to the details of the wave-function overlap in the quark model because of the large recoil momentum. In an extreme case with a relativistic recoil dependence in the overlap, we obtain $\varepsilon^{B\to K^*}(0) = 0.010$ and $\varepsilon^{B\to\rho}(0) = 0.043$ which gives $\mathcal{I} = 1.09$. On the other hand, using a milder nonrelativistic recoil dependence, we obtain $\varepsilon^{B\to K^*}(0) = 0.046$ and $\varepsilon^{B\to\rho}(0) = 0.087$ which gives $\mathcal{I} = 1.18$.

We also estimate the correction to $\mathcal{I}$ using the relativistic quark model of Bauer, Stech, and Wirbel (BSW) [12]. In this model, we have, at $q^2 = 0$,

$$T_1(0) = -\frac{m_b - m_Q}{m_B^2 - m_V^2}\, g_2 \ , \tag{34}$$

$$\alpha(0) = \left[(m_b - m_Q)\left(\frac{m_B + m_V}{m_B - m_V}\right) - (m_b + m_Q)\right] g_2 \ , \tag{35}$$

$$\beta(0) = \frac{4m_B}{m_B^2 - m_V^2}\,(m_Q g_2 - m_V g_1) \ , \tag{36}$$

where g_1 and g_2 are overlap integrals given by

$$g_1 = \int d\mathbf{p}_T \int_0^1 dx\, \varphi_V^*(\mathbf{p}_T, x)\varphi_B(\mathbf{p}_T, x) \ , \tag{37}$$

$$g_2 = \int d\mathbf{p}_T \int_0^1 \frac{dx}{x}\, \varphi_V^*(\mathbf{p}_T, x)\varphi_B(\mathbf{p}_T, x) \ . \tag{38}$$

In the BSW model, the orbital wave functions φ_B and φ_V are solutions to a relativistic scalar harmonic oscillator potential. For meson M with constituent quarks $Q\bar{q}$, the wave function φ_M is given by

$$\varphi_M(\mathbf{p}_T, x) = N_M\sqrt{x(1-x)}\exp\left(-\frac{\mathbf{p}_T^2}{2\omega^2}\right)\exp\left(-\frac{M^2}{2\omega^2}\left[x - \frac{1}{2} - \frac{m_Q^2 - m_q^2}{2M^2}\right]^2\right) \ , \tag{39}$$

where x and $\mathbf{p}_T$ denote the longitudinal momentum fraction and transverse momentum of the decaying quark Q in the infinite momentum frame. The term N_M is a normalization factor defined by $\int d\mathbf{p}_T \int_0^1 dx |\varphi(\mathbf{p}_T, x)|^2 = 1$. The wave function φ_M depends only on one free parameter ω, which determines the average transverse quark momentum $\langle \mathbf{p}_T^2 \rangle = \omega^2$. In the BSW model, the value of ω is taken to be $\omega = 0.4$ GeV as it fits most of the experimental data very well.

If we follow Eq. (25) and define the parameters ε and $\bar{\varepsilon}$ as $-\alpha/T_1 = (m_B + m_V)^2\varepsilon$ and $m_B\beta/T_1 = (m_B + m_V)^2\bar{\varepsilon}$ at $q^2 = 0$, we have

$$\varepsilon(0) = 1 - \left(\frac{m_B - m_V}{m_B + m_V}\right)\left(\frac{m_b + m_Q}{m_b - m_Q}\right) \ , \tag{40}$$

$$\bar{\varepsilon}(0) = \frac{1}{(m_B + m_V)^2}\frac{4m_B^2}{m_b - m_Q}\left(m_V\frac{g_1}{g_2} - m_Q\right) \ . \tag{41}$$

Notice that the expression for $\varepsilon(0)$ is independent of the overlapping effects in Eqs. (37) and (38). Numerically, we get $\varepsilon^{B\to K^*}(0) = 0.11$ and $\varepsilon^{B\to\rho}(0) = 0.15$. In Eq. (41), $\bar{\varepsilon}(0)$ depends on the ratio g_1/g_2 of the overlap integrals. It can be shown that the ratio g_1/g_2 is very stable with respect to the changes in ω and m_Q. For $\omega = 0.4$ GeV, we obtain $\bar{\varepsilon}^{B\to K^*}(0) = 0.15$ and $\bar{\varepsilon}^{B\to\rho}(0) = 0.22$. Thus, the sizes of ε and $\bar{\varepsilon}$ are relatively larger in the BSW model than in the nonrelativistic quark model. However, the overall correction to the symmetry relations in Eq. (17) is still less than 15%. In Eqs. (40) and (41), it is clear that the smallness of ε and $\bar{\varepsilon}$ is due to the subtraction between different mass terms (see also the case in the nonrelativistic quark model). For light m_Q, ε and $\bar{\varepsilon}$ both scale like $1/m_B$; this gives an extra $1/m_B$ suppression to the α and β terms, relative to the T_1 term, in the symmetry relations.

We can write $\mathcal{I}$ in terms of ε and $\bar{\varepsilon}$ as

$$\mathcal{I} = \frac{1 - \frac{m_B + m_{K^*}}{2m_B}\varepsilon^{B\to K^*}(0)}{1 + \frac{(m_B - m_\rho)(m_B + m_\rho)^2}{4m_B^2 m_\rho}\bar{\varepsilon}^{B\to\rho}(0) - \frac{m_B + m_\rho}{2m_\rho}\varepsilon^{B\to\rho}(0)} \ . \tag{42}$$

While the values of ε and $\bar{\varepsilon}$ are larger in the BSW model, the overall correction to $\mathcal{I}$ is still small with $\mathcal{I} = 1.12$ close to the value obtained by the nonrelativistic quark model. Thus, the uncertainty in calculating the branching ratio $R(B \to K^*\gamma)$ due to the recoil problem has now been reduced by an order of magnitude.

We have derived a relation between the branching ratio $R(B \to K^*\gamma)$ and the q^2 spectrum for $B \to \rho e\bar{\nu}$. Since the q^2 spectrum for $B \to \rho e\bar{\nu}$ does not vanish at $q^2 = 0$, this reduces the uncertainty in the measurement of the semileptonic decay in contrast to the case in Eq. (29). Now ARGUS has given the result [16] of $R(B^- \to \rho^0 l\bar{\nu}) = (11.3 \pm 3.6 \pm 2.7) \times 10^{-4}$, and by isospin symmetry, $\Gamma(\bar{B}^0 \to \rho^+ l\bar{\nu}) = 2\Gamma(B^- \to \rho^0 l\bar{\nu})$. This allows us to estimate $d\Gamma(B \to \rho e\bar{\nu})/dq^2$ at $q^2 = 0$ to be about 10^{-17} GeV^{-1}. Equation (32) then gives $R(B \to K^*\gamma)$ about 10^{-1}, which is quantitatively correct. A direct measurement of $d\Gamma(B \to \rho e\bar{\nu})/dq^2$ at $q^2 = 0$ can therefore provide reliable information for $R(B \to K^*\gamma)$.

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