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Nonperturbative approach to scalar field triviality

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We show how the nonlinear δ expansion correctly predicts when quantum fluctuations will screen the forces between scalar fields.

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For many years quantum field theorists had only the weak-coupling Feynman diagram expansion as a calculational tool. Because of the pointlike nature of the particles described by the quantum fields, individual diagrams in the series are beset by ultraviolet singularities. Simple power-counting arguments in momentum space show that our ability to control these singularities depends on the dimensionality of the coupling strengths. For example, consider the simplest theory of all, that of a single scalar field ϕ with interaction $\lambda(\phi^2)^n$ in d spacetime dimensions. Setting $n=1+\delta$, $\delta \geq 0$, the Lagrangian density is most conveniently written as

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\mu^2\phi^2+\lambda M^2\phi^2(\phi^2 M^{2-d})^{\delta} \ . \qquad (1)$$

In (1), μ is the bare mass, λ is the *dimensionless* coupling constant (in units in which $\hbar=c=1$) and M is a fixed parameter that allows the interaction to have the correct dimensions.

The dimensionful coupling constant is, from (1),

$$\bar{\lambda}=\lambda M^{2-\delta(d-2)} \ . \qquad (2)$$

There are four possibilities, displayed in Fig. 1.

$$\text{(i)}\ 2-\delta(d-2)>0 \ , \qquad (3)$$

for which power counting shows that it is not possible to redefine away ultraviolet singularities to arbitrary order. The theory is *perturbatively nonrenormalizable.*

$$\text{(ii)}\ 2-\delta(d-2)=0 \ , \qquad (4)$$

for which power counting shows that it is possible to eliminate ultraviolet singularities. The theory is *perturbatively renormalizable.*

$$\text{(iii)}\ 2-\delta(d-2)<0 \ , \qquad (5)$$

for which all ultraviolet singularities can again be eliminated. In fact, not all counterterms are formally infinite and the theory is *perturbatively super-renormalizable.*

$$\text{(iv)}\ \delta=0 \ , \qquad (6)$$

corresponding to the trivial case of a *free-field* Langrangian (1).

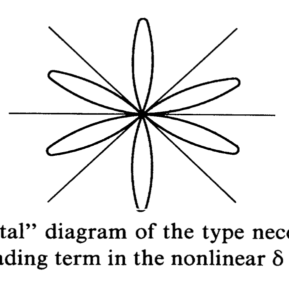

FIG. 1. A "petal" diagram of the type necessary for the calculation of the leading term in the nonlinear δ expansion.

In recent times our understanding of these different regions of parameter space has improved considerably as alternatives to the finite-order weak coupling have been devised. Made plausible by general arguments (e.g., Ref. [1]), the consensus is that scalar field theories satisfying (3) are *trivial*[1] (i.e., the renormalized coupling constant λ_R *vanishes*). That is, the unrenormalized interactions are too singular to implement, and the only way out is quantum suicide, whereby they do not exist on renormalization. For example, we know that, for $d>4$, the theory is trivial for $\delta=1$ and 2 [2,3].

The boundary line (4) is particularly interesting, since it is difficult to decide whether or not a theory on it is trivial. Simple calculations are often misleading. For example, mean-field theory predicts nontriviality when $d=4$ and $\delta=1$ (i.e. four-dimensional $\lambda(\phi^4)_4$ theory). In fact, $\lambda(\phi^4)_4$ is firmly believed to be trivial [4]. However, it seems likely that $\lambda\phi^6$ in $d=3$ dimensions is not trivial [12]. Other theories on the boundary line between renormalizability and nonrenormalizability are more problematical. The $\lambda\phi^3$ theory in $d=6$ dimensions is asymptotically free [5], often used as a trial ground for calculations with QCD in mind [6]. Although asymptotic freedom is usually seen as a sufficient condition for nontriviality, in this case the Hamiltonian is unbounded below, to give an unstable theory. We have avoided the problem by adopting (1) which, for $\delta=\frac{1}{2}$, is a $|\phi|^3$ theory, guaranteeing an acceptable but nonanalytic Hamiltonian.

Theories on the boundary (4) with large δ have been studied by Lipatov [6] who has calculated the β function as $\delta\to\infty$, $d\to 2$ in (1) in what is, essentially, a $1/\delta$ expansion. Simplification occurs because, at *each* order in the β function, only *one* diagram survives the $\delta\to\infty$ limit, in

[1]There are slight complications in that large-δ interactions generate lower-δ interactions (e.g., a $\lambda\phi^6$ theory will generate $g\phi^4$ terms of necessity). Triviality in this case requires that both renormalized couplings vanish.

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which all points are joined by an equal number of lines. (Of course, this requires that we work in fractional dimensions.) He has shown that, in the limit, there is at least one ultraviolet fixed point, $\lambda^* \neq 0$, implying a *nontrivial* theory. The original hope that this leading term was valid down to $\delta=1$ is not sustainable, as we noted earlier (although valid at $\delta=2$ [12]), but it is an ingenious attempt to tackle the highly nonlinear $\delta>1$ theories.

In this paper we adopt the other extreme and address the problem of the triviality of the theory (1) by expanding in δ about the *free theory,* $\delta=0$, for all d. Along the boundary line (4) this corresponds to a large-d limit.

The δ expansion of (1) was originally proposed by Bender *et al.* [7]. It is of no surprise that, as an expansion in $\ln|\phi|$, it is difficult to implement, practically impossible beyond terms $O(\delta^2)$. However, even at $O(\delta)$ it has been shown by Bender and Jones [8] that we can correctly predict the existence, or not, of interacting theories at $\delta=1$. This paper continues that work to the more general case, with particular emphasis on the boundary. We note that the $|\phi|^{2(1+\delta)}$ theory becomes singular for $\delta<-1$ and hence is unreliable for $|\delta|>1$. This is confirmed by simple calculations, but analytic continuation beyond $\delta=1$ is not precluded. Nonetheless, it is the antithesis to the Lipatov regime and can be understood as complementary to it.

We start by extending the work of Ref. [8]. Technically, the easiest quantities to calculate are zero-momentum Green's functions, rather than the β function. We compute the renormalized coupling constant G_R (at zero momentum) in terms of the bare mass μ and the bare coupling constant λ. As always, regularization is necessary. By insisting on a finite fixed value of the renormalized mass m_R, we investigate the behavior of G_R as the regularization is removed. If G_R is forced to vanish the theory is trivial. If not, it is not trivial. (If G_R is infinite we have adopted an incorrect or inappropriate regularization.)

In the general case of $(\lambda\phi^{2n})_d$ theories, the dimensionless renormalized coupling constant G_R is given by

$$G_R \equiv (-1)^{n+1} Z^n G^{(2n)}(0,0,0,\ldots,0) M_R^{-[d+n(2-d)]} , \quad (7)$$

where Z is the wave-function renormalization constant and M_R is the renormalized mass. Detailed calculations of Z, M_R^2, and G_R are straightforward to first order in δ. From the observation that

$$\ln\phi^2 = \partial(\phi^2)^\alpha/\partial\alpha|_{\alpha=0} ,$$

we require only the calculation of "petal" diagrams (see Fig. 1) with all possible numbers of petals and the appropriate number of external legs (prior to differentiation). Explicitly [8],

$$Z = 1 + O(\delta^2) , \quad (8)$$

$$M_R^2 = m^2 + 2\delta\lambda M^2 S + O(\delta^2) , \quad (9)$$

$$G_R^{(2n)} = \frac{(-1)^{n+1}\lambda\delta M^{d+n(2-d)} 2^n (n-2)!}{[M^{2-d}\Delta(0)]^{n-1}} + O(\delta^2) , \quad (10)$$

where $m^2 = \mu^2 + 2\lambda M^2$ and

$$S = 1 + \psi(\tfrac{3}{2}) + \ln[2\Delta(0) M^{2-d}] .$$

We have yet to set $n=\delta+1$. On doing so, to leading order we have

$$G_R = \frac{\lambda 2^{1+\delta}\Gamma(1+\delta)}{[M^{2-d}\Delta(0)]^\delta} , \quad (11)$$

$$M_R^2 = m^2 + 2\delta\lambda M^2 S . \quad (12)$$

In the above, $\Delta(x)$ represents the free propagator in d-dimensional coordinate space and can be expressed as the associated Bessel function

$$\Delta(x) = (2\pi)^{-d/2}(x/m)^{1-d/2} K_{1-d/2}(mx) . \quad (13)$$

We regulate our expressions by introducing a short-distance cutoff a; that is, the "petal" $\Delta(0)$ is replaced by $\Delta(a)$ where

$$\Delta(a) = (2\pi)^{-d/2}(a/m)^{1-d/2} K_{1-d/2}(ma) . \quad (14)$$

The *bare* parameter m is a function of the cutoff a. There are three cases to examine since we do not know *a priori* how ma should behave as a goes to zero.

Case 1: $ma \ll 1$ as $a \to 0$. Here, the Bessel function in (14) has

$$\Delta(a) \sim \frac{1}{4\pi}\Gamma\left[\frac{d}{2}-1\right](\pi a^2)^{1-d/2} \quad (15)$$

as its leading singularity when $\delta>2$.

Case 2: $ma \sim 1$ as $a \to 0$. In this case,

$$\Delta(a) \sim k m^{d-2} \sim k(a^2)^{1-d/2} , \quad (16)$$

as in case 1. When the prefactor is not important these two cases will be taken together.

Case 3: $ma \gg 1$ as $a \to 0$. Here, the asymptotic behavior of the Bessel function is

$$\Delta(a) \sim \frac{1}{2m}\left[\frac{2\pi a}{m}\right]^{(1-d)/2} e^{-ma} . \quad (17)$$

We renormalize by substituting (11) into (12), thereby eliminating the unrenormalized coupling constant λ in favor of the renormalized coupling constant G_R:

$$M_R^2 = m^2 + \frac{S G_R [M^{2-d}\Delta(a)]^\delta M^2}{2^\delta \Gamma(\delta)} . \quad (18)$$

Case 1. For $ma \ll 1$ as $a \to 0$, the substitution of (15) into (18) gives

$$M_R^2 = m^2 + \frac{(-1)^n S G_R M^{(2-d)\delta+2}}{2^\delta\Gamma(\delta)}\left[\frac{1}{4\pi}\Gamma\left[\frac{d}{2}-1\right]\right]^\delta \times (\pi a^2)^{(1-d/2)\delta} . \quad (19)$$

We now insist that the renormalized mass be finite. However, as $a \to 0$, the second term on the right-hand side of (19) becomes infinite for $d>2$. Therefore, both terms on the right-hand side must be infinite and of the same order of magnitude as $a \to 0$ to combine to give a finite result for M_R^2. On multiplying by a^2,

$$S G_R\left[\frac{1}{4\pi}\Gamma\left[\frac{d}{2}-1\right]\right]^\delta (Ma\sqrt{\pi})^{(2-d)\delta+2} \sim -(ma)^2 \ll 1 . \quad (20)$$

It follows that if

$$2-\delta(d-2)\le 0\ , \tag{21}$$

then G_R is driven to zero, as anticipated. Because of the logarithm in S, condition (23) includes the boundary (iii). That is, for small δ the theories on the boundary are free, in contrast with the large-δ results of Lipatov, for which the theories are nontrivial. We note that we cannot infer from (23) that the theory will be nontrivial in region (iii) for $\delta<1$, only that it will be trivial outside it.

Case 2 gives identical results.

Case 3. If we try to tune m as a function of a so that $(ma)\gg 1$ we have, upon substituting (17) into (18),

$$M_R^2=m^2+\frac{SG_RM^{(2-d)\delta+2}}{2^\delta\Gamma(\delta)}\left[\frac{1}{2m}\right]^\delta\left[\frac{2\pi a}{m}\right]^{(1-d)\delta/2}\times e^{-ma\delta}\ . \tag{22}$$

Following our previous line of argument, the second term on the right-hand side must be $\sim m^2$. However, solving for G_R always implies $G_R\to\infty$ as $ma\to\infty$. We must therefore assume that case 3 does not arise.

We now apply these techniques to investigate the triviality or otherwise of $(\lambda\phi^2)^{(1+\delta)}_{d\to\infty}$ scalar theory along the boundary. The limit $d\to\infty$ corresponds to the limit $\delta\to 0$, which, in turn, corresponds to the limit $n\to 1$ in the nonperturbative expansion method we are employing. In terms of $\lambda\phi^{(2n)}$ theory, we reexpress d of (4) as $d=2+2/\delta$. As a result, the renormalized mass can now be written as

$$M_R^2=m^2+\left[\frac{SG_R\delta\Delta(a)^\delta}{\Gamma(1+\delta)}\right] \tag{23}$$

in which

$$[\Delta(a)]^\delta=\{(2\pi)^{1+\delta}(m/a)[K_{-1/\delta}(ma)]^\delta\}\ . \tag{24}$$

As before, we examine, in turn, the three possibilities $ma\ll 1$, $ma\sim 1$, and $ma\gg 1$ as $a\to 0$.

Case 1. $ma\ll 1$, and so

$$K_{-1/\delta}\sim\frac{1}{2}\Gamma\left[\frac{1}{\delta}\right]\left[\frac{ma}{2}\right]^{-1/\delta}\ . \tag{25}$$

For a finite M_R^2 (infinite m^2) this implies

$$G_R\sim\lim_{\delta\to 0}\left[\frac{(ma)^2 2^\delta\Gamma(1+\delta)e}{2S(2\pi)^{\delta+1}}\right]\ . \tag{26}$$

Thus, $G_R\to 0$ as $d\to\infty$ suggesting that the theory remains trivial as its dimensionality becomes infinite along the boundary.

Case 2. $ma\sim 1$ and the free propagator goes like $\Delta(a)\sim km^{2/\delta}$. Since $S\sim(2/\delta)\ln(m/M)$, a finite M_R^2 requires

$$G_R\sim\lim_{\delta\to 0}\left[\frac{\Gamma(1+\delta)k^{-\delta}}{2\ln(m/M)}\right]\ . \tag{27}$$

That is, $G_R\to 0$ as $m\to\infty$, again suggesting a free theory.

Case 3. $ma\gg 1$ again gives nonsense.

In summary, the nonlinear δ expansion is successful (at leading order) at predicting the triviality of scalar theories when $\delta\ge 0$. The only caveat is that, although the constraint cannot be seen at leading order, we should restrict ourselves to $\delta\le 1$. Somehow, the expansion is selecting the relevant quantum fluctuations that can lead to charge shielding. The results are displayed in Fig. 2.

In this it is to be contrasted with the *linear δ expansion* which succeeded it [9]. Despite the terminology, this approach is very different in principle. The idea is to linearize the theory (1), now written as

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\mu^2\phi^2+\lambda M^{2-(n-1)(d-2)}(\phi^2)^n\ . \tag{28}$$

To this end we write

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\Omega^2\phi^2+\tfrac{1}{2}\delta(\mu^2-\Omega^2)\phi^2+\delta\lambda M^{2-(n-1)(d-2)}(\phi^2)^n\ , \tag{29}$$

where Ω^2 is a free parameter, to be fixed by variation. For $\delta=0$ we have a free theory, for $\delta=1$ the theory that we wish to solve.

A calculation of the effective potential $V(\phi)$ to order δ at $\delta=1$ again requires only "petal" diagrams. Specifically, the vacuum contribution to the effective potential (usually ignored, but now a variable) is

$$V_0=\int\frac{d^dp}{(2\pi)^d}\ln[p^2+\Omega^2-\delta(\Omega^2-m^2)]=\int\frac{d^dp}{(2\pi)^d}\ln(p^2+\Omega^2)-\delta\int\frac{d^dp}{(2\pi)^d}\frac{\Omega^2-m^2}{p^2-\Omega^2} \tag{30}$$

on expanding the log term to first order. The effective potential to first order is then given by

$$V_1=V_0+\frac{1}{2}m^2\phi^2+\delta\lambda M^d\sum_{m=1}^{n}\frac{(2n)!(M^{2-d}\phi^2)^m}{(2m)!C_{2m}}\times[M^{2-d}\Delta(0)]^{n-m}\ , \tag{31}$$

where $(n-m)$ is the number of loops (petals) $\Delta(0)$, $2m$ is the number of external legs, and C_{2m} is the appropriate symmetry factor for the diagram with $(n-m)$ loops. In (31) $\Delta(0)$ is now the petal for a scalar field of mass Ω.

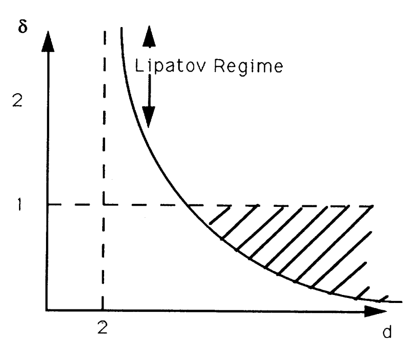

FIG. 2. The curve asymptotic to $d=2$ and $\delta=0$ is the triviality boundary $2-\delta(d-2)=0$. The nonlinear δ expansion (valid for $\delta\le 1$) shows triviality in the shaded region, and the boundary for $\delta\le 1$. Lipatov's calculation shows nontriviality along the boundary for large $\delta(\delta>1)$. The linear δ expansion to leading order disregards the boundary, predicting nontriviality on the boundary of the shaded region at least (e.g., $4<d<6$ for $\delta=1$).

We fix Ω by imposing the condition

$$\frac{\partial V(\phi)}{\partial \Omega^2}=0 \quad (32)$$

(the so-called strong principle of minimum sensitivity [10]). For $\delta=1$ this gives

$$\Omega^2=m^2+\lambda M^2 \sum_{m=1}^{n} \frac{(2n)!(M^{2-d}\phi^2)^m}{(2m)!C_{2m}}(n-m) \times[M^{2-d}\Delta(0)]^{(n-m-1)} . \quad (33)$$

The equation for the renormalized mass M_R is

$$M_R^2=\left.\frac{d^2V(\phi,\Omega^2(\phi))}{d\phi^2}\right|_{\phi=0} \quad (34)$$

subject to the equation of constraint (32) above. Since Ω^2 is itself a function of ϕ, the full derivative is given by

$$\frac{d^2V(\phi,\Omega^2(\phi))}{d\phi^2}=\frac{\partial^2 V}{\partial\phi^2}+2\frac{\partial^2 V}{\partial\Omega^2\partial\phi}\left[\frac{d\Omega^2}{d\phi}\right]+\frac{\partial^2 V}{\partial\Omega^2}\left[\frac{d\Omega^2}{d\phi}\right]^2+\frac{\partial V}{\partial\Omega^2}\left[\frac{d^2\Omega^2}{d\phi^2}\right] . \quad (35)$$

The calculation is greatly simplified by observing that

$$\left.\frac{d\Omega^2}{d\phi}\right|_{\phi=0}=0 , \quad (36)$$

whence

$$M_R^2=\left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=0} . \quad (37)$$

The only nonzero contribution from (33) occurs when $m=1$, giving the result ($C_{2m}=2n$)

$$M_R^2=\mu^2+2n\,(2n)!\lambda[M^{2-d}\Delta(0)]^{n-1}M^2 , \quad (38)$$

where $\Delta(0)$ is now the loop for a scalar field of mass M_R. That is, we have a self-consistent equation for M_R that corresponds to a summation of selective diagrams to all orders in λ, as can be seen by iterating the right-hand side of (38). Equation (38) essentially reproduces the large-N result (at $N=1$) of the O(N) extension of (1), in which $\phi^2\to\phi^2/N$ and $\lambda\to N\lambda$. This looks very much like (18), but for (1) the substitution of μ^2 for m^2, (2) the replacement of the renormalized coupling constant SG_R by the unrenormalized λ, and (3) the evaluation of $\Delta(0)$ as $\Delta(a)$ with mass M_R replacing m.

However, the similarity is very superficial. This becomes more apparent when the unrenormalized λ is replaced by the renormalized coupling G_R, now given by

$$\frac{1}{G_R}=\frac{1}{\lambda}+O(B\Delta(0)^{n-2}) , \quad (39)$$

where $B=-d\Delta(0)/dM_R^2$ is the zero-momentum bubble diagram. For $d<4$ dimensions B is finite; for $d=4$ dimensions B diverges logarithmically for small a, whence λ vanishes logarithmically for $n=2$ and remains finite and nonzero for $n\leq 2$. For $d>4$ dimensions $B=O((M_R a)^{4-d})$.

The constraints on d and n for G_R to be finite and nonzero are much less restrictive than in the nonlinear case. For example, for the canonical $\lambda(\phi)^4$ in $d=4$ dimensions the logarithmic a dependence now occurs in the denominator of the second term of (38), rather than in the numerator. As a consequence there are no intimations that the theory is trivial in this approximation. This is equally true for the quartic theory in $4<d<6$ dimensions, for which, because of its equivalence to the large-N limit, the linear δ-expansion permits finite, *nonzero* G_R [11]. Only for $d\geq 6$ dimensions do we recover the correct triviality of the theory. Admittedly, these results are proven only at leading order in the δ expansion. However, in the $1/N$ expansion we know that nontriviality at leading order implies seeming nontriviality to all orders. As a rearrangement of diagrams, we might expect the same for the δ expansion.

Perhaps we should not be surprised at the inability of the linear δ expansion to predict triviality correctly. By definition triviality is all but invisible in perturbation theory and the linear δ expansion provides too simple a resummation of diagrams for the essential nonlinearity to be exposed, despite the optimization condition.

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[1] J. R. Klauder, Phys. Rev. D **24**, 2599 (1981).
[2] M. Aizenmann, Phys. Rev. Lett. **47**, 1 (1981).
[3] K. Kondo, Report No. DPNU84 25 (unpublished).
[4] J. Frohlich, Nucl. Phys. **B200**, 281 (1982).
[5] J. C. Collins, *Renormalisation* (Cambridge University Press, Cambridge, England, 1983).
[6] L. N. Lipatov, Pis'ma Zh. Eksp. Teor. Fiz. **24**, 179 (1976), [JETP Lett. **24**, 157 (1976)].
[7] C. M. Bender *et al.*, Phys. Rev. D **37**, 1472 (1988).
[8] C. M. Bender and H. F. Jones, Phys. Rev. D **38**, 2526 (1988).
[9] A. Duncan and M. Moshe, Phys. Lett. B **215**, 352 (1988).
[10] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
[11] J. M. Ebbutt and R. J. Rivers, J. Phys. A **15**, 3285 (1982).
[12] K. G. Wilson, Phys. Rev. D **6**, 419 (1972).