# Schrödinger fields on the plane with $[\mathrm{U}(1)]^N$ Chern-Simons interactions and generalized self-dual solitons

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A general nonrelativistic field theory on the plane with couplings to an arbitrary number of Abelian Chern-Simons gauge fields is considered. Elementary excitations of the system are shown to exhibit fractional and mutual statistics. We identify the self-dual systems for which certain classical and quantal aspects of the theory can be studied in a much simplified mathematical setting. Then, specializing to the general self-dual system with two Chern-Simons gauge fields (and nonvanishing mutual statistics parameter), we present a systematic analysis for the static vortexlike classical solutions, with or without a uniform background magnetic field. Relativistic generalizations are also discussed briefly.

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## I. INTRODUCTION

In two-dimensional space, we can have particles obeying fractional statistics [1], and in the field-theoretic context a similar effect is generated by introducing the Chern-Simons (CS) term [2] in the action. The CS field theory was then found to be useful in describing the fractional quantum Hall states [3]. According to some recent suggestions [4], a suitable generalization of it may in fact provide a unified mathematical approach to the long-distance physics of various quantum topological fluids. The construction involves a set of Abelian gauge fields $a_\mu^I$ $(I = 1, 2, \ldots, N)$ with the CS-type term

$$\mathcal{L}_g = \sum_{I,J=1}^{N} \frac{1}{2}\kappa_{IJ}\epsilon^{\mu\nu\lambda} a_\mu^I(x)\partial_\nu a_\lambda^J(x) \tag{1.1}$$

[here $(\kappa_{IJ})$ is a real symmetric matrix] as the sole kinetic-energy density for them. By considering matter fields with generic gauge-invariant couplings to the $a_\mu^I$'s, we obtain a compound or multilayered system which exhibits fractional statistics for the exchange of indistinguishable particles and mutual "statistical" interactions between particles belonging to different species (or layers). Models of a similar nature have been considered also in Ref. [5], and as these authors emphasize, parity need not be broken in a field theory with an even number of CS gauge fields. See Ref. [6] for an application to the quantum Hall effects in the double-layer electron system.

The purpose of this paper is twofold. First, we clarify the precise nature of a nonrelativistic quantum field theory incorporating the above form of CS-type interaction. In particular, we find explicitly the corresponding first-quantized description in the general $n$-body sector. In the latter description, fractional and mutual statistics for the particles are the manifestations of the Aharonov-Bohm effect involving a combination of fictitious charges and localized fluxes affixed to them. This is described in Sec. II. (For a complementary discussion on mutual statistics from the braid group viewpoint, readers are referred to Ref. [7].)

Second, in Sec. III, we identify the corresponding *self-dual* system (with an arbitrary background magnetic field), which has a simpler mathematical structure than the generic case due to hidden supersymmetry [8]. For instance, thanks to the supersymmetry, one can construct the exact many-body ground-state wave function in this case. (See Ref. [9] for related discussions.) In this paper, however, we concentrate on the analysis of static soliton solutions to the classical field equations that follow, i.e., look for a generalization of the Jackiw-Pi solutions [10,11]. More general types of vortex solitons, which are likely to exhibit fractional and mutual statistics themselves, are found. The self-duality equations for our model share certain common elements with those for self-dual non-Abelian CS vortices discussed recently [12,13]. For instance, the Toda-type equation

$$\nabla^2 \ln|\phi_p(\mathbf{r})|^2 = -K_{pp'}|\phi_{p'}(\mathbf{r})|^2 \tag{1.2}$$

has a prominent role in both cases. But note that, in our model, $K = (K_{pp'})$ need *not* be equal to the Car-

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tan matrix of a certain Lie algebra and very little is known for this case. Concentrating on the case of two CS gauge fields, we will present a fairly detailed study on the nature of possible self-dual vortex solutions, with or without uniform background magnetic field. For the case with nonzero magnetic field the self-duality equations were also touched upon in Ref. [6], but there is only a minimal overlap between the latter work and ours.

Fully relativistic self-dual systems, including several Abelian CS gauge fields, are also possible, and we briefly discuss them in Sec. IV. The nature of static soliton solutions is discussed in the special case of these models. Section V contains a summary and discussion of our work. There are two appendixes. In the first we provide the derivations of certain formulas appearing in Sec. II. The second appendix contains the index-theorem analysis for the self-dual system treated in this paper.

## II. NONRELATIVISTIC QUANTUM FIELD THEORY

Choosing the basis where the matrix $(\kappa_{IJ})$ [see Eq. (1.1)] is diagonal, let us consider the nonrelativistic CS gauge-field theory defined by the Lagrangian density[1]

$$\mathcal{L} = \sum_{I=1}^{N} \frac{1}{2}\kappa_I \epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^I + \sum_{p=1}^{M}\left\{ i\hbar\Psi_p^\dagger \left(\frac{\partial}{\partial t} - \frac{i}{\hbar}\sum_{I=1}^{N} q_p^I a_0^I\right)\Psi_p - \frac{\hbar^2}{2m_p}\left|\left(\nabla - \frac{i}{\hbar c}\sum_{I=1}^{N} q_p^I \mathbf{a}^I - \frac{i}{\hbar c} e_p \mathbf{A}^{\rm ex}\right)\Psi_p\right|^2\right\} - U(\Psi^\dagger,\Psi), \quad (2.1)$$

where $\Psi_p$ ($p$=1,...,$M$) denote $M$ different bosonic (fermionic) fields satisfying the equal-time (anti)commutation relations

$$[\Psi_p(\mathbf{r},t),\Psi_{p'}(\mathbf{r}',t)]_\mp = [\Psi_p^\dagger(\mathbf{r},t),\Psi_{p'}^\dagger(\mathbf{r}',t)]_\mp = 0,$$
$$[\Psi_p(\mathbf{r},t),\Psi_{p'}^\dagger(\mathbf{r}',t)]_\mp = \delta_{pp'}\delta^2(\mathbf{r}-\mathbf{r}'). \quad (2.2)$$

[The subscript $-$ $(+)$ refers to the commutator (anticommutator).] This system possesses $[\mathrm{U}(1)]^N$ local gauge invariance in connection with $N$ independent CS gauge fields $a_\mu^I$, and we have included the external electromagnetic field $\mathbf{A}^{\rm ex}$ for the sake of generality. Excluding the pathological cases, we will below assume that all $\kappa_I$'s are nonzero and $M \geq N$; i.e., the number of the CS fields does not exceed that of the matter fields. Also, for definiteness, we shall take the potential $U(\Psi^\dagger,\Psi)$ to have the general form

$$U = \sum_p V_p(\mathbf{r},t)\Psi_p^\dagger(\mathbf{r},t)\Psi_p(\mathbf{r},t) + \frac{1}{2}\sum_{p,p'}\int d^2\mathbf{r}'\,\Psi_p^\dagger(\mathbf{r},t)\Psi_{p'}^\dagger(\mathbf{r}',t)V_{pp'}(\mathbf{r}-\mathbf{r}') \times\Psi_{p'}(\mathbf{r}',t)\Psi_p(\mathbf{r},t), \quad (2.3)$$

where $V_{pp'}(\mathbf{r}-\mathbf{r}') = V_{p'p}(\mathbf{r}'-\mathbf{r})$.

The stationary action principle for varying $a_0^I$ yields the Gauss laws $(i,j = 1 \text{ or } 2)$

$$b^I \equiv \epsilon^{ij}\nabla_i a_j^I = -\frac{1}{\kappa_I}\sum_p q_p^I \Psi_p^\dagger\Psi_p\,. \quad (2.4)$$

These, together with the Coulomb gauge conditions $\nabla_i a_i^I = 0$, then determine $\mathbf{a}^I(\mathbf{r},t)$ in terms of the matter densities $\rho_p(\mathbf{r},t) \equiv \Psi_p^\dagger(\mathbf{r},t)\Psi_p(\mathbf{r},t)$,

$$a_i^I(\mathbf{r},t) = \epsilon^{ij}\nabla_j \frac{1}{\kappa_I}\sum_p q_p^I \int d^2\mathbf{r}'\,G(\mathbf{r}-\mathbf{r}')\rho_p(\mathbf{r}',t), \quad (2.5)$$

where $G$ is the Green's function for the 2-dimensional (2D) Laplacian, i.e.,

$$\nabla^2 G(\mathbf{r}-\mathbf{r}') = \delta^2(\mathbf{r}-\mathbf{r}'),$$
$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|. \quad (2.6)$$

Now the Hamiltonian operator of the system can be identified with

$$H = \int d^2\mathbf{r}\left\{\sum_{p=1}^{M}\frac{\hbar^2}{2m_p}\mathbf{\Pi}_p^\dagger(\mathbf{r},t)\cdot\mathbf{\Pi}_p(\mathbf{r},t) + U(\Psi^\dagger,\Psi)\right\}, \quad (2.7)$$

where

$$\mathbf{\Pi}_p(\mathbf{r},t) = \left[\nabla - \frac{i}{\hbar c}\sum_I q_p^I \mathbf{a}^I(\mathbf{r},t) - \frac{i}{\hbar c}e_p\mathbf{A}^{\rm ex}(\mathbf{r},t)\right]\Psi_p(\mathbf{r},t) \equiv \mathbf{D}\Psi_p(\mathbf{r},t), \quad (2.8)$$

with the fields $\mathbf{a}^I$ expressed in terms of $\Psi^\dagger$ and $\Psi$ through Eq. (2.5).

What we have in Eq.(2.7) is the *properly ordered* Hamiltonian, and the corresponding operator field equations are

[1]Note that $\mu,\nu,\lambda = 0, 1,$ or 2 and our (2+1)-dimensional metric is given as $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1)$.

$$\begin{aligned} i\hbar\frac{\partial\Psi_p(\mathbf{r},t)}{\partial t} &= [\Psi_p(\mathbf{r},t),H] \\ &\equiv \sum_I q_p^I a_0^I(\mathbf{r},t)\Psi_p(\mathbf{r},t) - \frac{\hbar^2}{2m_p}\mathbf{D}^2\Psi_p(\mathbf{r},t) + V_p(\mathbf{r},t)\Psi_p(\mathbf{r},t) \\ &\quad + \int d^2\mathbf{r}'\,\Psi_{p'}^\dagger(\mathbf{r}',t)V_{pp'}(\mathbf{r}-\mathbf{r}')\Psi_{p'}(\mathbf{r}',t)\Psi_p(\mathbf{r},t) + \mathcal{R}, \end{aligned} \tag{2.9}$$

where $\mathcal{R}$, the quantum correction from operator ordering as first discussed in Ref. [11], is specified as

$$\mathcal{R} = \sum_{p'} \frac{1}{2m_{p'}c^2}\left(\sum_I \frac{q_{p'}^I q_p^I}{\kappa_I}\right)^2 \int d^2\mathbf{r}'\left(\frac{1}{4\pi^2}\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\right)\rho_{p'}(\mathbf{r}',t)\Psi_p(\mathbf{r},t) \tag{2.10}$$

and the operators $a_0^I(\mathbf{r},t)$ represent the solutions to the equations

$$\kappa_I\epsilon^{ij}\nabla_j a_0^I(\mathbf{r},t) + i\hbar\sum_p \frac{q_p^I}{2m_pc}\left[(D_i\Psi_p)^\dagger(\mathbf{r},t)\Psi_p(\mathbf{r},t) - \Psi_p^\dagger(\mathbf{r},t)(D_i\Psi_p)(\mathbf{r},t)\right] = 0 \tag{2.11}$$

or, more explicitly,

$$\begin{aligned} a_0^I(\mathbf{r},t) &= \sum_p \frac{q_p^I}{\kappa_I c}\epsilon^{ij}\nabla_j\int d^2\mathbf{r}'\,G(\mathbf{r}-\mathbf{r}')J_{pi}(\mathbf{r}',t)\,, \\ J_{pi}(\mathbf{r},t) &\equiv \frac{i\hbar}{2m_p}\left[(D_i\Psi_p)^\dagger(\mathbf{r},t)\Psi_p(\mathbf{r},t) - \Psi_p^\dagger(\mathbf{r},t)(D_i\Psi_p)(\mathbf{r},t)\right]\,. \end{aligned} \tag{2.12}$$

From the (anti)commutation relations (2.2) it follows that

$$[\Psi_p(\mathbf{r}',t), a_i^I(\mathbf{r},t)] = \epsilon^{ij}\nabla_j\frac{1}{\kappa_I}q_p^I G(\mathbf{r}-\mathbf{r}')\Psi_p(\mathbf{r}',t), \tag{2.13}$$

and in evaluating the commutator needed to derive Eq. (2.9), we have taken (following Ref. [11]) that

$$[\Psi_{p'}(\mathbf{r}',t), a_i^I(\mathbf{r},t)]|_{\mathbf{r}'=\mathbf{r}} = 0\,; \tag{2.14}$$

i.e., the quantity $\epsilon^{ij}\nabla_j G(\mathbf{r}-\mathbf{r}')$ at the coincidence limit $\mathbf{r}'=\mathbf{r}$ has been prescribed to be zero. Now we denote

$$\sum_I \frac{q_p^I q_{p'}^I}{\kappa_I} \equiv \beta_{pp'}(=\beta_{p'p}) \tag{2.15}$$

and then

$$D_i\Psi_p(\mathbf{r},t) = \left\{\nabla_i - \frac{i}{\hbar c}\epsilon^{ij}\nabla_j\sum_{p'}\beta_{pp'}\int d^2\mathbf{r}'\,G(\mathbf{r}-\mathbf{r}')\rho_{p'}(\mathbf{r}',t) - \frac{i}{\hbar c}e_p\mathbf{A}^{\mathrm{ex}}(\mathbf{r},t)\right\}\Psi_p(\mathbf{r},t), \tag{2.16}$$

$$\sum_I q_p^I a_0^I(\mathbf{r},t) = \sum_{p'}\frac{\beta_{pp'}}{c}\epsilon^{ij}\nabla_j\int d^2\mathbf{r}'\,G(\mathbf{r}-\mathbf{r}')J_{p'i}(\mathbf{r}',t), \tag{2.17}$$

showing that the parameters $q_p^I$'s and $\kappa_I$'s enter the field equations only through the $\beta_{pp'}$'s. From this we infer that theories with different $q_p^I$'s and $\kappa_I$'s but with the same values for the $\beta_{pp'}$'s are physically equivalent. A simple physical interpretation for the $\beta_{pp'}$'s will be given later.

To see clearly the physical content of the above non-relativistic quantum field theory, we will now derive the equivalent first-quantized description. For the two-particle sector of a one-layer system (i.e., a single matter field), this has been explicitly performed in Ref. [11]. The general $n$-body sector with the Hamiltonian (2.7) will be considered here.[2] Let $|\Phi\rangle$ denote any Heisenberg-picture state vector with the total number of particles equal to $n=\sum_{p=1}^M n_p$.

[2]After the completion of this paper we learned that C.-L. Ho and Y. Hosotani [14] previously considered the Schrödinger equation for $n$ anyons (on a torus), starting from the corresponding CS field theory. Ours is more explicit and also deals with a more general Hamiltonian, and so we include this discussion for the sake of completeness.

Then the corresponding many-particle Schrödinger wave function is given by

$$\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_1}^{(1)},\ldots,\mathbf{r}_1^{(M)},\ldots,\mathbf{r}_{n_M}^{(M)},t) = \left\langle 0 \left| \prod_{p=1}^{M} \frac{1}{\sqrt{n_p!}} \Psi_p(\mathbf{r}_1^{(p)},t)\cdots\Psi_p(\mathbf{r}_{n_p}^{(p)},t) \right| \Phi \right\rangle , \tag{2.18}$$

where the vacuum $|0\rangle$ satisfies the conditions $\Psi_p(\mathbf{r},t)|0\rangle = 0$ for any $p = 1,\ldots,M$. Now, using the field equations (2.9), we have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_1}^{(1)},\ldots,\mathbf{r}_1^{(M)},\ldots,\mathbf{r}_{n_M}^{(M)},t) &= \sum_p \left\langle 0 \right| \left\{ \frac{1}{\sqrt{n_1!}} \Psi_1(\mathbf{r}_1^{(1)},t)\cdots\Psi_1(\mathbf{r}_{n_1}^{(1)},t) \right\} \\ &\quad \times\cdots\left\{ \sum_{k=1}^{n_p} \frac{1}{\sqrt{n_p!}} \Psi_p(\mathbf{r}_1^{(p)},t)\cdots\left( i\hbar\frac{\partial}{\partial t}\Psi_p(\mathbf{r}_k^{(p)},t)\right)\cdots\Psi_p(\mathbf{r}_{n_p}^{(p)},t)\right\} \\ &\quad \times\cdots\left\{ \frac{1}{\sqrt{n_M!}} \Psi_M(\mathbf{r}_1^{(M)},t)\cdots\Psi_M(\mathbf{r}_{n_M}^{(M)},t)\right\} \left| \Phi \right\rangle \\ &= A + B + C + D, \end{aligned} \tag{2.19}$$

with

$$\begin{aligned} A = \sum_p \left\langle 0 \right| \cdots \left\{ \sum_{k=1}^{n_p} \frac{1}{\sqrt{n_p!}} \Psi_p(\mathbf{r}_1^{(p)},t) \right. \\ \left. \times\cdots\left( \sum_{p'} \frac{\beta_{pp'}}{c} \epsilon^{ij} \nabla_j^{(p,k)} \int d^2\mathbf{r}'\, G(\mathbf{r}_k^{(p)} - \mathbf{r}') J_{p'i}(\mathbf{r}',t) \right) \Psi_p(\mathbf{r}_k^{(p)},t)\cdots\Psi_p(\mathbf{r}_{n_p}^{(p)},t) \right\} \cdots \left| \Phi \right\rangle , \end{aligned} \tag{2.20}$$

$$\begin{aligned} B = \sum_p \left\langle 0 \right| \cdots \left\{ \sum_{k=1}^{n_p} \frac{1}{\sqrt{n_p!}} \Psi_p(\mathbf{r}_1^{(p)},t) \right. \\ \times\cdots\left( -\frac{\hbar^2}{2m_p} \left[ \nabla_i^{(p,k)} - \frac{i}{\hbar c} \epsilon^{ij} \nabla_j^{(p,k)} \sum_{p'} \beta_{pp'} \int d^2\mathbf{r}'\, G(\mathbf{r}_k^{(p)} - \mathbf{r}') \rho_{p'}(\mathbf{r}',t) \right. \right. \\ \left. \left. \left. - \frac{i}{\hbar c} e_p A_i^{\mathrm{ex}}(\mathbf{r}_k^{(p)},t) \right]^2 \Psi_p(\mathbf{r}_k^{(p)},t) \right) \cdots \Psi_p(\mathbf{r}_{n_p}^{(p)},t) \right\} \cdots \left| \Phi \right\rangle , \end{aligned} \tag{2.21}$$

$$\begin{aligned} C = \sum_p \left\langle 0 \right| \cdots \left\{ \sum_{k=1}^{n_p} \frac{1}{\sqrt{n_p!}} \Psi_p(\mathbf{r}_1^{(p)},t) \cdots \left( V_p(\mathbf{r}_k^{(p)},t)\Psi_p(\mathbf{r}_k^{(p)},t) + \sum_{p'} \int d^2\mathbf{r}'\, \Psi_{p'}^{\dagger}(\mathbf{r}',t) V_{pp'}(\mathbf{r}_k^{(p)} - \mathbf{r}') \Psi_{p'}(\mathbf{r}',t) \Psi_p(\mathbf{r}_k^{(p)},t) \right) \right. \\ \left. \times\cdots\Psi_p(\mathbf{r}_{n_p}^{(p)},t) \right\} \cdots \left| \Phi \right\rangle , \end{aligned} \tag{2.22}$$

$$\begin{aligned} D = \sum_p \left\langle 0 \right| \cdots \left\{ \sum_{k=1}^{n_p} \frac{1}{\sqrt{n_p!}} \Psi_p(\mathbf{r}_1^{(p)},t) \right. \\ \left. \times\cdots\left( \sum_{p'} \frac{1}{2m_{p'}c^2} \beta_{pp'}^2 \int d^2\mathbf{r}'\, \frac{1}{4\pi^2} \frac{1}{|\mathbf{r}_k^{(p)} - \mathbf{r}'|^2} \rho_{p'}(\mathbf{r}',t) \Psi_p(\mathbf{r}_k^{(p)},t) \right) \cdots \Psi_p(\mathbf{r}_{n_p}^{(p)},t) \right\} \cdots \left| \Phi \right\rangle . \end{aligned} \tag{2.23}$$

where $\nabla_i^{(p,k)}$ denotes the derivative with respect to $\mathbf{r}_k^{(p)}$.

The contributions designated as $C$ and $D$ above have the structure of the standard one-body and two-body interaction terms. So we may immediately rewrite them as [15]

$$C = \left\{ \sum_{(p,k)} V_p(\mathbf{r}_k^{(p)},t) + \frac{1}{2} \sum_{(p,k)} \sum_{(p',k')\neq(p,k)} V_{pp'}(\mathbf{r}_k^{(p)} - \mathbf{r}_{k'}^{(p')}) \right\} \Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t) , \tag{2.24}$$

$$D = \left\{ \frac{1}{2} \sum_{(p,k)} \sum_{(p',k')\neq(p,k)} \frac{1}{2m_{p'}c^2} \beta_{pp'}^2 \frac{1}{4\pi^2} \frac{1}{|\mathbf{r}_k^{(p)} - \mathbf{r}_{k'}^{(p')}|^2} \right\} \Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t) , \tag{2.25}$$

where the sum $\frac{1}{2}\sum_{(p,k)}\sum_{(p',k')\neq(p,k)}$ has the effect of taking in once every different set of pairs $(p,k)$ and $(p',k')$, excluding the case with $p'=p$ and $k'=k$. On the other hand, we show in Appendix A that the contributions $B$ and $A$ above can in fact be expressed as

$$B=\sum_{(p,k)}\left(-\frac{\hbar^2}{2m_p}\right)\left[\nabla_i^{(p,k)}-\frac{i}{\hbar c}\epsilon^{ij}\nabla_j^{(p,k)}\sum_{(p',k')<(p,k)}\beta_{pp'}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})-\frac{i}{\hbar c}e_pA_i^{\mathrm{ex}}(\mathbf{r}_k^{(p)},t)\right]^2 \times\Phi(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)},t)\,, \tag{2.26}$$

$$A=\sum_{(p,k)}\sum_{(p',k')<(p,k)}\left(-\frac{i\hbar}{m_{p'}c}\right)\beta_{pp'}\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')}) \times\left[\nabla_i^{(p',k')}-\frac{i}{\hbar c}\epsilon^{il}\nabla_l^{(p',k')}\sum_{\substack{(p'',k'')<(p,k)\\(p'',k'')\neq(p',k')}}\beta_{p'p''}G(\mathbf{r}_{k'}^{(p')}-\mathbf{r}_{k''}^{(p'')})-\frac{i}{\hbar c}e_{p'}A_i^{\mathrm{ex}}(\mathbf{r}_{k'}^{(p')},t)\right] \times\Phi(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)},t), \tag{2.27}$$

where $\sum_{(p',k')<(p,k)}$ denotes the sum over all indices $(p',k')$ that appear on the left of $\mathbf{r}_k^{(p)}$ in the arrangement $(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_1}^{(1)},\dots,\mathbf{r}_1^{(p)},\dots,\mathbf{r}_k^{(p)},\dots,\mathbf{r}_{n_p}^{(p)},\dots,\mathbf{r}_1^{(M)},\dots,\mathbf{r}_{n_M}^{(M)})$. In Appendix A it is further shown (after a bit involved manipulations) that the contributions $A$, $B$, and $D$ above combine to give a surprisingly simple expression, namely,

$$A+B+D=\sum_{(p,k)}\left(-\frac{\hbar^2}{2m_p}\right)\left[\nabla_i^{(p,k)}-\frac{i}{\hbar c}\epsilon^{ij}\nabla_j^{(p,k)}\left(\sum_{(p',k')\neq(p,k)}\beta_{pp'}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right) -\frac{i}{\hbar c}e_pA_i^{\mathrm{ex}}(\mathbf{r}_k^{(p)},t)\right]^2\Phi(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)},t). \tag{2.28}$$

Using this result in Eq. (2.19), we then find that the appropriate many-particle Schrödinger equation reads

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)},t)=\left\{\sum_{(p,k)}\left(-\frac{\hbar^2}{2m_p}\right)\left[\nabla^{(p,k)}-\frac{i}{\hbar c}\mathcal{A}_{(p,k)}(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)})-\frac{i}{\hbar c}e_p\mathbf{A}^{\mathrm{ex}}(\mathbf{r}_k^{(p)},t)\right]^2 +\sum_{(p,k)}V_p(\mathbf{r}_k^{(p)},t)+\frac{1}{2}\sum_{(p,k)}\sum_{(p',k')\neq(p,k)}V_{pp'}(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right\}\Phi(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)},t), \tag{2.29}$$

where we have defined

$$\begin{aligned}\mathcal{A}_{(p,k)i}(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)})&=\epsilon^{ij}\nabla_j^{(p,k)}\left(\sum_{(p',k')\neq(p,k)}\beta_{pp'}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right)\\&=\epsilon^{ij}\nabla_j^{(p,k)}\left(\frac{1}{2}\sum_{(p',k')}\sum_{(p'',k'')\neq(p',k')}\beta_{p'p''}G(\mathbf{r}_{k'}^{(p')}-\mathbf{r}_{k''}^{(p'')})\right).\end{aligned} \tag{2.30}$$

The wave function $\Phi$ should be single valued with respect to every particle coordinate and satisfy the symmetry requirement appropriate to bosons or fermions,

$$\Phi(\dots,\mathbf{r}_{k_1}^{(p)},\dots,\mathbf{r}_{k_2}^{(p)},\dots)=\pm\Phi(\dots,\mathbf{r}_{k_2}^{(p)},\dots,\mathbf{r}_{k_1}^{(p)},\dots). \tag{2.31}$$

The Schrödinger equation (2.29) provides an equivalent first-quantized description for the quantum field theory defined by the ordered Hamiltonian operator (2.7). The entire effect of the CS interactions now enters through the vector potentials $\mathcal{A}_{(p,k)}$, which can be related to the induced Aharonov-Bohm-type interactions between the charge-flux composites of suitable nature. In particular, the form (2.30) implies that the Aharonov-Bohm interaction strength between the two particles carrying the labels $p$ and $p'$ is equal to $-\beta_{pp'}$. A straightforward interpretation of this is as follows. In view of the fact that we had $N$ Abelian CS gauge fields $a_\mu^I$ $(I=1,\dots,N)$, we associate with a type-$p$ particle an $N$-tuple of charges $(q_p^1,\dots,q_p^N)$ and also an $N$-tuple of corresponding fluxes $(-\frac{q_p^1}{\kappa_1},\dots,-\frac{q_p^N}{\kappa_N})$ in accordance with the Gauss laws (2.4).

Then, given two particles from type-$p$ and type-$p'$ each, the expected strength of the Aharonov-Bohm interaction will be $\sum_I q_p^I(-\frac{q_{p'}^I}{\kappa_I}) = -\beta_{pp'}$, in agreement with the observation we just made. [Actually, with the Aharonov-Casher interaction [16] taken as an *additional* effect to the Aharonov-Bohm interaction, the flux affixed to a $p$-type particle will have to read $(-\frac{q_p^1}{2\kappa_1},\ldots,-\frac{q_p^N}{2\kappa_N})$. Note that, in interpreting the many-particle Schrödinger equation (2.29), we are in no way bound by the Gauss laws (2.4).]

We here remark that the above interpretation, based on $N$ *distinct U(1) charges and corresponding fluxes,* is not the only possible. We will illustrate this phenomenon through a closer look at the $N=1$ and $N=2$ cases. With $N=1$ (i.e., one CS gauge field only) but arbitrary number of matter fields, we may write $\beta_{pp'}=\frac{q_p q_p'}{\kappa}$ and so assign flux $-\frac{q_p}{\kappa}$ to a particle of charge $q_p$; here, the charge-flux ratio is necessarily the same for all particle species. With $N=2$ (i.e., two CS gauge fields), on the other hand, we have the formula

$$\beta_{pp'}=\frac{q_p^1 q_{p'}^1}{\kappa_1}+\frac{q_p^2 q_{p'}^2}{\kappa_2} \qquad (p,p'=1,2,\ldots,M), \tag{2.32}$$

from which we immediately derive the results

$$\begin{aligned}\beta_{pp'}^2 &\le \beta_{pp}\beta_{p'p'} \quad \text{if } \kappa_1\kappa_2>0,\\ \beta_{pp'}^2 &\ge \beta_{pp}\beta_{p'p'} \quad \text{if } \kappa_1\kappa_2<0,\end{aligned} \tag{2.33}$$

for any given $p$, $p'$. Equation (2.33) puts a restriction on the possible values of the $\beta_{pp'}$ for $M>2$ [to be realizable by a U(1)×U(1) CS field theory], and here the sign of $\kappa_1\kappa_2$ matters also. As we described in the previous paragraph, this system may be related to that of composites carrying appropriate two-vector charges and corresponding two-vector fluxes. But, for the case with $\kappa_1\kappa_2<1$, an alternative, in some sense simpler, interpretation is also available. Specifically, we assign (scalar) charge $\widetilde{q}_p=\frac{q_p^1}{\sqrt{|\kappa_1|}}-\frac{q_p^2}{\sqrt{|\kappa_2|}}$ and flux $\widetilde{\phi}_p=\pm(\frac{q_p^1}{2\sqrt{|\kappa_1|}}+\frac{q_p^2}{2\sqrt{|\kappa_2|}})$ to a type-$p$ particle, and then the Aharonov-Bohm and Aharonov-Casher interactions between two particles belonging to the type-$p$ and type-$p'$, respectively, will have the net strength

$$\widetilde{q}_p\widetilde{\phi}_{p'}+\widetilde{q}_{p'}\widetilde{\phi}_p=\pm\left[\frac{q_p^1 q_{p'}^1}{|\kappa_1|}-\frac{q_p^2 q_{p'}^2}{|\kappa_2|}\right], \tag{2.34}$$

i.e., equal to $\beta_{pp'}$ [see Eq. (2.32)] under the restriction $\kappa_1\kappa_2<0$. This shows that a multicomponent system of charge-flux composites, with *different* charge-flux ratios for individual components, can equivalently be represented by a U(1)×U(1) CS field theory. We have an exceptional situation if

$$\beta_{pp'}^2-\beta_{pp}\beta_{p'p'}=0 \qquad (\text{for every } p,\, p'), \tag{2.35}$$

and for this case the charge-flux ratios $\widetilde{q}_p/\widetilde{\phi}_p$ become independent of $p$. When the $\beta_{pp'}$'s satisfy the conditions (2.35), the equivalent U(1)×U(1) CS theory constructed according to the above correspondence is effectively reduced to a U(1) CS theory; i.e., from the two CS gauge fields, it is only their particular linear combination that has a dynamical role. [Incidentally, if $\kappa_1\kappa_2<0$, we will always be allowed to set $\kappa_1=-\kappa_2=\kappa$ thanks to the rescaling freedom of the CS fields. Then we may introduce new CS fields $v_\mu^{(1)}$, $v_\mu^{(2)}$ by

$$a_{1\mu}=\frac{1}{\sqrt{2}}(v_\mu^{(1)}+v_\mu^{(2)}), \qquad a_{2\mu}=\frac{1}{\sqrt{2}}(v_\mu^{(1)}-v_\mu^{(2)}), \tag{2.36}$$

and in terms of these fields the CS Lagrangian becomes

$$\mathcal{L}_g=\kappa\epsilon^{\mu\nu\lambda}v_\mu^{(1)}\partial_\nu v_\lambda^{(2)}\,. \tag{2.37}$$

The Lagrangian of this form has been considered recently by Wilczek [5].] It should be an interesting mathematical exercise to extend the above discussion to the case of general $N$, but it is not pursued in this paper.

As is well known, the Aharonov-Bohm interaction affects the statistical character of the particles involved. To see this, observe that the vector potential in Eq. (2.30) is locally a pure gauge:

$$\begin{aligned}\mathcal{A}_{(p,k)i}(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)}) &= -\nabla_i^{(p,k)}\Lambda\,,\\ \Lambda &= \frac{1}{2}\sum_{(p',k')}\ \sum_{(p'',k'')\neq(p',k')}\beta_{p'p''}\frac{1}{2\pi}\\ &\qquad\times\arctan\left(\frac{y_{k'}^{(p')}-y_{k''}^{(p'')}}{x_{k'}^{(p')}-x_{k''}^{(p'')}}\right).\end{aligned} \tag{2.38}$$

So, by redefining the single-valued wave function $\Phi$ according to

$$\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t)=e^{-\frac{i}{\hbar c}\Lambda}\bar{\Phi}(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t)\,, \tag{2.39}$$

we may have the gauge potentials $\mathcal{A}_{(p,k)}$ disappear in the Schrödinger equation for $\bar{\Phi}$. But the gauge function $\Lambda$ here being multivalued in general, $\bar{\Phi}$ will have to be multivalued (for a single-valued $\Phi$) and so satisfy highly nontrivial boundary conditions that are sensitive to the $\beta_{p'p''}$'s. To be explicit, consider exchanging the positions of two identical particles, say, $\mathbf{r}_{k_1}^{(p)}$ and $\mathbf{r}_{k_2}^{(p)}$, with the function $\bar{\Phi}(\ldots,\mathbf{r}_{k_1}^{(p)},\ldots,\mathbf{r}_{k_2}^{(p)},\ldots)$ along a certain closed path $C$ (see Fig. 1). Then, from Eqs. (2.31) and (2.39), the resulting expression should differ from the original by the phase factor $\pm e^{\frac{i}{2\hbar c}\beta_{pp}}e^{\frac{i}{\hbar c}\gamma(C)}$, where $\gamma(C)$ is equal to the sum of the $\beta_{pp'}$'s over the index set $(p',k')$ associated with the $\mathbf{r}_{k'}^{(p')}$'s in the interior of $C$. [Here an implicit assumption is that no position variable other than $\mathbf{r}_{k_1}^{(p)}$ and $\mathbf{r}_{k_2}^{(p)}$ takes values on $C$.] Also note that as we allow a specific position variable $\mathbf{r}_k^{(p)}$ to be taken along a closed path $C$, the initial and final expressions of $\bar{\Phi}$ should differ by a phase $e^{\frac{i}{\hbar c}\gamma(C)}$, with $\gamma(C)$ defined as above. If all $\beta_{pp'}$'s with $p\neq p'$ vanish, the system is that of $M$ species of fractional-statistics-obeying particles. In the terminology of Wilczek [5], the $\beta_{pp'}$'s with $p\neq p'$ are relevant for *mutual* statistics, while the $\beta_{pp}$'s are responsible for more

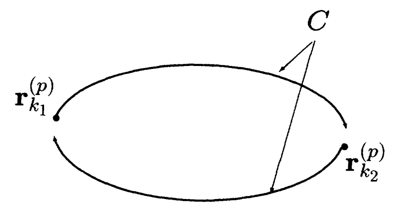


FIG. 1. Closed curve $C$ formed as the positions of two identical particles, $\mathbf{r}^{(p)}_{k_1}$ and $\mathbf{r}^{(p)}_{k_2}$, are exchanged.

usual fractional statistics. We close this section with the remark that, because of the highly nontrivial nature of the boundary conditions satisfied by $\bar{\Phi}$, the regular gauge description based on a single-valued function $\Phi$ should be preferred for all more explicit studies.

## III. SELF-DUAL SYSTEMS AND SOLITON SOLUTIONS

The system described by the field theory of Sec. II is highly nontrivial, and it is extremely difficult to obtain any concrete information on its behavior. Naturally, one might then ask whether there exists a certain special choice of the potential $U(\Psi^\dagger, \Psi)$ for which the mathematical treatment of the system becomes more tractable. This leads us to consider the so-called self-dual system, which is based on the potential

$$U = \sum_p \left(-\frac{\hbar}{2m_p c}\sigma_p e_p B^{\rm ex}(\mathbf{r})\right) \Psi_p^\dagger(\mathbf{r},t)\Psi_p(\mathbf{r},t) + \frac{1}{2}\sum_{p,p'}\left(\frac{\hbar}{2m_p c}\sigma_p + \frac{\hbar}{2m_{p'} c}\sigma_{p'}\right)\beta_{pp'}\Psi_p^\dagger(\mathbf{r},t) \times\Psi_{p'}^\dagger(\mathbf{r},t)\Psi_{p'}(\mathbf{r},t)\Psi_p(\mathbf{r},t), \tag{3.1}$$

where $B^{\rm ex}(\mathbf{r}) \equiv \epsilon^{ij}\nabla_i A_j^{\rm ex}(\mathbf{r})$ and $\sigma_p = 1$ or $-1$ (for each $p$, independently). For this choice of the potential, the many-particle Schrödinger equation (2.29) can be cast into the form

$$i\hbar\frac{\partial\Phi}{\partial t} = H_{(1\mathrm{st})}\Phi ,$$
$$H_{(1\mathrm{st})} = \sum_{(p,k)}\left(-\frac{\hbar^2}{2m_p}\right)\left[\nabla^{(p,k)} - \frac{i}{\hbar c}\mathcal{A}_{(p,k)}(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)}) - \frac{i}{\hbar c}e_p\mathbf{A}^{\rm ex}(\mathbf{r}_k^{(p)},t)\right]^2 + \sum_{(p,k)}\left(-\frac{\hbar}{2m_p c}\sigma_p\right)\epsilon^{ij}\nabla_i^{(p,k)}\left\{e_p A_j^{\rm ex}(\mathbf{r}_k^{(p)}) + \mathcal{A}_{(p,k)j}(\mathbf{r}_1^{(1)},\dots,\mathbf{r}_{n_M}^{(M)})\right\} , \tag{3.2}$$

since $\beta_{pp'} = \beta_{p'p}$ and we have, thanks to Eqs. (2.30) and (2.6),

$$\epsilon^{ij}\nabla_i^{(p,k)}\mathcal{A}_{(p,k)j} = -\sum_{(p',k')\neq(p,k)}\beta_{pp'}\delta^2(\mathbf{r}_k^{(p)} - \mathbf{r}_{k'}^{(p')}) . \tag{3.3}$$

Note that $H_{(1\mathrm{st})}$ in Eq. (3.2) has the form of the nonrelativistic (many-body) Pauli Hamiltonian on the plane, with spins $\sigma_p = +1$ or $-1$. There is a hidden supersymmetry in the system [8,9] which can be exploited to find the exact many-body ground state. But, in this paper, we shall direct our interest to the static solutions of the corresponding classical field theory. That is, we consider $\Psi_p(\mathbf{r},t)$ to be classical $c$-number fields and $\Psi_p^\dagger(\mathbf{r},t)$ the corresponding complex conjugates $\Psi_p^*(\mathbf{r},t)$. The Hamiltonian is as in Eq. (2.7) [with the potential $U$ given by Eq. (3.1)], but the classical equations of motion do not include the operator ordering term in Eq. (2.9), i.e.,

$$i\hbar\frac{\partial\Psi_p(\mathbf{r},t)}{\partial t} = \sum_I q_p^I a_0^I(\mathbf{r},t)\Psi_p(\mathbf{r},t) - \frac{\hbar^2}{2m_p}\mathbf{D}^2\Psi_p(\mathbf{r},t) - \frac{\hbar}{2m_p c}\sigma_p e_p B^{\rm ex}(\mathbf{r})\Psi_p(\mathbf{r},t) + \sum_{p'}\left(\frac{\hbar}{2m_p c}\sigma_p + \frac{\hbar}{2m_p' c}\sigma_{p'}\right)\beta_{pp'} \times\Psi_{p'}^*(\mathbf{r},t)\Psi_{p'}(\mathbf{r},t)\Psi_p(\mathbf{r},t) , \tag{3.4}$$

where the CS gauge fields $a_\mu^I(\mathbf{r},t)$ are of course supposed to satisfy Eqs. (2.5) and (2.12). As we will show, this classical field system under suitable restrictions admits a class of interesting, vortex-type, static solutions, carrying nontrivial characteristics endowed upon them by the CS interactions. Our work generalizes Refs. [11] and [17], where the case of a single matter field with $B^{\rm ex} = 0$ (Ref. [11]) or $B^{\rm ex} \neq 0$ (Ref. [17]) was analyzed. It is conceivable that the solitonlike solutions discussed here may have significant physical roles as regards the nature of various topological fluids within the effective field-theory approach [3,4].

To proceed, note that the choice of the potential as in Eq. (3.1) allows us to write the static energy functional in the form

$$E = \int d^2\mathbf{r}\sum_p\frac{\hbar^2}{2m_p}\left|D_1\Psi_p + i\sigma_p D_2\Psi_p\right|^2 ,$$
$$D_i\Psi_p \equiv \left(\nabla_i - \frac{i}{\hbar c}\sum_I q_p^I a_i^I - \frac{i}{\hbar c}e_p A_i^{\rm ex}\right)\Psi_p, \qquad i = 1,2, \tag{3.5}$$

dropping unimportant surface terms. This is an immediate consequence of the identity

$$|\mathbf{D}\Psi_p|^2 = |(D_1 + i\sigma_p D_2)\Psi_p|^2 + \frac{1}{\hbar c}\sigma_p(e_p B^{\rm ex} + \sum_I q_p^I b^I)|\Psi_p|^2 - i\sigma_p\epsilon^{ij}\nabla_i(\Psi_p^* D_j\Psi_p) \tag{3.6}$$

and the relation (2.4). Hence any configuration satisfying the *self-duality equations*

$$D_1\Psi_p = -i\sigma_p D_2\Psi_p \quad (\sigma_p = +1 \text{ or } -1), \tag{3.7a}$$

$$\kappa_I \epsilon^{ij}\nabla_i a_j^I = -\sum_p q_p^I \Psi_p^* \Psi_p \tag{3.7b}$$

will have the lowest possible energy, i.e., $E = 0$. A solution of these equations should solve the classical field equations (3.4) automatically, but there is of course no guarantee that one can always find a nontrivial solution to Eqs. (3.7a) and (3.7b). Suppose that a nontrivial solution exists. Then we may conveniently write (i.e., work in the Coulomb gauge)

$$a_i^I(\mathbf{r}) = -\epsilon^{ij}\nabla_j U^I(\mathbf{r}), \qquad A_i^{\text{ex}}(\mathbf{r}) = -\epsilon^{ij}\nabla_j V^{\text{ex}}(\mathbf{r}), \tag{3.8}$$

and introduce the functions $f_p(\mathbf{r})$ $(p = 1, \ldots, M)$ by

$$\Psi_p(\mathbf{r}) = e^{-\frac{1}{\hbar c}\sigma_p\{\sum_I q_p^I U^I(\mathbf{r}) + e_p V^{\text{ex}}(\mathbf{r})\}} f_p(\mathbf{r}). \tag{3.9}$$

For the functions $f_p(\mathbf{r})$, Eq. (3.7a) now implies that

$$(\nabla_1 + i\sigma_p\nabla_2) f_p(\mathbf{r}) = 0, \tag{3.10}$$

and therefore $f_p(\mathbf{r})$ should be restricted to an entire function of $z_{(\sigma_p)} \equiv x + i\sigma_p y$, viz., $f_p = f_p(z_{(\sigma_p)})$. Clearly, the function $\Psi_p(\mathbf{r})$ may vanish only at the zeros of $f_p$ and let these zeros be at $(\mathbf{R}_1^{(p)}, \ldots, \mathbf{R}_{n_p}^{(p)})$. Also, from Eq. (3.9), we have

$$\sum_I q_p^I U^I(\mathbf{r}) + e_p V^{\text{ex}}(\mathbf{r}) = -\sigma_p \frac{\hbar c}{2} \ln\left[|\Psi_p(\mathbf{r})|^2 / |f_p(z_{(\sigma_p)})|^2\right], \tag{3.11}$$

and combining these with Eq. (3.7b) then yields the equations

$$\nabla^2 \ln|\Psi_p|^2 = \sigma_p \frac{2}{\hbar c}\left\{\sum_{p'} \beta_{pp'} |\Psi_{p'}|^2 - e_p B^{\text{ex}}\right\} + 4\pi \sum_{r=1}^{n_p} \delta^2(\mathbf{r} - \mathbf{R}_r^{(p)}) \, (p = 1, \ldots, M), \tag{3.12}$$

where we have used the definition (2.15) and the relation $\nabla^2 \ln|f_p|^2 = 4\pi\sum_{r=1}^{n_p}\delta^2(\mathbf{r} - \mathbf{R}_r^{(p)})$.

Our problem is now reduced to the study of the coupled nonlinear equations (3.12), satisfied by the matter densities $|\Psi_p(\mathbf{r})|^2$. Note that, in connection with the CS interactions, only the parameters $\beta_{pp'}$ enter Eq. (3.12); this is natural in view of our general observation made in Sec. II. While the equations of this form appear very frequently, say, in the study of various integrable models, there is no systematic mathematical method for constructing the solutions for generic $(\beta_{pp'})$, $M \geq 2$. Nonetheless, the solution space is expected to have a rich structure. With that in mind, we shall below study in some detail the nature of solutions allowed when there are just two independent matter fields, i.e., $M = 2$, and the external magnetic field $B^{\text{ex}}$ is zero or at most uniform. Assuming this, we now write $(\Psi_1, \Psi_2) \equiv (\phi, \chi)$ and present Eq. (3.12) in the form

$$\begin{aligned} \nabla^2 \ln|\phi|^2 &= \sigma_1(\bar\beta_{11}|\phi|^2 + \bar\beta_{12}|\chi|^2 - \bar e_1 B^{\text{ex}}) + 4\pi\sum_{r=1}^{n_1}\delta^2(\mathbf{r} - \mathbf{R}_r), \\ \nabla^2 \ln|\chi|^2 &= \sigma_2(\bar\beta_{12}|\phi|^2 + \bar\beta_{22}|\chi|^2 - \bar e_2 B^{\text{ex}}) + 4\pi\sum_{r=1}^{n_2}\delta^2(\mathbf{r} - \mathbf{R}'_r), \end{aligned} \tag{3.13}$$

where we have denoted $\frac{2}{\hbar c}\beta_{pp'} = \bar\beta_{pp'}$ and $\frac{2}{\hbar c}e_p = \bar e_p$. Here it is not difficult to guess that the system under nonzero $B^{\text{ex}}$ behaves differently from that for $B^{\text{ex}} = 0$. Also, a rather different behavior is expected when the condition $\bar\beta_{11}\bar\beta_{22} = \bar\beta_{12}^2$ is satisfied. [See Eq. (2.35).] Hence, for the respective cases decided by these factors, we will separately look for the solutions to Eq. (3.13) below.

### A. The case with $B^{\text{ex}} = 0$ and $\Delta \equiv \bar\beta_{11}\bar\beta_{22} - \bar\beta_{12}^2 \neq 0$

In this case, Eq. (3.13) may be written in the form[3]

$$\nabla^2 \begin{pmatrix} \ln|\phi|^2 \\ \ln|\chi|^2 \end{pmatrix} = -K \begin{pmatrix} |\phi|^2 \\ |\chi|^2 \end{pmatrix}, \tag{3.14}$$

where

$$K = \begin{pmatrix} -\sigma_1\bar\beta_{11} & -\sigma_1\bar\beta_{12} \\ -\sigma_2\bar\beta_{12} & -\sigma_2\bar\beta_{22} \end{pmatrix} \equiv \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}. \tag{3.15}$$

This is a Toda-type equation [18], and we are here interested in the regular solutions with $Q_\phi \equiv \int d^2\mathbf{r}\,|\phi|^2 < \infty$ and $Q_\chi \equiv \int d^2\mathbf{r}\,|\chi|^2 < \infty$. As we shall see below, the characters of this equation depend very much on the properties of the matrix $K$.

The above system is integrable if $K$ has certain specific forms. To discuss this case, we rescale the matter fields as $|\phi|^2 = |\widetilde\phi|^2/a$ and $= |\widetilde\chi|^2/b$ ($a$, $b$: positive constants) so that Eq. (3.14) may assume the form

$$\begin{aligned} \nabla^2 \begin{pmatrix} \ln|\widetilde\phi|^2 \\ \ln|\widetilde\chi|^2 \end{pmatrix} &= -\widetilde K \begin{pmatrix} |\widetilde\phi|^2 \\ |\widetilde\chi|^2 \end{pmatrix}, \\ \widetilde K &= \begin{pmatrix} -\sigma_1\bar\beta_{11}/a & -\sigma_1\bar\beta_{12}/b \\ -\sigma_2\bar\beta_{12}/a & -\sigma_2\bar\beta_{22}/b \end{pmatrix}. \end{aligned} \tag{3.16}$$

This equation will be integrable [19] if the matrix $\widetilde K$ is identified with one of the Cartan matrices of the classical Lie algebra. We here recall the following rank-2 Cartan matrices:

[3]Here and henceforth, we will often omit the $\delta$-function terms in the equation, which are significant only at the zeros of $\phi$ or $\chi$.

$$A_2:\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix},\quad B_2:\begin{pmatrix}2 & -2\\ -1 & 2\end{pmatrix},\quad G_2:\begin{pmatrix}2 & -1\\ -3 & 2\end{pmatrix}. \tag{3.17}$$

Comparing $\widetilde{K}$ with these expressions, we conclude that our system is integrable if $\sigma_1=\sigma_2=\sigma(=\pm 1)$ and $K$ belongs to one of the following one-parameter families:

$$K=a\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix},\quad a\begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix}, \tag{3.18}$$

or

$$a\begin{pmatrix}-2 & 3\\ 3 & -6\end{pmatrix}.$$

[Note that, in Eq. (3.18), the constant $b$ appeared above has been set to $a$, $\frac{1}{2}a$, or $\frac{1}{3}a$.] These special cases are known to be also relevant for non-abelian self-dual CS systems first discussed in Ref. [12]. We here have $\Delta\equiv\bar\beta_{11}\bar\beta_{22}-\bar\beta_{12}^2>0$, and hence, according to Eq. (2.33), they all belong to the case with $\kappa_1\kappa_2>0$. A constructive method for the corresponding exact solutions is described in Ref. [13]. An interesting property of the solutions when $\widetilde{K}$ is equal to the Cartan matrix of $A_2$ (and probably $B_2$ and $G_2$ as well) is that the charges $Q_\phi$, $Q_\chi$ are quantized in such a way that the fluxes [see Eq. (2.4)]

$$\begin{aligned}\Phi_\phi&\equiv\int d^2\mathbf{r}\,\epsilon^{ij}\nabla_i(q_1^1a_j^1+q_1^2a_j^2)\\&=-(\beta_{11}Q_\phi+\beta_{12}Q_\chi),\\ \Phi_\chi&\equiv\int d^2\mathbf{r}\,\epsilon^{ij}\nabla_i(q_2^1a_j^1+q_2^2a_j^2)\\&=-(\beta_{21}Q_\phi+\beta_{22}Q_\chi)\end{aligned} \tag{3.19}$$

may become integer multiples of $2\pi\hbar c$.

For a generic matrix $K$, Eq. (3.14) has been studied little so far. For example, there is no known criterion for the existence of a regular solution to the equation. Therefore, we will first try to narrow down the range of the parameters $\bar\beta_{pp'}$ (for given $\sigma_1$, $\sigma_2$) in which a regular solution with finite charges might be available and then give some explicit solutions for certain specific cases. First of all, we recall that a solution to Eqs. (3.7a) and (3.7b), if it exists, should have zero energy. Hence, in view of Eq. (2.7), there will be no regular self-dual solution if the potential $U$ is positive definite. The potential is, from Eq. (3.1), given by (here it suffices to set $m_1=m_2=m$ since the self-duality equations are independent of the masses)

$$\frac{4m}{\hbar^2}U=\sigma_1\bar\beta_{11}|\phi|^4+(\sigma_1+\sigma_2)\bar\beta_{12}|\phi|^2|\chi|^2+\sigma_2\bar\beta_{22}|\chi|^4\,. \tag{3.20}$$

When both $\kappa_1$ and $\kappa_2$ are positive, we have $\bar\beta_{11}>0$, $\bar\beta_{22}>0$, and $\bar\beta_{11}\bar\beta_{22}-\bar\beta_{12}^2>0$ and so conclude that $U$ is positive definite if $\sigma_1=\sigma_2=1$. With $\kappa_1<0$ and $\kappa_2<0$, $U$ will be positive definite if $\sigma_1=\sigma_2=-1$. On the other hand, with $\kappa_1>0$, and $\kappa_2<0$ (and therefore $\bar\beta_{11}\bar\beta_{22}-\bar\beta_{12}^2<0$), we will have a positive-definite potential if all the coefficients appearing in the right-hand side of Eq. (3.20) are positive, viz., no self-dual solution if $K_{11}<0$, $K_{22}<0$ and (for the case of $\sigma_1\sigma_2=1$) $K_{12}<0$. The parameter range is also restricted by the fact that if the Laplacian of a function $f$ is positive for large $|\mathbf{r}|$, $f$ is asymptotically increasing. We may apply this with $f=-\ln|\phi|^2$ or $-\ln|\chi|^2$, noting that $|\phi|^2$ and $|\chi|^2$ should approach zero as $|\mathbf{r}|\to\infty$ (and hence the functions $-\ln|\phi|^2$ and $-\ln|\chi|^2$ increase indefinitely) to obtain a configuration with finite charges. Then, in view of Eq. (3.14), this cannot be realized if the matrix $K$ is strictly negative; hence, no solution if $K_{pp'}(=-\sigma_p\bar\beta_{pp'})<0$ for every $p$, $p'$.

For further restrictions on the parameters, it is useful to note that Eq. (3.14), now including the $\delta$-function terms, can be cast as

$$\begin{aligned}\nabla^2\ln\left(\frac{\{|\chi|^2/\prod_{r=1}^{n_2}|z-Z'_r|\}^{K_{12}}}{\{|\phi|^2/\prod_{r=1}^{n_1}|z-Z_r|\}^{K_{22}}}\right)&=(\det K)|\phi|^2\,,\\ \nabla^2\ln\left(\frac{\{|\phi|^2/\prod_{r=1}^{n_1}|z-Z_r|\}^{K_{21}}}{\{|\chi|^2/\prod_{r=1}^{n_2}|z-Z'_r|\}^{K_{11}}}\right)&=(\det K)|\chi|^2\,.\end{aligned} \tag{3.21}$$

If $\det K=\sigma_1\sigma_2\{\bar\beta_{11}\bar\beta_{22}-(\bar\beta_{12})^2\}$ is positive, these relations show that both

$$\frac{\{|\chi|^2/\prod_{r=1}^{n_2}|z-Z'_r|\}^{K_{12}}}{\{|\phi|^2/\prod_{r=1}^{n_1}|z-Z_r|\}^{K_{22}}}$$

and

$$\frac{\{|\phi|^2/\prod_{r=1}^{n_1}|z-Z_r|\}^{K_{21}}}{\{|\chi|^2/\prod_{r=1}^{n_2}|z-Z'_r|\}^{K_{11}}}$$

should be asymptotically increasing; but this cannot be the case if $K_{12}>0$ and $K_{22}<0$ or if $K_{21}>0$ and $K_{11}<0$. With $\det K<0$, we encounter a similar inconsistency if $K_{12}<0$ and $K_{22}>0$ or if $K_{21}<0$ and $K_{11}>0$. Based on this discussion, we conclude that *no* solution can be found in the following parameter ranges: (i) ($\sigma_1=\sigma_2=+1$, $\bar\beta_{12}<0$) or ($\sigma_1=-\sigma_2$, $\bar\beta_{12}>0$), with $\kappa_1>0$ and $\kappa_2>0$, (ii) ($\sigma_1=\sigma_2=-1$, $\bar\beta_{12}>0$) or ($\sigma_1=-\sigma_2$, $\bar\beta_{12}<0$), with $\kappa_1<0$ and $\kappa_2<0$, (iii) $\sigma_1=\sigma_2=\sigma(=\pm1)$, $K_{12}=-\sigma\bar\beta_{12}<0$, and at least one between $K_{11}(=-\sigma\bar\beta_{11})$ and $K_{22}(=-\sigma\bar\beta_{22})$ is positive, with $\kappa_1\kappa_2<0$, (iv) $\sigma_1\sigma_2=-1$, $\bar\beta_{12}\bar\beta_{22}>0$ or $\bar\beta_{12}\bar\beta_{11}>0$, with $\kappa_1\kappa_2<0$.

We have summarized our findings in Fig. 2. The shaded regions are those excluded on the basis of the above arguments; viz., there does not exist a nontrivial solution satisfying the self-duality equations if the parameters lie in these regions. Note that exactly a half of the parameter space is excluded.

Let us now discuss some explicit solutions to the self-

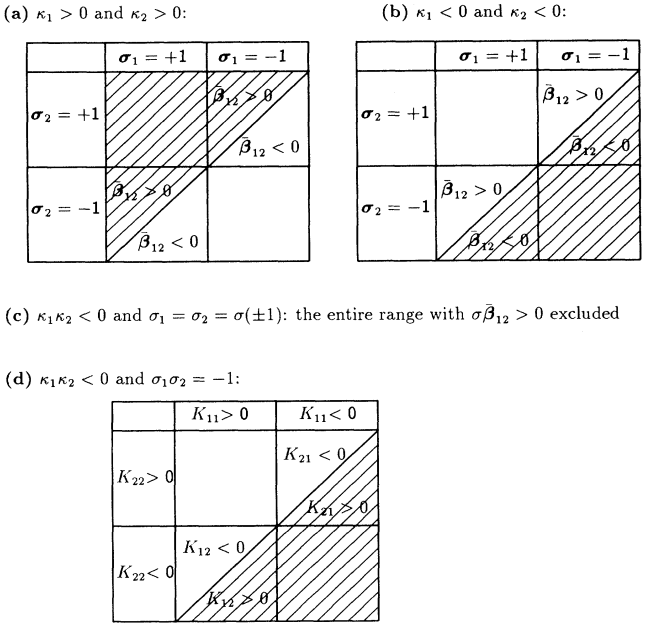


FIG. 2. For parameters lying in the shaded regions, there exists no solution to the self-duality equations.

duality equations in question. A glance at Eq. (3.14) shows that, with the ansatz $|\chi(\mathbf{r})|^2 = \gamma|\phi(\mathbf{r})|^2$ ($\gamma$: a positive constant), it can be reduced to a single equation for $|\phi|^2$ as long as $\gamma$ is chosen suitably. Indeed, as we make the choice

$$\gamma = \frac{K_{11} - K_{21}}{K_{22} - K_{12}}\,, \tag{3.22}$$

what follows from Eq. (3.14) is just the Liouville equation for $|\phi|^2$:

$$\nabla^2 \ln|\phi|^2 = \frac{(\det K)}{K_{12} - K_{22}}|\phi|^2\,. \tag{3.23}$$

To have nontrivial solutions, the coefficient of $|\phi|^2$ on the right-hand side of Eq. (3.23) should be negative as well as $\gamma > 0$. For these Liouville-type solutions, the fluxes are quantized, i.e., $\Phi_\phi = \Phi_\chi =$ (integer multiples of $2\pi\hbar c$). We note here that, because of the restrictions mentioned above, the parameter range allowing these Liouville-type solutions does not cover the entire unshaded region in Fig. 2. One may then ask the following questions.

(i) When the Liouville-type solutions exist, are there additional solutions distinct from the Liouville-type? If so, should they have quantized flux values always?

(ii) When the parameters are such that no Liouville-type solution exists, will there be some solutions to Eq. (3.14) after all?

These issues are discussed below.

On question (i), the existence of more general types of solutions is confirmed by the index theorem, and our numerical study further shows that there are also solutions with nonquantized flux values. The result of the index theorem analysis (see Appendix B for details) is as follows: The number of free parameters in the general solution, with the field $\phi$ ($\chi$) having vorticity $n_1$ ($n_2$) and the asymptotic behavior $\sim \frac{1}{r^{\alpha_1}}$ ($\sim \frac{1}{r^{\alpha_2}}$), is equal to $2(n_1+\hat{\alpha}_1)+2(n_2+\hat{\alpha}_2)$, where $\hat{\alpha}$ denotes the largest integer less than $\alpha$. Here the 'vorticity' is equal to the total number of zeros in a specific matter field. By applying this to the case when the Liouville-type solutions appropriate to the values $n_1 = n_2 = n$ and $\alpha_1 = \alpha_2 = n+2$ are allowed, we immediately see that the corresponding general solution should contain $8(n+1)$ free parameters. But this is precisely twice the number of free parameters entering the Liouville-type solutions [20]. A plausible conjecture is that, in a general solution, we no longer have the restriction (satisfied by all Liouville-type solutions) that the zeros of $\phi$ be also the zeros of $\chi$. Furthermore, we found numerically that there exist also solutions with the asymptotic behaviors not given by integral power falloff; i.e., $\alpha_1$ and $\alpha_2$ above need not be integers but vary continuously. For an example, see the next-to-next paragraph [especially the discussion after (3.26)]. In view of the relations $\Phi_p = \pm 2\pi\hbar c(n_1+\alpha_1)$ and $\Phi_\chi = \pm 2\pi\hbar c(n_2+\alpha_2)$, these solutions will then have *nonquantized* fluxes.[4] [In this regard, see also the comment immediately after Eq. (3.36).]

In the parameter range where no Liouville-type solution exists, analytical means are not available at present and we resorted to numerical analysis (assuming the rotationally symmetric form). For some values of parameters we succeeded in finding solutions while, for other values, no acceptable solution could be found. It seems that solutions exist in a large portion of this parameter range also, but the precise criterion on the parameters to have solutions is not clear yet. We also note that if a particular non-Liouville-type solution, say,

$$\begin{pmatrix} |\tilde{\phi}(z,z^*)| \\ |\tilde{\chi}(z,z^*)| \end{pmatrix},$$

is known, a family of new solutions may be obtained by considering its conformal transformation [12], i.e.,

$$\begin{pmatrix} |\varphi'(z)||\tilde{\phi}(\varphi(z),\varphi^*(z^*))| \\ |\varphi'(z)||\tilde{\chi}(\varphi(z),\varphi^*(z^*))| \end{pmatrix}$$

for a polynomial function $\varphi(z)$.

Now, as a specific example, we will discuss solutions in the self-dual system with $\beta_{11} = \beta_{22} = 0$ but nonzero $\beta_{12}$ (i.e., keep only mutual statistical interaction). This can be realized by the choice

$$\kappa_1 = -\kappa_2 \equiv \kappa\,, \qquad q_1^1 = q_1^2 \equiv \frac{1}{\sqrt{2}}q\,, \qquad q_2^1 = -q_2^2 \equiv q'\,, \tag{3.24}$$

and then $\beta_{12} = \frac{qq'}{\kappa}$. The self-duality equations now read

[4]Recently, some authors [21] argued that the flux (or charge) for the solutions of the Liouville equation is quantized because of the inversion symmetry. This is misleading, however. Rather, we might as well say that the inversion transformation generates *nonsingular* solutions because the charges happen to be quantized (and, correspondingly, the field has integral power falloff asymptotically). No useful information is gained by considering the inversion symmetry in our case.

$$\nabla^2 \ln|\phi|^2 = -\frac{2}{\hbar c}\frac{qq'}{\kappa}|\chi|^2 ,$$
$$\nabla^2 \ln|\chi|^2 = -\frac{2}{\hbar c}\frac{qq'}{\kappa}|\phi|^2 , \tag{3.25}$$

choosing $\sigma_1 = \sigma_2 = -1$ so that solutions may exist. Note that if the theory is rewritten in terms of the fields $v_\mu^{(1)}$ and $v_\mu^{(2)}$ defined by Eq. (2.36), this system is described by the Lagrangian density

$$\mathcal{L} = \kappa\epsilon^{\mu\nu\lambda} v_\mu^{(1)}\partial_\nu v_\lambda^{(2)} + i\hbar\phi^* \left(\frac{\partial}{\partial t} - \frac{i}{\hbar} q v_0^{(1)}\right)\phi + i\hbar\chi^*\left(\frac{\partial}{\partial t} - \frac{i}{\hbar}q' v_0^{(2)}\right)\chi$$
$$-\frac{\hbar^2}{2m_1}\left|\left(\nabla - \frac{i}{\hbar c} q\mathbf{v}^{(1)}\right)\phi\right|^2 - \frac{\hbar^2}{2m_2}\left|\left(\nabla - \frac{i}{\hbar c}q'\mathbf{v}^{(2)}\right)\chi\right|^2 + \left(\frac{\hbar}{2m_1 c} + \frac{\hbar}{2m_2 c}\right)\frac{qq'}{\kappa}|\phi|^2|\chi|^2 . \tag{3.26}$$

Equations (3.25) clearly admit the Liouville-type solutions, which are based on the ansatz $|\phi(\mathbf{r})| = |\chi(\mathbf{r})|$. In addition, we have found some non-Liouville-type solutions numerically, assuming that the fields $\phi(\mathbf{r})$ and $\chi(\mathbf{r})$ are functions of $r$ only. These are shown in Fig. 3. In Fig. 3(a), a plot is given for a solution corresponding to $n_1 = n_2 = 0$ (i.e., zero vorticity for both $\phi$ and $\chi$), the asymptotic behavior of which is determined as $|\phi| \sim r^{-1.34}$ and $|\chi| \sim r^{-3.95}$. Evidently, the fluxes are not quantized for this solution. Another plot now for a solution corresponding to $n_1 = 0$ and $n_2 = 1$ is given in Fig. 3(b).

Finally, note that the charges $(Q_\phi, Q_\chi)$ of a vortex soliton with given fluxes $(\Phi_\phi, \Phi_\chi)$ are determined by Eq. (3.19), and they will be in general nonzero. This prompts one to make a conjecture that the vortices described above experience fractional and also mutual statistical interactions analogous to those experienced by the elementary quanta in the theory. The most direct approach to settle this question is to derive the effective Lagrangian which is relevant to the slow dynamics of these vortices [22]. But, in our case, this program is nontrivial especially because a multivortex-type solution cannot be interpreted in an unambiguous way as representing a collection of some elementary vortices here. So it remains as an open problem.

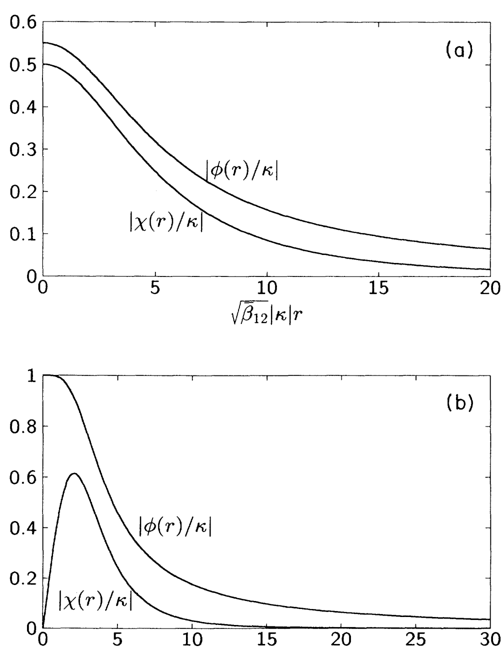


FIG. 3. Non-Liouville-type solutions in the case of $\beta_{11} = \beta_{22} = 0$. (a) The plot of a solution with vorticities $n_1 = n_2 = 0$. We have here chosen $\phi(0) = 0.55$ and $\chi(0) = 0.5$. (b) The plot of a solution with vorticities $n_1 = 0$ and $n_2 = 1$.

### B. The case with $B^{\text{ex}} = 0$ and $\Delta = \bar\beta_{11}\bar\beta_{22} - \bar\beta_{12}^2 = 0$

If $\bar\beta_{11}\bar\beta_{22}$ is equal to $\bar\beta_{12}^2$, two CS gauge fields are no longer independent and hence we are led to a self-dual system with two matter fields coupled to a single CS gauge field. In fact, we may now write $D_i\phi = (\nabla_i - \frac{i}{\hbar c}q_1' a_i)\phi$, $D_i\chi = (\nabla_i - \frac{i}{\hbar c}q_2' a_i)\chi$, and $\bar\beta_{ij} = \frac{2}{\hbar c}\frac{q_i' q_j'}{\kappa}$ $(i, j = 1, 2)$. Here we find it convenient to write Eq. (3.13) using the couplings $q_i \equiv \sqrt{\frac{2}{\hbar c|\kappa|}}q_i'$ rather than $\bar\beta_{ij}$, viz.,

$$\nabla^2 \ln|\phi|^2 = -\sigma_1 q_1(q_1|\phi|^2 + q_2|\chi|^2) ,$$
$$\nabla^2 \ln|\chi|^2 = -\sigma_2 q_2(q_1|\phi|^2 + q_2|\chi|^2) , \tag{3.27}$$

where we have chosen $\kappa < 0$ (with no loss of generality). Evidently, if $q_1$ and $q_2$ are of the same sign (i.e., $q_1 q_2 > 0$), these equations will admit a bounded solution only for $\sigma_1 = \sigma_2 = +1$. For $q_1 q_2 < 0$, on the other hand, one can have a solution only with $\sigma_1\sigma_2 = -1$. To show this, suppose that $\sigma_1 = \sigma_2 = 1$. Then, from Eq. (3.27) (with the suppressed $\delta$-function terms put in), we have

$$\nabla^2 \ln|\phi|^2 - \frac{q_1}{q_2}\nabla^2\ln|\chi|^2 = 4\pi\sum_{r=1}^{n_1}\delta^2(\mathbf{r} - \mathbf{R}_r)$$
$$-4\pi\left(\frac{q_1}{q_2}\right)\sum_{r=1}^{n_2}\delta^2(\mathbf{r} - \mathbf{R}_r') , \tag{3.28}$$

and hence

$$\frac{\phi}{\chi^{q_1/q_2}} = C\frac{\prod_{r=1}^{n_1}(z-Z_r)}{\prod_{r=1}^{n_2}(z-Z'_r)^{q_1/q_2}}$$
$$(C: \text{a complex number}). \quad (3.29)$$

But with $q_1q_2 < 0$, this is impossible: The left-hand side of Eq. (3.29) vanishes asymptotically, while its right-hand side clearly does not. The situation is analogous for $\sigma_1 = \sigma_2 = -1$. Hence the choice $\sigma_1\sigma_2 = -1$ is appropriate in the case of $q_1q_2 < 0$.

Incidentally, the above consideration is sufficient to show that there is no solution with finite charge if the matrix $\widetilde{K}$, defined in Eq. (3.16), is equal to the $2\times 2$ affine Cartan matrix $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 2 & -1 \\ -4 & 2\end{smallmatrix}\right)$. [This corresponds to the case for which Eq. (3.16) can be reduced to the sinh-Gordon or Bullough-Dodd equation, as noted in Ref. [12].] Indeed, for these particular forms for $\widetilde{K}$, we have $\sigma_1\sigma_2 > 0$, and then, by the above consideration, a bounded solution may be possible only with $\sigma_1 = \sigma_2 = +1$. So choose $\sigma_1 = \sigma_2 = +1$ and then we find the matrix $(\bar\beta_{pp'})$ described by the one-parameter family of singular matrices, $a\left(\begin{smallmatrix} -2 & 2 \\ 2 & -2\end{smallmatrix}\right)$ or $a\left(\begin{smallmatrix} -2 & 1 \\ 1 & -\frac{1}{2}\end{smallmatrix}\right)$. When translated into the above notation, this matrix form leads to $q_1q_2 < 0$ (i.e., the wrong sign); hence, there should not be any bounded solution.

Let us now study soliton solutions for some representative cases. First we consider the system with $q_1 = q_2 \equiv q$ (and $\sigma_1 = \sigma_2 = +1$, of course), so that Eq. (3.27) may assume the form

$$\begin{aligned}\nabla^2 \ln|\phi|^2 &= -q^2(|\phi|^2 + |\chi|^2)\,,\\ \nabla^2 \ln|\chi|^2 &= -q^2(|\phi|^2 + |\chi|^2)\,.\end{aligned} \quad (3.30)$$

The corresponding system has additional global SU(2) symmetry and may be viewed as a nonrelativistic version of a special self-dual model considered by Kim [23]. By exploiting this global SU(2) symmetry, a series of exact solutions, which are different from the Liouville-type solutions (obtained under the ansatz $|\chi|^2 = \gamma|\phi|^2$), can be obtained. Specifically, we found that the coupled equations in Eq. (3.30) have also the solutions of the type

$$\begin{pmatrix}\phi \\ \chi\end{pmatrix} = \frac{\sqrt{12}(P(z)Q'(z) - Q(z)P'(z))}{|q|(|P(z)|^2 + |Q(z)|^2)^{3/2}}\begin{pmatrix}P(z) \\ Q(z)\end{pmatrix}, \quad (3.31)$$

where $P(z)$ and $Q(z)$ are arbitrary polynomials of $z$ under the restriction that these two functions share no common zero. Note that this solution in Eq. (3.31) does not satisfy the Liouville equation and includes no rotationally symmetric configuration since $P(z)$ and $Q(z)$ have no common zero. The fluxes $\Phi_\phi$ and $\Phi_\chi$ for these solutions are quantized as in the Liouville-type solutions; in detail, for the solution (3.31), we have $\Phi_\phi = \Phi_\chi = 2\pi\hbar c(3n_P + 3n_Q)$ if $P(z)$ $[Q(z)]$ is an $n_P$ $(n_Q)$th order polynomial. [Since the vorticities of $\phi$ and $\chi$ are equal to $n_1 = 2n_P + n_Q - 1$ and $n_2 = n_P + 2n_Q - 1$, this may also be written as $\Phi_\phi = \Phi_\chi = 2\pi\hbar c(n_1 + n_2 + 2)$.] However, using the index theorem argument (see Appendix B), we know that there must be solutions other than these two types. We are also not sure whether or not the fluxes are necessarily quantized for *all* bounded solutions to Eq. (3.30).

As another case, we choose $q_2/q_1 = -2$ and $\sigma_1 = -\sigma_2 = +1$. Then, according to the same procedure which led to Eq. (3.29), we have

$$\frac{\chi^*}{\phi^2} = C^*\frac{\prod_{r=1}^{n_2}(z-Z'_r)}{\prod_{r=1}^{n_1}(z-Z_r)^2} \quad (C\text{: a complex number}) \quad (3.32)$$

and inserting this into Eq. (3.27) yields a single equation for $|\phi|$:

$$\nabla^2 \ln|\phi|^2 = -q_1^2\left\{|\phi|^2 - 2|C|^2\frac{\prod_{r=1}^{n_2}|z-Z'_r|^2}{\prod_{r=1}^{n_1}|z-Z_r|^4}|\phi|^4\right\}. \quad (3.33)$$

If we here restrict ourselves to the special case

$$\frac{\chi^*}{\phi^2} = C^*\,, \quad (3.34)$$

Eq. (3.33) is simplified as

$$\nabla^2 \ln|\phi|^2 = -q_1^2|\phi|^2(1 - 2|C|^2|\phi|^2)\,. \quad (3.35)$$

This is identical to the equation encountered in the relativistic self-dual CS Higgs system of Refs. [24,25]. For the latter system, there are now rigorous existence proofs [26] for both topological and nontopological soliton solutions. Only the nontopological ones are relevant in our case, for the other class leads to infinite charge. In particular, a rotationally symmetric nontopological soliton solution has the behaviors

$$|\phi(\mathbf{r})| \sim \begin{cases} r^n & \text{(for some non-negative integer } n\text{)} \\ & \text{as } r \to 0, \\ \frac{1}{r^\alpha} & \text{(with } \alpha > n+2\text{) as} \quad r\to\infty, \end{cases} \quad (3.36)$$

and for this solution we find the (nonquantized) fluxes $\Phi_\phi = 2\pi\hbar c(n+\alpha)$ and $\Phi_\chi = -2\pi\hbar c(2n + 2\alpha)$. This is another evidence for our assertion that quantized flux values are not to be expected generally.

### C. The case with $B^{\text{ex}} \neq 0$

As a uniform external magnetic field is turned on, a spontaneously broken vacuum becomes possible and correspondingly we might then have nontrivial solutions to Eq. (3.13) in the form of *topological* solitons. With a single matter field, an analogous phenomenon was noticed in Ref. [17]. Assuming $\Delta \equiv \bar\beta_{11}\bar\beta_{22} - \bar\beta_{12}^2 \neq 0$ (the $\Delta = 0$ case is considered later), the asymptotic values of $|\phi(\mathbf{r})|$ and $|\chi(\mathbf{r})|$ for a topological soliton solution should be equal to

$$v_\phi = \sqrt{B^{\text{ex}}(\bar\beta_{22}\bar e_1 - \bar\beta_{12}\bar e_2)/\Delta}\,,$$
$$v_\chi = \sqrt{B^{\text{ex}}(-\bar\beta_{12}\bar e_1 + \bar\beta_{11}\bar e_2)/\Delta}\,. \tag{3.37}$$

Using these vacuum values, we may now rewrite the self-duality equations as [cf. Eq. (3.14)]

$$\nabla^2 \begin{pmatrix} \ln|\phi|^2 \\ \ln|\chi|^2 \end{pmatrix} = -K \begin{pmatrix} |\phi|^2 - v_\phi^2 \\ |\chi|^2 - v_\chi^2 \end{pmatrix}, \tag{3.38}$$

with the same matrix $K$ as in Eq. (3.15). In case a topological soliton solution is allowed, it will be subject to the topological quantization conditions of the form

$$\int d^2\mathbf{r}\ \left[\epsilon^{ij}\nabla_i(q_1^1 a_j^1 + q_1^2 a_j^2) + e_1 B^{\text{ex}}\right] = 2\pi\hbar c n_1\,,$$
$$\int d^2\mathbf{r}\ \left[\epsilon^{ij}\nabla_i(q_2^1 a_j^1 + q_2^2 a_j^2) + e_2 B^{\text{ex}}\right] = 2\pi\hbar c n_2\,, \tag{3.39}$$

where $n_1$ and $n_2$ are integers.

In view of Eq. (3.37), one may hope to find a topological soliton solution only when the parameters of the theory satisfy certain restrictions; namely, for $\Delta > 0$ (and hence $\kappa_1\kappa_2 > 0$), one must have

$$B^{\text{ex}}(\bar\beta_{22}\bar e_1 - \bar\beta_{12}\bar e_2) > 0\,, \qquad B^{\text{ex}}(-\bar\beta_{12}\bar e_1 + \bar\beta_{11}\bar e_2) > 0\,, \tag{3.40}$$

while, for $\Delta < 0$ (and hence $\kappa_1\kappa_2 < 0$), the inequality signs in Eq. (3.40) should be reversed. Aside from these, there must be some conditions involving the elements of the matrix $K$ mainly. We can obtain such conditions by studying Eq. (3.38) in the asymptotic region. For this asymptotic analysis, we set

$$f_\phi(\mathbf{r}) = \frac{1}{v_\phi^2}(v_\phi^2 - |\phi(\mathbf{r})|^2)\,, \quad f_\chi(\mathbf{r}) = \frac{1}{v_\chi^2}(v_\chi^2 - |\chi(\mathbf{r})|^2)\,, \tag{3.41}$$

and may instead study the linearized form of Eq. (3.38), i.e.,

$$\nabla^2 \begin{pmatrix} f_\phi \\ f_\chi \end{pmatrix} = -L \begin{pmatrix} f_\phi \\ f_\chi \end{pmatrix}, \quad L = \begin{pmatrix} v_\phi^2 K_{11} & v_\chi^2 K_{12} \\ v_\phi^2 K_{21} & v_\chi^2 K_{22} \end{pmatrix}. \tag{3.42}$$

Now suppose that the $2\times2$ matrix $L$ can be diagonalized, i.e., $SLS^{-1} = \left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$ for some nonsingular matrix $S$. Then Eq. (3.42) can be cast as

$$\nabla^2 \begin{pmatrix} \tilde f_\phi \\ \tilde f_\chi \end{pmatrix} = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde f_\phi \\ \tilde f_\chi \end{pmatrix},$$
$$\begin{pmatrix} \tilde f_\phi \\ \tilde f_\chi \end{pmatrix} \equiv S \begin{pmatrix} f_\phi \\ f_\chi \end{pmatrix}. \tag{3.43}$$

Here the eigenvalues $\lambda_1$ and $\lambda_2$, which have crucial importance in determining the asymptotic behaviors, are the roots of the secular equation

$$\begin{aligned} 0 &= \det(L - \lambda I) \\ &= \lambda^2 - (v_\phi^2 K_{11} + v_\chi^2 K_{22})\lambda + v_\phi^2 v_\chi^2 (\det K)\,. \end{aligned} \tag{3.44}$$

Now take the case of $\kappa_1\kappa_2 > 0$ (and so $\Delta > 0$). In this case, it is easy to show that Eq. (3.44) has necessarily two real roots. Then note that, for an acceptable soliton solution, $f_\phi$ and $f_\chi$ above should approach zero asymptotically in such a way that the resulting soliton may have finite energy. This requires both roots to be negative (i.e., $\lambda_1 < 0$ and $\lambda_2 < 0$), and in view of Eq. (3.44), this translates into the conditions

$$v_\phi^2 K_{11} + v_\chi^2 K_{22} \equiv v_\phi^2 \sigma_1 \beta_{11} + v_\chi^2 \sigma_2 \beta_{22} < 0\,,$$
$$\det K = \sigma_1\sigma_2\Delta > 0\,. \tag{3.45}$$

Based on these, we find that the appropriate choice for a nontrivial soliton solution is

$$\sigma_1 = \sigma_2 = -1 \quad \text{if } \kappa_1 > 0 \text{ and } \kappa_2 > 0,$$
$$\sigma_1 = \sigma_2 = +1 \quad \text{if } \kappa_1 < 0 \text{ and } \kappa_2 < 0\,. \tag{3.46}$$

A similar analysis may be repeated for the case of $\kappa_1\kappa_2 < 0$. For the latter case, however, the roots of Eq. (3.44) are not always real and this introduces a certain uncertain feature in the analysis. Nevertheless, for $\kappa_1\kappa_2 < 0$, we can make the following definite statement: *no* solution exists with $\sigma_1\sigma_2 = +1$.

In addition to the above, certain (plausible) conditions can also be derived on the basis of the conjecture that a regular soliton solution, if it exists, is likely to satisfy the inequalities

$$|\phi(\mathbf{r})| - v_\phi < 0\,, \qquad |\chi(\mathbf{r})| - v_\chi < 0, \tag{3.47}$$

at least for sufficiently large $r$. Then, accepting this behavior, it is possible to apply the same reasoning as in the case of $B^{\text{ex}} = 0$ (see Sec. III A). For instance, Eq. (3.38) will be inconsistent with the assumed asymptotic behavior of $|\phi(\mathbf{r})|$ and $|\chi(\mathbf{r})|$ if the matrix $K$ is strictly positive; hence, no solution to Eq. (3.38) if $K_{pp'} = \sigma_p\bar\beta_{pp'} > 0$ for every $p$, $p'$. Also, by proceeding as in Eq. (3.21), we expect that a nontrivial solution may exist only under the conditions

$$(\det K)\{n_1 K_{22} - n_2 K_{12}\} < 0\,,$$
$$(\det K)\{-n_1 K_{21} + n_2 K_{11}\} < 0\,, \tag{3.48}$$

where $n_1$ ($n_2$) denotes the vorticity of the field $\phi$ ($\chi$).

We do not know of any analytic method developed to study the system in Eq. (3.38), even for some special $K$. We thus looked for numerical solutions to Eq. (3.38), while assuming the rotationally symmetric field configurations. Here it should suffice to say that, at least for certain choices of parameters (and vorticities) which satisfy the conditions given above, we did confirm the existence of regular topological soliton solutions. Note that, in view of the index theorem (see Appendix B), the existence of a particular solution with vorticities $n_1$ and $n_2$ actually implies the existence of a $2(n_1+n_2)$ parameter family of soliton solutions.

If $\Delta$ happens to vanish, we again have a self-dual system with two matter fields coupled to a single CS gauge field. Using the same notation as in Sec. III B, it will be possible to rewrite Eq. (3.38) (with the choice $\kappa<0$) as

$$\nabla^2 \ln|\phi|^2 = -\sigma_1\left[q_1(q_1|\phi|^2+q_2|\chi|^2)-\bar{e}_1B^{\rm ex}\right], \qquad \nabla^2 \ln|\chi|^2 = -\sigma_2\left[q_2(q_1|\phi|^2+q_2|\chi|^2)-\bar{e}_2B^{\rm ex}\right]. \tag{3.49}$$

A precondition to have a topological soliton solution is the existence of a nontrivial (uniform) vacuum solution, and in the present case this will be true only when the coupling parameters satisfy the relation

$$\frac{e_1}{q_1}=\frac{e_2}{q_2}. \tag{3.50}$$

Here a particularly simple case is obtained for $e_1=e_2=e$, and then, thanks to Eq. (3.50), $q_1=q_2=q$. For this special case, the system has in fact a global SU(2) symmetry and this is also manifest in the self-duality equations

$$\nabla^2 \ln|\phi|^2 = -\sigma_1\left[q^2(|\phi|^2+|\chi|^2)-\bar{e}B^{\rm ex}\right], \qquad \nabla^2 \ln|\chi|^2 = -\sigma_2\left[q^2(|\phi|^2+|\chi|^2)-\bar{e}B^{\rm ex}\right]. \tag{3.51}$$

This system admits a topological soliton solution if we choose $\sigma_1=\sigma_2=-1$ and $eB^{\rm ex}>0$. Then Eq. (3.51) becomes identical to the self-duality equations found in the relativistic self-dual Ginzburg-Landau model with the so-called semilocal symmetry [27]. General solutions for this model are discussed in those papers to which readers are referred.

## IV. RELATIVISTIC GENERALIZATION

Here we will introduce the relativistic self-dual U(1)×U(1) CS system (with no external magnetic field, for simplicity) and then study the static soliton solutions in the model. If one wishes, one may view this investigation as a direct generalization of the model considered in Ref. [24]. Our model contains two complex scalar fields $\phi$ and $\chi$, and is described by the Lagrangian density

$$\mathcal{L}=\frac{\kappa_1}{2}\epsilon^{\mu\nu\lambda}a^1_\mu\partial_\nu a^1_\lambda+\frac{\kappa_2}{2}\epsilon^{\mu\nu\lambda}a^2_\mu\partial_\nu a^2_\lambda -|D_\mu\phi|^2-|D_\mu\chi|^2-U(\phi,\chi), \tag{4.1}$$

where

$$D_\mu\phi\equiv[\partial_\mu-i(q_1^1a^1_\mu+q_1^2a^2_\mu)]\phi, \qquad D_\mu\chi\equiv[\partial_\mu-i(q_2^1a^1_\mu+q_2^2a^2_\mu)]\chi, \tag{4.2}$$

and $U(\phi,\chi)$ is to be chosen shortly. (In this section, we set $c=\hbar=1$.) The Gauss laws read

$$b^1\equiv\epsilon^{ij}\nabla_i a^1_j=-\frac{1}{\kappa_1}(q_1^1J^0_\phi+q_2^1J^0_\chi), \qquad b^2\equiv\epsilon^{ij}\nabla_i a^2_j=-\frac{1}{\kappa_2}(q_1^2J^0_\phi+q_2^2J^0_\chi), \tag{4.3}$$

with the currents $J^\mu_\phi\equiv -i[\phi^*D^\mu\phi-(D^\mu\phi)^*\phi]$ and $J^\mu_\chi\equiv -i[\chi^*D^\mu\chi-(D^\mu\chi)^*\chi]$.

The static energy functional is given by

$$\begin{aligned}E&=\int d^2\mathbf{r}\,\{|D_0\phi|^2+|D_0\chi|^2+|D_i\phi|^2+|D_i\chi|^2+U(\phi,\chi)\}\\&=\int d^2\mathbf{r}\left\{|D_i\phi|^2+|D_i\chi|^2+\frac{1}{4|\phi|^2}\left[\frac{q_2^2\kappa_1b^1-q_2^1\kappa_2b^2}{q_1^1q_2^2-q_2^1q_1^2}\right]^2+\frac{1}{4|\chi|^2}\left[\frac{q_1^2\kappa_1b^1-q_1^1\kappa_2b^2}{q_1^1q_2^2-q_2^1q_1^2}\right]^2+U(\phi,\chi)\right\},\end{aligned} \tag{4.4}$$

where, on the second line, we have used the following relations [derived from the Gauss laws (4.3), assuming time-independent fields]:

$$q_1^1a_0^1+q_1^2a_0^2=-\frac{(q_2^2\kappa_1)b^1-(q_2^1\kappa_2)b^2}{2(q_1^1q_2^2-q_2^1q_1^2)|\phi|^2}, \qquad q_2^1a_0^1+q_2^2a_0^2=-\frac{(q_1^2\kappa_1)b^1-(q_1^1\kappa_2)b^2}{2(q_1^1q_2^2-q_2^1q_1^2)|\chi|^2}. \tag{4.5}$$

Now suppose that the potential $U$ has the form

$$U=|\phi|^2\left\{\sigma_1\beta_{11}(|\phi|^2-c^2)+\sigma_2\beta_{12}(|\chi|^2-c'^2)\right\}^2 +|\chi|^2\left\{\sigma_1\beta_{21}(|\phi|^2-c^2)+\sigma_2\beta_{22}(|\chi|^2-c'^2)\right\}^2, \tag{4.6}$$

where $\sigma_p=+1$ or $-1$ $(p=1,2)$, the $\beta_{pp'}$'s are defined as in Eq. (2.15) [i.e., $\beta_{11}=\frac{(q_1^1)^2}{\kappa_1}+\frac{(q_1^2)^2}{\kappa_2}$, $\beta_{12}=\frac{q_1^1q_2^1}{\kappa_1}+\frac{q_1^2q_2^2}{\kappa_2}$, etc.], and $c$, $c'$ are arbitrary constants. Then, with the help of the identities analogous to Eq. (3.6), it is straightforward to show that the energy functional in Eq. (4.4) can be rewritten as

$$\begin{aligned}E=\int d^2\mathbf{r}\Bigg\{&\frac{1}{4|\phi|^2}\left[\left(\frac{q_2^2\kappa_1b^1-q_2^1\kappa_2b^2}{q_1^1q_2^2-q_2^1q_1^2}\right)+2|\phi|^2\left\{\sigma_1\beta_{11}(|\phi|^2-c^2)+\sigma_2\beta_{12}(|\chi|^2-c'^2)\right\}\right]^2\\&+\frac{1}{4|\chi|^2}\left[\left(\frac{q_1^2\kappa_1b^1-q_1^1\kappa_2b^2}{q_1^1q_2^2-q_2^1q_1^2}\right)+2|\chi|^2\left\{\sigma_1\beta_{21}(|\phi|^2-c^2)+\sigma_2\beta_{22}(|\chi|^2-c'^2)\right\}\right]^2\\&+|D_1\phi+i\sigma_1D_2\phi|^2+|D_1\chi+i\sigma_2D_2\chi|^2+\sigma_1c^2(q_1^1b^1+q_1^2b^2)+\sigma_2c'^2(q_2^1b^1+q_2^2b^2)\Bigg\}.\end{aligned} \tag{4.7}$$

Hence, for the theory defined by the Lagrangian density (4.1) with the potential (4.6), there exists a Bogomol'nyi-type bound [28] for the static energy (which is non-negative):

$$E \geq \sigma_1 c^2 \Phi_\phi + \sigma_2 c'^2 \Phi_\chi , \tag{4.8}$$

where the fluxes $\Phi_\phi$ and $\Phi_\chi$ are defined as in Eq. (3.19). Since $E$ must be non-negative, this bound is meaningful only for a positive value of $\sigma_1 c^2 \Phi_\phi + \sigma_2 c'^2 \Phi_\chi$.

Of particular interest here are the solutions saturating the above Bogomol'nyi bound, which are realized if and only if all squared expressions appearing in the integrand of Eq. (4.7) vanish identically. This gives rise to the following self-duality equations

$$D_1\phi + i\sigma_1 D_2\phi = 0\,, \qquad D_1\chi + i\sigma_2 D_2\chi = 0 \tag{4.9}$$

and

$$\begin{aligned} b^1 = &-\frac{2}{\kappa_1} q_1^1 |\phi|^2 [\sigma_1\beta_{11}(|\phi|^2 - c^2) + \sigma_2\beta_{12}(|\chi|^2 - c'^2)] \\ &-\frac{2}{\kappa_2} q_2^1 |\chi|^2 [\sigma_1\beta_{21}(|\phi|^2 - c^2) + \sigma_2\beta_{22}(|\chi|^2 - c'^2)]\,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} b^2 = &-\frac{2}{\kappa_1} q_1^2 |\phi|^2 [\sigma_1\beta_{11}(|\phi|^2 - c^2) + \sigma_2\beta_{12}(|\chi|^2 - c'^2)] \\ &-\frac{2}{\kappa_2} q_2^2 |\chi|^2 [\sigma_1\beta_{21}(|\phi|^2 - c^2) + \sigma_2\beta_{22}(|\chi|^2 - c'^2)]\,. \end{aligned}$$

We expect that, for the parameters in some range at least, this system of equations admit both topological and nontopological soliton solutions just as in the model of Refs. [24,25]. To analyze these equations, we may again write $a_i^I(\mathbf{r})$ as in Eq. (3.8), and then, for the functions $f_1(\mathbf{r})$ and $f_2(\mathbf{r})$ defined by

$$\begin{aligned} \phi(\mathbf{r}) &= e^{-\sigma_1\{q_1^1 U^1 + q_1^2 U^2\}} f_1(\mathbf{r})\,, \\ \chi(\mathbf{r}) &= e^{-\sigma_2\{q_2^1 U^1 + q_2^2 U^2\}} f_2(\mathbf{r})\,, \end{aligned} \tag{4.11}$$

Eq. (4.9) reduces to the statement of complex analyticity, i.e., $f_1 = f_1(z_{(\sigma_1)})$ and $f_2 = f_2(z_{(\sigma_2)})$ with $z_{(\sigma_p)} = x + i\sigma_p y$. At the same time, we use Eq. (4.11) to express $b^1(\mathbf{r})$ and $b^2(\mathbf{r})$ in terms of $|\phi|$ and $|\chi|$, and then combine them with Eq. (4.10). The results are the equations[5]

$$\begin{aligned} \nabla^2 \ln|\phi|^2 = 4\sigma_1 \Big\{ &\beta_{11}|\phi|^2[\sigma_1\beta_{11}(|\phi|^2 - c^2) + \sigma_2\beta_{12}(|\chi|^2 - c'^2)] \\ &+\beta_{12}|\chi|^2[\sigma_1\beta_{21}(|\phi|^2 - c^2) + \sigma_2\beta_{22}(|\chi|^2 - c'^2)] \Big\} + 4\pi\sum_{r=1}^{n_1}\delta^2(\mathbf{r} - \mathbf{R}_r)\,, \\ \nabla^2 \ln|\chi|^2 = 4\sigma_2 \Big\{ &\beta_{21}|\phi|^2[\sigma_1\beta_{11}(|\phi|^2 - c^2) + \sigma_2\beta_{12}(|\chi|^2 - c'^2)] \\ &+\beta_{22}|\chi|^2[\sigma_1\beta_{21}(|\phi|^2 - c^2) + \sigma_2\beta_{22}(|\chi|^2 - c'^2)] \Big\} + 4\pi\sum_{r=1}^{n_2}\delta^2(\mathbf{r} - \mathbf{R}'_r)\,. \end{aligned} \tag{4.12}$$

We have assumed here that the fields $\phi$ and $\chi$ have zeros at $(\mathbf{R}_1, \ldots, \mathbf{R}_{n_1})$ and $(\mathbf{R}'_1, \ldots, \mathbf{R}'_{n_2})$, respectively. Note that there is a certain similarity between Eq. (4.12) and Eq. (3.13). This is not surprising since one can recover the model discussed in Sec. III as the nonrelativistic limit [11] of this relativistic theory (restricted to the nontopological soliton sector).

A general investigation on possible solutions to Eq. (4.12) is beyond the scope of this paper. We will below concentrate on a particularly interesting special case, the self-dual system with $\beta_{11} = \beta_{22} = 0$ and $\beta_{12} \neq 0$. Note that we studied the nonrelativistic model under the same condition in Eqs. (3.24)–(3.26). Choosing the parameters as in Eq. (3.24), Eq. (4.12) assumes a much simpler form, viz.,

$$\begin{aligned} \nabla^2 \ln|\phi|^2 &= \frac{4q^2q'^2}{\kappa^2}|\chi|^2(|\phi|^2 - c^2)\,, \\ \nabla^2 \ln|\chi|^2 &= \frac{4q^2q'^2}{\kappa^2}|\phi|^2(|\chi|^2 - c'^2)\,, \end{aligned} \tag{4.13}$$

with the $\delta$-function terms not written out explicitly. The Lagrangian density for this system reads

$$\begin{aligned} \mathcal{L} = &\kappa\epsilon^{\mu\nu\lambda} v_\mu^{(1)} \partial_\nu v_\lambda^{(2)} \\ &-|(\partial_\mu - iqv_\mu^{(1)})\phi|^2 - |(\partial_\mu - iq'v_\mu^{(2)})\chi|^2 \\ &-\frac{q^2q'^2}{\kappa^2}|\phi|^2(|\chi|^2 - c'^2)^2 \\ &-\frac{q^2q'^2}{\kappa^2}|\chi|^2(|\phi|^2 - c^2)^2, \end{aligned} \tag{4.14}$$

with the Gauss laws (for time-independent fields) given by

[5]Note that the $q$'s, $\kappa_1$, and $\kappa_2$ enter Eq. (4.12) only through the quantities $\beta_{pp'}$. This is an expected result even in the present relativistic case, for the given equations should remain the same under suitable linear transformations on the CS fields.

$$\epsilon^{ij}\nabla_i v_j^{(1)} = -\frac{q'}{\kappa}J_\chi^0 = \frac{2q'^2}{\kappa}v^{(2)0}|\chi|^2\,,$$
$$\epsilon^{ij}\nabla_i v_j^{(2)} = -\frac{q'}{\kappa}J_\phi^0 = \frac{2q^2}{\kappa}v^{(1)0}|\phi|^2\,. \tag{4.15}$$

There are two distinct classes of solutions to the self-duality equations. The first is a topological soliton with the asymptotic behavior

$$r\to\infty:\quad \frac{|\phi(\mathbf{r})|}{c}\to 1,\quad \frac{|\chi(\mathbf{r})|}{c'}\to 1, \tag{4.16}$$

and the fluxes are quantized for this solution, i.e., $\Phi_\phi = q\int d^2\mathbf{r}\,\epsilon^{ij}\nabla_i v_j^{(1)} = 2\pi n_1$ and $\Phi_\chi = q'\int d^2\mathbf{r}\,\epsilon^{ij}\nabla_i v_j^{(2)} = 2\pi n_2$ ($n_1$, $n_2$: integers). The second class is a nontopological soliton with the asymptotic behavior

$$r\to\infty:\quad |\phi(\mathbf{r})|\to\frac{\text{const}}{r^{\alpha_1}},\quad |\chi(\mathbf{r})|\to\frac{\text{const}}{r^{\alpha_2}} \tag{4.17}$$

($\alpha_1$ and $\alpha_2$ are real numbers larger than 1). For this nontopological soliton the fluxes are not quantized: We have here the formulas $\Phi_\phi = 2\pi(n_1+\alpha_1)$ and $\Phi_\chi = 2\pi(n_2+\alpha_2)$, when the field $\phi(\chi)$ has vorticity $n_1(n_2)$. A topological soliton with $\Phi_\phi = 2\pi n_1$ and $\Phi_\chi = 2\pi n_2$ may conveniently be visualized as an assembly of $|n_1|$ "$\phi$ vortices" with respective centers at the zeros of $\phi$ and $|n_2|$ "$\chi$ vortices" with respective centers at the zeros of $\chi$.

Note that, for the above soliton configurations, the charges $Q_\phi \equiv \int d^2\mathbf{r}\,J_\phi^0$ and $Q_\chi \equiv \int d^2\mathbf{r}\,J_\chi^0$ are simply related to the fluxes as

$$\Phi_\phi = -\frac{qq'}{\kappa}Q_\chi\,,\qquad \Phi_\chi = -\frac{qq'}{\kappa}Q_\phi\,, \tag{4.18}$$

due to the Gauss laws (4.15). This relationship suggests the existence of *mutual* statistical interaction between $\phi$ vortices and $\chi$ vortices; but an assembly of $\phi$ vortices (or $\chi$ vortices) only will show no peculiar statistical effect. This conclusion is further supported by calculating the angular momentum $J\equiv\int d^2\mathbf{r}\,\epsilon^{ij}x_iT^{0j}$, where $T^{0j}$ denotes the momentum density in the theory. In fact, at least for a spherically symmetric solution based on the form

$$\phi(\mathbf{r}) = f(r)e^{in_1\theta}\,,$$
$$\chi(\mathbf{r}) = g(r)e^{in_2\theta}\,,$$
$$qv_i^{(1)}(\mathbf{r}) = \epsilon^{ij}\frac{x^j}{r^2}[h^{(1)}(r)-n_1]\,,$$
$$q'v_i^{(2)}(\mathbf{r}) = \epsilon^{ij}\frac{x^j}{r^2}[h^{(2)}(r)-n_2], \tag{4.19}$$

with

$$h^{(1)}(\infty) = h^{(2)}(\infty) = 0\quad\text{(topological soliton)},$$
$$h^{(1)} = \alpha_2\,,\quad h^{(2)}(\infty) = \alpha_1\quad\text{(nontopological soliton)}, \tag{4.20}$$

a simple calculation gives the result

$$J = \begin{cases} -\dfrac{\pi\kappa}{qq'}n_1n_2 & \text{(topological)},\\[2mm] -\dfrac{\pi\kappa}{qq'}(n_1n_2-\alpha_1\alpha_2) & \text{(nontopological)}\,. \end{cases} \tag{4.21}$$

Thus individual $\phi$ or $\chi$ vortices do not carry angular momentum. On the other hand, a composite of a $\phi$ vortex and a $\chi$ vortex each has a nonvanishing $J$, and this is an anticipated result in a system with mutual statistical interaction.

The general solution to the given self-duality equations is difficult to obtain, although certain subclasses of solutions can be readily identified in terms of those of the previously known self-dual system. By evaluating the index of the differential operator associated with the appropriate fluctuation equation, the number of free parameters entering the general solution with given values of $\Phi_\phi$ and $\Phi_\chi$ is determined as

$$N = \begin{cases} 2n_1+2n_2 & \text{(topological)},\\ 2n_1+2\hat\alpha_1+2n_2+2\hat\alpha_2 & \text{(nontopological)}\,. \end{cases} \tag{4.22}$$

Here the general topological soliton solution with $n_2=0$ but $n_1\neq 0$ (or, if one wishes, with $n_1=0$ but $n_2\neq 0$) is easy to describe—one may set $\chi(\mathbf{r})=c'$ and, in view of Eq. (4.13), just choose $\phi(\mathbf{r})$ to be a solution to the familiar equation from the study of the Ginzburg-Landau-type model [28]:

$$\nabla^2\ln|\phi|^2 = \frac{4q^2q'^2c'^2}{\kappa^2}(|\phi|^2-c^2)\,. \tag{4.23}$$

In this case, it follows from Eq. (4.15) that $v^{(1)0}(\mathbf{r}) = v_j^{(2)}(\mathbf{r}) = 0$, while $v^{(2)0} = \frac{\kappa}{2q'^2c'}\epsilon^{ij}\nabla_i v_j^{(1)}(\mathbf{r})\neq 0$. Another subclass of topological or nontopological soliton solutions are obtained by setting $|\chi(\mathbf{r})|^2 = \left(\frac{c'}{c}\right)^2|\phi(\mathbf{r})|^2$, and this of course corresponds to the case with $n_1=n_2$ (and $\alpha_1=\alpha_2$). For the latter, the two equations in Eq. (4.13) collapse to one, namely,

$$\nabla^2\ln|\phi|^2 = \frac{4q^2q'^2c'^2}{\kappa^2c^2}|\phi|^2(|\phi|^2-c^2)\,, \tag{4.24}$$

the form of which matches precisely the corresponding equation encountered in the study of the 'minimal' self-dual CS Higgs model [24]. But this does not comprise the full general solution in the sector specified by $n_1=n_2$ (and $\alpha_1=\alpha_2$). The number of free parameters which enter the solution based on Eq. (4.24) (as calculated in Ref. [25]) is just a half of the value given in Eq. (4.22). This may be understood by observing that the basic unit in a solution satisfying the condition $|\chi|^2 = \left(\frac{c'}{c}\right)^2|\phi|^2$ is assumed by "a $\phi$ vortex *on top of* a $\chi$ vortex," while the index theorem suggests the existence of more general solution in which $\phi$ vortices and $\chi$ vortices serve as separate units.

## V. SUMMARY AND DISCUSSION

The precise nature of the Schrödinger quantum field theory with general $[\mathrm{U}(1)]^N$ CS interactions has been clarified, a novel feature being the existence of mutual statistical interactions between distinguishable particles. Then, for the corresponding self-dual models with two matter fields, we investigated the structure of classical soliton-type solutions to the static equations of motion, with or without uniform external magnetic field. In particular, to obtain a system which admits nontrivial soliton solutions satisfying the self-duality equations, we derived a set of necessary conditions for the parameters of the theory. While our self-duality equations reduce to the Toda-type equations or their generalizations, the matrix $K$ in the equations is not necessarily equal to the Cartan matrix of a certain Lie algebra. For some special cases we exhibited soliton solutions in a more explicit way. Soliton solutions in a relativistic self-dual system with two CS gauge fields were also discussed briefly. We conjectured that these solitons exhibit mutual (as well as fractional) statistics.

Some comments are in order. First of all, it is intriguing that the Toda equation retains some of its interesting mathematical properties (e.g., the existence of multisoliton solutions) even if its structure gets suitably modified. Aside from the fact that the matrix $K$ in Eq. (1.2) need not have a group-theoretical origin, we saw this phenomenon realized when we add constant terms on the right-hand side [as in Eq. (3.38)] and also quadratic terms in the densities [as in Eq. (4.10)]. Quite possibly, certain universal mathematical structures might exist behind all these models. Also desirable will be to clarify further various physical properties (e.g., statistics) of the vortex solitons discussed in this paper and to study their possible roles in the *real* physics of multilayered Hall media. Another fruitful line of research is the quantum-mechanical investigation of our model Hamiltonian in Eq. (3.2). We noted already that, by exploiting the supersymmetry in this system, it should be possible to find the corresponding *exact* ground state and also their degeneracy. Just as in the case of a one-layer system [9], this investigation might yield some valuable insight in understanding the multilayered fractional quantum Hall effects.

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## APPENDIX A

Here we will first explain how Eqs. (2.26) and (2.27) are derived and then go on to establish the expression (2.28). For the contribution $B$ defined by Eq. (2.21), one may repeatedly use the relations such as

$$\begin{aligned}\Psi_p(\mathbf{r}^{(p)}_{k-1},t)&\left(-\frac{\hbar^2}{2m_p}\left[\nabla_i^{(p,k)}-\frac{i}{\hbar c}\epsilon^{ij}\nabla_j^{(p,k)}\sum_{p'}\beta_{pp'}\int d^2\mathbf{r}'\,G(\mathbf{r}^{(p)}_k-\mathbf{r}')\rho_{p'}(\mathbf{r}',t)-\frac{i}{\hbar c}e_pA_i^{\mathrm{ex}}(\mathbf{r}^{(p)}_k,t)\right]^2\Psi_p(\mathbf{r}^{(p)}_k,t)\right)\\&=-\frac{\hbar^2}{2m_p}\left[\nabla_i^{(p,k)}-\frac{i}{\hbar c}\epsilon^{ij}\nabla_j^{(p,k)}\sum_{p'}\beta_{pp'}\int d^2\mathbf{r}'\,G(\mathbf{r}^{(p)}_k-\mathbf{r}')\rho_{p'}\right.\\&\qquad\left.-\frac{i}{\hbar c}\epsilon^{ij}\nabla_j^{(p,k)}\beta_{pp}G(\mathbf{r}^{(p)}_k-\mathbf{r}^{(p)}_{k-1})-\frac{i}{\hbar c}e_pA_i^{\mathrm{ex}}(\mathbf{r}^{(p)}_k,t)\right]^2\Psi_p(\mathbf{r}^{(p)}_{k-1},t)\Psi_p(\mathbf{r}^{(p)}_k,t),\end{aligned}\tag{A1}$$

which follows from the noncommutativity of $\rho_{p'}(\mathbf{r}',t)$ and $\Psi_p(\mathbf{r}^{(p)}_{k-1},t)$. Once all the field operators on the left of the squared differential operator are relocated to its right by this procedure, one readily recognizes the expression in Eq. (2.26) as a consequence of the definition (2.18) and the fact that $\langle 0|\rho_{p'}(\mathbf{r}',t)=0$. For the contribution $A$, more steps are necessary to derive the result (2.27) from the form in Eq. (2.20). Here, using the commutation relations (2.2) and (2.13), we first observe that

$$\begin{aligned}\Psi_p(\mathbf{r}^{(p)}_{k-1},t)&\left(\sum_{p'}\frac{\beta_{pp'}}{c}\epsilon^{ij}\nabla_j^{(p,k)}\int d^2\mathbf{r}'\,G(\mathbf{r}^{(p)}_k-\mathbf{r}')J_{p'i}(\mathbf{r}',t)\right)\\&=\left(-\frac{i\hbar}{m_pc}\beta_{pp}\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}^{(p)}_k-\mathbf{r}^{(p)}_{k-1})D_i^{(p,k-1)}\right.\\&\qquad-\sum_{p'}\frac{1}{m_{p'}c^2}\beta_{pp'}^2\epsilon^{ij}\nabla_j^{(p,k)}\int d^2\mathbf{r}'\,G(\mathbf{r}^{(p)}_k-\mathbf{r}')\epsilon^{il}\nabla_l^{(\mathbf{r}')}G(\mathbf{r}^{(p)}_{k-1}-\mathbf{r}')\Psi^\dagger_{p'}(\mathbf{r}',t)\Psi_{p'}(\mathbf{r}',t)\\&\qquad\left.+\sum_{p'}\frac{\beta_{pp'}}{c}\epsilon^{ij}\nabla_j^{(p,k)}\int d^2\mathbf{r}'\,G(\mathbf{r}^{(p)}_k-\mathbf{r}')J_{p'i}(\mathbf{r}',t)\right)\Psi_p(\mathbf{r}^{(p)}_{k-1},t),\end{aligned}\tag{A2}$$

where $D_i^{(p,k)}$ is defined as in the case of $\nabla_i^{(p,k)}$. If we further let the operator $\Psi_p(\mathbf{r}_{k-2}^{(p)},t)$ act on the expression (A2) from the left, the result can then be written as

$$\begin{aligned}&\left(-\frac{i\hbar}{m_pc}\beta_{pp}\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k-1}^{(p)})D_i^{(p,k-1)}-\frac{i\hbar}{m_pc}\beta_{pp}\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k-2}^{(p)})D_i^{(p,k-2)}\right.\\&-\frac{i\hbar}{m_pc^2}\beta_{pp}^2\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k-1}^{(p)})\epsilon^{il}\nabla_l^{(p,k-1)}G(\mathbf{r}_{k-1}^{(p)}-\mathbf{r}_{k-2}^{(p)})\\&-\frac{i\hbar}{m_pc^2}\beta_{pp}^2\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k-2}^{(p)})\epsilon^{il}\nabla_l^{(p,k-2)}G(\mathbf{r}_{k-2}^{(p)}-\mathbf{r}_{k-1}^{(p)})\\&-\sum_{p'}\frac{1}{m_{p'}c^2}\beta_{pp'}^2\epsilon^{ij}\nabla_j^{(p,k)}\int d^2\mathbf{r}'\,G(\mathbf{r}_k^{(p)}-\mathbf{r}')\epsilon^{il}\nabla_l^{(\mathbf{r}')}[G(\mathbf{r}_{k-1}^{(p)}-\mathbf{r}')+G(\mathbf{r}_{k-2}^{(p)}-\mathbf{r}')]\Psi_{p'}^\dagger(\mathbf{r}',t)\Psi_{p'}(\mathbf{r}',t)\\&\left.+\sum_{p'}\frac{\beta_{pp'}}{c}\epsilon^{ij}\nabla_j^{(p,k)}\int d^2\mathbf{r}'\,G(\mathbf{r}_k^{(p)}-\mathbf{r}')J_{p'i}(\mathbf{r}',t)\right)\Psi_p(\mathbf{r}_{k-2},t)\Psi_p(\mathbf{r}_{k-1}^{(p)},t),\end{aligned}\tag{A3}$$

where we have again used Eq. (2.13). Now it is not difficult to infer that as analogous steps are repeated all the way, the final result should be the expression (2.27).

To show that the sum of $A$, $B$, and $D$ can be expressed as in Eq. (2.28), we proceed as follows. We begin with the trivial observation

$$\sum_{(p',k')\neq(p,k)}=\sum_{(p',k')<(p,k)}+\sum_{(p',k')>(p,k)}$$

to cast the right-hand side of Eq. (2.28) into the form

$$\begin{aligned}B&+\sum_{(p,k)}\frac{i\hbar}{m_pc}\epsilon^{ij}\nabla_j^{(p,k)}\left(\sum_{(p',k')>(p,k)}\beta_{pp'}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right)\\&\times\left[\nabla_i^{(p,k)}-\frac{i}{\hbar c}\epsilon^{il}\nabla_l^{(p,k)}\left(\sum_{(p'',k'')<(p,k)}\beta_{pp''}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k''}^{(p'')})\right)-\frac{i}{\hbar c}e_pA_i^{\mathrm{ex}}(\mathbf{r}_k^{(p)},t)\right]\\&\times\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t)\\&+\sum_{(p,k)}\sum_{(p',k')>(p,k)}\sum_{(p'',k'')>(p,k)}\frac{1}{2m_pc^2}\beta_{pp'}\beta_{pp''}\left(\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right)\\&\times\left(\epsilon^{il}\nabla_l^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k''}^{(p'')})\right)\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t).\end{aligned}\tag{A4}$$

The second term in this expression can then be rewritten as

$$\begin{aligned}-\sum_{(p,k)}\sum_{(p',k')<(p,k)}&\frac{i\hbar}{m_{p'}c}\epsilon^{ij}\nabla_j^{(p,k)}\beta_{pp'}G(\mathbf{r}_{k'}^{(p')}-\mathbf{r}_k^{(p)})\\&\times\left[\nabla_i^{(p',k')}-\frac{i}{\hbar c}\epsilon^{ij}\nabla_j^{(p',k')}\left(\sum_{(p'',k'')<(p',k')}\beta_{p'p''}G(\mathbf{r}_k^{(p')}-\mathbf{r}_{k''}^{(p'')})\right)-\frac{i}{\hbar c}e_pA_i^{\mathrm{ex}}(\mathbf{r}_{k'}^{(p')},t)\right]\\&\times\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t),\end{aligned}\tag{A5}$$

while the last term is equal to

$$\begin{aligned}&\sum_{(p,k)}\sum_{(p',k')<(p,k)}\frac{1}{2m_{p'}c^2}\beta_{p'p}^2\left[\epsilon^{ij}\nabla_j^{(p',k')}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right]^2\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t)\\&-\sum_{(p,k)}\sum_{(p',k')<(p,k)}\sum_{\substack{(p'',k'')>(p',k')\\(p'',k'')>(p,k)}}\frac{1}{m_{p'}c^2}\beta_{p'p}\beta_{p'p''}\left(\epsilon^{ij}\nabla_j^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k'}^{(p')})\right)\\&\qquad\times\left(\epsilon^{il}\nabla_l^{(p,k)}G(\mathbf{r}_k^{(p)}-\mathbf{r}_{k''}^{(p'')})\right)\Phi(\mathbf{r}_1^{(1)},\ldots,\mathbf{r}_{n_M}^{(M)},t).\end{aligned}\tag{A6}$$

The first term in Eq. (A6) evidently coincides with the contribution $D$ shown in Eq. (2.25). On the other hand, the second term in Eq. (A6) and the expression (A5) combine to yield the contribution $A$ in Eq. (2.27). Equation (2.28) has been established now.

## APPENDIX B

As regards the solutions to the self-duality equations (3.7a) and (3.7b), we will here present the index-theorem analysis. Our immediate concern is to count the free parameters entering the general soliton solution, with given flux values $\sum_I q_p^I \int d^2\mathbf{r}\, b^I(\mathbf{r}) = \sigma_p 2\pi\hbar c(n_p + \alpha_p)$ $(p = 1, \ldots, M)$ and the asymptotic behaviors

$$r \longrightarrow \infty: \quad |\Psi_p(\mathbf{r})| \longrightarrow \frac{\text{const}}{r^{\alpha_p}} \quad (\alpha_p > 1)\,. \qquad \text{(B1)}$$

Thanks to Eq. (3.11), the integer $n_p$ here coincides with the number of zeros for the field $\Psi_p(\mathbf{r})$. The number of free parameters in question is equal to the number of normalizable zero modes of the small fluctuation equations given in the background of any specific soliton solution. From Eqs. (3.7a) and (3.7b), the fluctuation equations are of the form

$$(D_1 + i\sigma_p D_2)\delta\Psi_p - \frac{i}{\hbar c}\Psi_p \sum_I q_p^I(\delta a_1^I + i\sigma_p\delta a_2^I) = 0\,, \qquad \text{(B2a)}$$

$$\sum_I q_p^I(\nabla_1\delta a_2^I - \nabla_2\delta a_1^I)\sum_{P'}\beta_{pp'}(\Psi_p^*\delta\Psi_p + \Psi_p\delta\Psi_p^*) = 0\,. \qquad \text{(B2b)}$$

[Here $(q_p^I)$ is supposed to be a nonsingular $M \times M$ matrix; but note that our formula (B9) is valid for more general $(q_p^I)$.] Then, to eliminate superficial zero modes related to the freedom of local gauge transformations, we may generalize the real equation (B2b) to the complex equation

$$(\nabla_1 - i\sigma_p\nabla_2)\sum_I q_p^I(\sigma_p a_1^I + i\sigma_p\delta a_2^I) + 2i\sigma_p\sum_{p'}\beta_{pp'}\Psi_{p'}^*\delta\Psi_{p'} = 0\,. \qquad \text{(B3)}$$

Taking the imaginary part of this equation reproduces Eq. (B2b), while the real part can be viewed as the gauge condition.

Equations (B2a) and (B3) are represented by a single matrix equation

$$\mathcal{D}\begin{pmatrix} \delta\Psi_1 \\ \vdots \\ \delta\Psi_M \\ [\sum_I q_1^I\delta a_1^I] + i\sigma_1[\sum_I q_1^I\delta a_2^I] \\ \vdots \\ [\sum_I q_M^I\delta a_1^I] + i\sigma_M[\sum_I q_M^I\delta a_2^I] \end{pmatrix} = 0\,, \qquad \text{(B4)}$$

where

$$\mathcal{D} = \begin{pmatrix} D_1 + i\sigma_1 D_2 & & 0 & -\frac{i}{\hbar c}\Psi_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & D_1 + i\sigma_M D_2 & 0 & & -\frac{i}{\hbar c}\Psi_M \\ 2i\sigma_1\bar\beta_{11}\Psi_1^* & \cdots & 2i\sigma_1\bar\beta_{1M}\Psi_M^* & \nabla_1 - i\sigma_1\nabla_2 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 2i\sigma_M\bar\beta_{M1}\Psi_1^* & \cdots & 2i\sigma_M\bar\beta_{MM}\Psi_M^* & 0 & & \nabla_1 - i\sigma_M\nabla_2 \end{pmatrix}. \qquad \text{(B5)}$$

The index of $\mathcal{D}$ is defined as

$$\text{Index}\,(\mathcal{D}) = \dim(\text{kernel}\,\mathcal{D}) - \dim(\text{kernel}\,\mathcal{D}^\dagger)\,. \qquad \text{(B6)}$$

To calculate this index, it is convenient to consider the quantity [29]

$$I(M^2) = \text{Tr}\left[\frac{M^2}{\mathcal{D}^\dagger\mathcal{D} + M^2}\right] - \text{Tr}\left[\frac{M^2}{\mathcal{D}\mathcal{D}^\dagger + M^2}\right], \qquad \text{(B7)}$$

which can be shown to be independent of $M^2$. Naively, one may expect to recover the above index in the $M^2 \to 0$ limit. On the other hand, a straightforward evaluation of $I(M^2)$ in the limit $M^2 \to \infty$ gives

$$I(M^2) = \frac{1}{2\pi\hbar c}\sum_{p,I}\sigma_p\int d^2\mathbf{r}\, q_p^I b^I = \sum_p(n_p + \alpha_p)\,. \qquad \text{(B8)}$$

Note that this is not integer valued in general, while the index defined by Eq. (B6) is necessarily an integer. This discrepancy is due to the continuum spectrum extending to zero, which gives rise to a nonzero contribution to $I(M^2)$. Subtracting this contribution from the value in

Eq. (B8) (see Refs. [20,30]), the correct value for the index reads

$$\text{Index}\,(\mathcal{D}) = \sum_p (n_p + \hat{\alpha}_p)\,, \tag{B9}$$

where $\hat{\alpha}_p$ denotes the largest integer less than $\alpha$. Further, by the manipulations analogous to those described in Refs. [20,25], it is not difficult to show that $\dim(\text{kernel } \mathcal{D}^\dagger) = 0$. So the kernel of $\mathcal{D}$ has the (complex) dimension $\sum_p (n_p + \hat{\alpha}_p)$. Based on this, we now conclude that the total number of (real) free parameters in the general solution of the given character is equal to $2\sum_p (n_p + \hat{\alpha}_p)$. Also, in the case of topological soliton solutions which are allowed in the presence of nonzero uniform externel magnetic field, it suffices to delete the $\hat{\alpha}_p$ term in this result.


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