Induced quantum numbers in a (2+1)-dimensional electron gas

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A gas of electrons confined to a plane is examined in both the relativistic and nonrelativistic case. Using a (0+1)-dimensional effective theory, a remarkably simple method is proposed to calculate the spin density induced by a uniform magnetic background field. The physical properties of possible fluxon excitations are determined. It is found that while in the relativistic case they can be considered as half-fermions (semions) in that they carry half a fermion charge and half the spin of a fermion, in the nonrelativistic case they should be thought of as fermions, having the charge and spin of a fermion.

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I. INTRODUCTION

Planar electron systems display peculiar phenomena which originate from the Abelian nature of the rotation group SO(2) in two spatial dimensions. Since the angular momentum is not quantized, quantum statistics is allowed which continuously interpolate between Bose-Einstein and Fermi-Dirac statistics [1]. Particles obeying such fractional statistics are called anyons [2]. Their existence in the context of the fractional quantum Hall effect (FQHE) is generally accepted. The quasihole excitations of the Laughlin ground state carry fractional charge and obey fractional statistics [3,4].

An alternative approach to the FQHE due to Jain [5] relates it to the integer quantum Hall effect (IQHE). The basic concept of this construction is that of a "composite particle," consisiting of an electron bound to an even number of flux units. The IQHE of such composite particles turns out to be equivalent to the FQHE of electrons. All experimentally observed filling fractions are predicted in this way. Also the observed hierarchy instability of the various states is naturally explained.

Whereas these systems involve nonrelativistic Landau levels, relativistic levels, related to the Dirac Hamiltonian in an external magnetic field, show up in a certain type of doped two-dimensional (2D) semimetals—materials with so-called diabolic points, where the valence and conduction bands intersect [6,7].

These facts motivated us to study a planar gas of electrons occupying an integer number of Landau levels in an uniform magnetic field, in both the relativistic and nonrelativistic frameworks. In our treatment we focus on induced quantum numbers such as the fermion charge and spin. Based on our results, physical properties of possible fluxon excitations are asessed. In 2+1 dimensions fluxons are pointlike objects carrying one magnetic flux unit $2\pi/e$, where e is the electric charge of the charge carriers in the system. A fluxon may be pictured as the object in the spatial plane that is obtained when this plane is pierced by a magnetic flux tube.

In Sec. II we consider the relativistic electron gas. We extend a method recently proposed by one of us [6] in order to calculate the spin induced by a magnetic background field with arbitrary strength, thus generalizing the vacuum result of Paranjape [8]. We find a close connection between induced fermion charge and induced spin, reflecting the fact that spin and charge are not separated. It is argued that a fluxon is, in fact, a half-fermion (semion) having spin $\frac{1}{4}$ and fermion charge $\frac{1}{2}$. In the nonrelativistic case, discussed in Sec. III, a fluxon has spin $\frac{1}{2}$ and fermion charge 1 and is, thus, a genuine fermion. The close connection between induced spin and induced fermion charge which we found in the relativistic case is lost. In the last section we explain that this is due to the fact that, contrary to the relativistic case, in the nonrelativistic system the spin degree of freedom is independent of the dynamics. We show that the induced spin in the nonrelativistic electron gas is not related to a Chern-Simons term, but to a so-called mixed Chern-Simons term, involving two different gauge potentials, viz., the electromagnetic potential and one which describes the spin degree of freedom.

II. RELATIVISTIC COMPUTATIONS

We consider a relativistic electron gas in two spatial dimensions in the presence of an uniform magnetic field, as described by a massive Dirac field at finite, positive chemical potential. In 2+1 dimensions the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{1}$$

may be represented in terms of the Pauli matrices. We choose the representation $\gamma^0 = \sigma^3, \gamma^k = i\sigma^k$, with $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1,-1,-1)$ the metric tensor of Minkowski space.

The eigenvalues of the Dirac Hamiltonian play an important role in our calculations. In order to make the discussion self-contained, we provide a brief account following Johnson and Lippmann [9]. The Dirac equation reads

$$(i\gamma^\mu D_\mu - m)\Psi = 0, \tag{2}$$

where Ψ is a two-component Dirac spinor field with mass

m and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, assuring a minimal coupling to the electromagnetic field with coupling constant e, the electric charge. We describe the uniform magnetic field B by the vector potential $A^0 = A^1 = 0; A^2 = Bx^1$. Separating the time variable by setting $\Psi(x^0, \boldsymbol{x}) = \psi(\boldsymbol{x}) \exp(-iEx^0)$, we write Eq. (2) as an eigenvalue equation for $\psi(\boldsymbol{x})$:

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta m)\psi = E\psi. \tag{3}$$

The operator in parentheses is the Dirac Hamiltonian H_D; it involves the matrices $\alpha^k = \gamma^0\gamma^k$, $\beta = \gamma^0$, and the gauge-invariant momentum $\pi^k = iD^k$. The idea is to look for the eigenvalues of the squared Hamiltonian

$$H_D^2 = \boldsymbol{\pi}^2 - eB\sigma^3 + m^2, \tag{4}$$

in which one recognizes the Schrödinger Hamiltonian H_S of a spinless particle with mass $\frac{1}{2}$ and charge e in a uniform magnetic field:

$$H_S = \boldsymbol{\pi}^2. \tag{5}$$

This operator has the well-known oscillator eigenvalues

$$\epsilon_n = \Omega(n + \tfrac{1}{2}), \quad n = 0, 1, 2 \ldots, \tag{6}$$

where $\Omega = 2|eB|$ is the cyclotron frequency. The energy eigenvalues corresponding to stationary solutions of the Dirac equation (2) now follow immediately. They are given by the relativistic Landau levels [9]

$$E_{\pm n} = \pm\sqrt{m^2 + 2|eB|(n + \tfrac{1}{2}) - 2eBS_\pm}, \tag{7}$$

where the plus and minus signs correspond to positive and negative energy spinors, respectively. The quantity $S_\pm$ denotes the eigenvalues of the spin operator Σ; they are given by [10,11]

$$S_\pm = \pm\tfrac{1}{2}\mathrm{sgn}(m). \tag{8}$$

This can be easily shown by observing that the energy eigenvalue equation (3) written in the rest frame without a magnetic field,

$$\beta m\psi = \pm|m|\psi, \tag{9}$$

can be transformed into an eigenvalue equation for the rest-frame spin operator $\frac{1}{2}\sigma^3$ since $\beta = \sigma^3$. This gives Eq. (8) for the spin of a particle in the rest frame. But the spin is a pseudoscalar with respect to the Lorentz group [SO(2,1) in 2+1 dimensions], and so a Lorentz boost leaves it unchanged. We conclude that the expression (8) gives the spin of a particle in an arbitrary frame.

The smallest energy eigenvalue is either

$$E_{+0} = +\sqrt{m^2 + |eB| - 2eBS_+}$$

or

$$E_{-0} = -\sqrt{m^2 + |eB| - 2eBS_-};$$

it has the value $m\,\mathrm{sgn}(eB)$.

We next turn to the problem of the electron gas. Our starting point is the Lagrangian

$$\mathcal{L} = \Psi^\dagger(i\partial_0 + \mu - H_D)\Psi + b\Psi^\dagger\Sigma\Psi, \tag{10}$$

where Ψ is a Dirac spinor with two anticommuting (Grassmann) components representing the positive and negative energy spinors, μ is a positive chemical potential which accounts for the finite density, and b is an external source which couples to the spin density operator $\Psi^\dagger\Sigma\Psi$. This last term enables us to compute the induced spin density, i.e., the ground-state expectation value of the spin density operator. It should be emphasized that b has nothing to do with the magnetic field B, i.e., the second term in (10) is not a Zeeman term which would appear in a nonrelativistic approximation.

Integrating out the fermionic degrees of freedom, one finds the one-loop effective action

$$S_{\text{eff}} = \int d^3x\, \mathcal{L}_{\text{eff}} = -i \ln \mathrm{Det}\,(i\partial_0 + \mu - H_D + b\Sigma)\,, \tag{11}$$

where Det stands for a functional determinant. Employing the identity $\ln \mathrm{Det} = \mathrm{Tr} \ln$, we obtain a functional trace, which can be written in the energy representation as

$$\mathcal{L}_{\text{eff}} = \frac{|eB|}{2\pi} \sum_{n=0}^{\infty} \int \frac{dk_0}{2\pi i} \left\{ \ln\left[k_0 + \mu - E_{+n} + \frac{b}{2}\mathrm{sgn}(m)\right] + \ln\left[k_0 + \mu - E_{-n} - \frac{b}{2}\mathrm{sgn}(m)\right]\right\}, \tag{12}$$

where $E_{\pm n}$ are the energy eigenvalues (7) and $\pm\frac{1}{2}\mathrm{sgn}(m)$ are the eigenvalues (8) of the spin operator. We note that all the information about the system (except for the degeneracy of a Landau level per unit area, $|eB|/2\pi$) is contained in a (0+1)-dimensional theory, i.e., ordinary quantum mechanics [6]. This is due to the fact that the system is translation invariant (up to a gauge transformation), and so it suffices to study the system in a single point.

In terms of the effective action one can express the ground-state expectation values of the fermion number density operator $\Psi^\dagger\Psi$ and that of the spin density operator $\frac{1}{2}\Psi^\dagger\sigma^3\Psi$ as

$$\rho = \left.\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\mu}\right|_{b=0}, \quad s = \left.\frac{\partial\mathcal{L}_{\text{eff}}}{\partial b}\right|_{b=0}. \tag{13}$$

In this way we obtain, from (12) [12,6],

$$\rho = \frac{|eB|}{2\pi}(N + \tfrac{1}{2})\theta(\mu - |m|) - \frac{eB}{4\pi}\mathrm{sgn}(m)\theta(|m| - \mu) \tag{14}$$

and

$$s = s_\sim + \frac{|eB|}{4\pi}\text{sgn}(m)(N+\tfrac{1}{2})\theta(\mu - |m|) - \frac{eB}{8\pi}\theta(|m|-\mu), \tag{15}$$

with θ the Heaviside unit step function. The integer N denotes the number of filled Landau levels:

$$N = \left[\frac{\mu^2 - m^2}{2|eB|}\right], \tag{16}$$

where the integer-part function $[x]$ denotes the largest integer less than x. We assume that the value of the chemical potential does not coincide with one of the Landau levels, thus avoiding the points in which the integer-part function is discontinuous. These points correspond to a partially filled Landau level. In deriving (14) and (15) we employed the integral

$$\int \frac{dk_0}{2\pi i}\,\frac{1}{k_0 + \xi + ik_0\delta} = \frac{1}{2}\text{sgn}(\xi), \tag{17}$$

where we have introduced the usual "causal" path-defining factor $ik_0\delta$, with δ a small positive number. The first term in expression (15) for the induced spin density stands for the infinite contribution stemming from negative energy states in the Dirac sea:

$$s_\sim = -\frac{1}{2}\frac{|eB|}{2\pi}\text{sgn}(m)\sum_{n=0}^{\infty}\theta(-E_{-n}). \tag{18}$$

A similar infinite term is not present in the expression for the induced fermion number density. There, because of the spectral symmetry $E_{n+1} = -E_{-n}$ [$\text{sgn}(eBm) > 0$] or $E_{n-1} = -E_{-n}$ [$\text{sgn}(eBm) < 0$], only Landau levels with $|E_{\pm n}| < \mu$ contribute. The contibutions to ρ from levels outside this energy interval cancel. We renormalize s by subtracting the infinite spin (18) of the Dirac sea. This will be justified in a moment. It then follows that the induced spin density is half the induced fermion number density up to a sign $\text{sgn}(m)$:

$$s = \tfrac{1}{2}\text{sgn}(m)\rho. \tag{19}$$

This result is reasonable. It shows that charge and spin are not separated; both are induced in a ratio that reflects the fact that these quantum numbers are carried by a single particle, viz., the electron with fermion charge 1 and spin $\frac{1}{2}$.

The low-density limit, which corresponds to a chemical potential smaller than the fermion mass ($\mu < |m|$), deserves particular scrutiny. This case is basically equivalent to vacuum (2+1)-dimensional QED (QED_{2+1}). In this limit only the last term in (14) survives, so that the fermion number density induced into the vacuum by the background magnetic field is [13,14,12,6]

$$\rho_{\text{vac}} = -\frac{eB}{4\pi}\text{sgn}(m) \tag{20}$$

and, ignoring the contribution $s_\sim$ due to states inside the Dirac sea,

$$s_{\text{vac}} = -\frac{eB}{8\pi} = \frac{1}{2}\text{sgn}(m)\rho_{\text{vac}}. \tag{21}$$

The above results, which were derived for a constant background field, also apply to cases where the magnetic field has a specific profile, e.g., corresponding to a fluxon, the essential physics being captured by the number of flux units that penetrate the spatial plane [15]. The vacuum result (21) restricted to a single fluxon carrying one magnetic flux unit $2\pi/e$ shows that it acquires fractional spin $S_\otimes = -\frac{1}{4}$. It was pointed out in Ref. [16] that this is in accord with the Chern-Simons term which is generated at the quantum level when the system is placed in an external *electromagnetic* field. This term is easily constructed from (20) by realizing that on account of Lorentz covariance the induced fermion number current density $\langle j^\mu\rangle$ in such a field, described by the field strength $F^{\mu\nu}$, is

$$\langle j^\mu\rangle = \frac{e}{8\pi}\text{sgn}(m)\epsilon^{\mu\nu\lambda}F_{\nu\lambda}. \tag{22}$$

This corresponds to a Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{\theta e^2}{2}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \tag{23}$$

in the effective Euler-Heisenberg Lagrangian, with $\theta = -\text{sgn}(m)/(4\pi)$. A Chern-Simons term imparts a spin

$$S_\otimes = \text{sgn}(m)\pi\theta \tag{24}$$

to a fluxon. In our case this yields a spin $S_\otimes = -\frac{1}{4}$, in agreement with the result (21). One may think of a fluxon as a half-fermion because it carries half the spin and half the charge of a fermion.

To make the connection with Paranjape's work [8], who considered the total angular momentum $J = S + L$ induced into the vacuum by N_ϕ fluxons, rather than the induced spin $S = S_\otimes N_\Phi = -\frac{1}{4}N_\Phi$, we note that the exclusion principle forbids two anyons to be in the same angular momentum state. So, when considering a state with N_ϕ semions, these objects have to be put in succesive *orbital* angular momentum states and [17]

$$L = 2S_\otimes\sum_{n=1}^{N_\Phi}(n-1) = S_\otimes N_\phi(N_\phi - 1). \tag{25}$$

In this way, J becomes

$$J = S + L = S_\otimes N_\Phi^2 = -\tfrac{1}{4}N_\Phi^2, \tag{26}$$

which is, apart from a sign $\text{sgn}(m)$, Paranjape's result [8].

From the full result (14) for the induced fermion number density we obtain the Chern-Simons coefficient

$$\theta = \frac{1}{2\pi}\text{sgn}(eB)(N+\tfrac{1}{2})\theta(\mu - |m|) - \frac{1}{4\pi}\text{sgn}(m)\theta(|m|-\mu), \tag{27}$$

which, according to (24), leads to a spin for a single fluxon given by

$$S_\otimes = \tfrac{1}{2}\mathrm{sgn}(eBm)(N+\tfrac{1}{2})\theta(\mu-|m|) - \tfrac{1}{4}\theta(|m|-\mu). \tag{28}$$

This yields, when multiplied with the density of fluxons, $eB/(2\pi)$, the previous result (15) with the contribution $s_\sim$ of states in the Dirac sea omitted.

We next provide a further justification of our renormalization of the expectation value of the spin density operator, which consisted of subtracting the contributions stemming from negative energy states in the Dirac sea. Since these contributions are independent of the chemical potential, we may set without loss of generality $\mu = 0$ in our analysis. In this limit s was given by (21), implying a spin magnetic moment, or magnetization M,

$$M = g_0\mu_B s = -\frac{e^2}{8\pi|m|}B, \tag{29}$$

with $\mu_B = e/(2|m|)$ the Bohr magneton and $g_0 = 2$ the electron g factor. The corresponding spin susceptibility χ_P is

$$\chi_P = \frac{\partial M}{\partial B} = -\frac{e^2}{8\pi|m|}, \tag{30}$$

where one should bear in mind that in 2+1 dimensions e^2 has the dimension of mass. We shall rederive this result, which hinges on our premise that the first term $s_\sim$ in (15) is to be omitted, in an alternative way involving the "proper-time" regularization developed by Schwinger [18].

To this end we carry out the k_0 integration in the effective Lagrangian (12) with $\mu = b = 0$ to obtain

$$\mathcal{L}_{\text{eff}} = \frac{|eB|}{2\pi}\frac{1}{2}\sum_n |E_n| \tag{31}$$

and introduce the "proper-time" representation of the square root [19,20]

$$\sqrt{a} = -\int_0^\infty \frac{d\tau}{(\pi\tau)^{1/2}}\frac{d}{d\tau}\exp(-a\tau). \tag{32}$$

After a partial integration and after subtracting the B-independent part, which corresponds to the free-electron contribution, one easily finds

$$\mathcal{L}_{\text{eff}} = -\frac{1}{8\pi^{3/2}}\int_0^\infty \frac{d\tau}{\tau^{3/2}}e^{-m^2\tau} \times\left(|eB|\frac{\cosh(2eB\sigma\tau)}{\sinh(|eB|\tau)} - \frac{1}{\tau}\right). \tag{33}$$

Here, the sinh factor stems from the fact that the planar orbits of a charged particle moving in a background magnetic field are quantized, while the cosh factor, with $\sigma = \frac{1}{2}$, arises from the magnetic moment and, thus, from the spin of the electrons. To obtain the magnetic susceptibility we expand the effective Lagrangian (33) to second order in the magnetic field. This gives

$$\mathcal{L}_{\text{eff},2} = \tfrac{1}{2}\chi B^2, \tag{34}$$

with χ the magnetic susceptibility,

$$\chi = (-1)^{2\sigma}\frac{e^2}{8\pi|m|}\left[(2\sigma)^2 - \frac{1}{3}\right]. \tag{35}$$

(We have written this formula in a general form valid for spin $\sigma = 0, \frac{1}{2}, 1$.) The first term, with $\sigma = \frac{1}{2}$, is the spin contribution which precisely yields the previous result (30). This justifies the renormalization procedure we adopted.

Incidentally, it follows from (35) that for relativistic spin-$\frac{1}{2}$ particles in two (and also in three) space dimensions the spin contribution is 3 times as large as the orbital contribution. The same ratio is found for a nonrelativistic electron gas at small magnetic fields in three space dimensions, where

$$\chi = (-1)^{2\sigma+1}2\mu_B^2\,\nu_{3\text{D}}(0)\left[(2\sigma)^2 - \tfrac{1}{3}\right], \tag{36}$$

with $\nu_{3\text{D}}(0) = mk_F/2\pi^2$ the three-dimensional density of states per spin degree of freedom at the Fermi sphere. However, whereas usually the spin contribution is paramagnetic ($\chi > 0$) and the orbital contribution is diamagnetic ($\chi < 0$), Eq. (35) reveals exactly the opposite behavior. Instead of screening the external field, the planar motion of relativistic electrons is such as to enhance the field. Since the diamagnetic (*sic*) spin contribution dominates, the overall effect in vacuum QED_{2+1} is nevertheless a screening of external fields ($\chi < 0$).

III. NONRELATIVISTIC CALCULATIONS

In this section we treat a nonrelativistic electron gas confined to a plane. We expect that some new qualitative features arise from the fact that in this case the spin degree of freedom is not enslaved by the dynamics. We continue to use a relativistic notation with $\partial_\mu = (\partial_0, \boldsymbol{\nabla}$, $\partial^\mu = (\partial_0, -\boldsymbol{\nabla})$, where $\boldsymbol{\nabla}$ is the gradient operator, and $A^\mu = (A^0, \boldsymbol{A})$.

Let us consider the Lagrangian

$$\mathcal{L} = \Psi^\dagger(i\partial_0 + \mu - H_P)\Psi + b\Psi^\dagger\frac{\sigma^3}{2}\Psi, \tag{37}$$

which governs the dynamics of the Pauli spinor field Ψ, with Grassmann components $\psi_\uparrow$ and $\psi_\downarrow$ describing the electrons with spin $\uparrow$ and $\downarrow$. The role of the chemical potential μ and the spin source b is the same as in the previous calculation. The Pauli Hamiltonian

$$H_P = \frac{1}{2m}(i\boldsymbol{\nabla} + e\boldsymbol{A})^2 - g_0\mu_B\frac{\sigma^3}{2}B + eA_0, \tag{38}$$

with $\mu_B = e/2m$ the Bohr magneton and g_0 the electron g factor, contains a Zeeman term which couples the electron spins to the background magnetic field. Usually this term is omitted. The reason is that in realistic systems the g factor is much larger than 2, the value for a free electron. In strong magnetic fields relevant to the QHE the energy levels of spin-$\downarrow$ electrons are too high and cannot be occupied; the system is spin polarized, and the electron spin is irrelevant to the problem. Setting again $A^0 = A^1 = 0$, $A^2 = Bx^1$, one finds, as eigenvalues for H_P,

$$E_{n,\pm} = \frac{|eB|}{m}\left(n + \frac{1}{2}\right) - \frac{eB}{m} S_\pm, \tag{39}$$

with $S_\pm = \pm\frac{1}{2}$ for spin-$\uparrow$ and spin-$\downarrow$ electrons, respectively. We note that in the nonrelativistic limit, corresponding to taking $m \to +\infty$, the relativistic Landau levels (7) reduce to

$$E_{+n} \to \text{const} + \frac{|eB|}{m}\left(n + \frac{1}{2}\right) - \frac{eB}{2m}, \tag{40}$$

where we omitted the negative energy levels which have no meaning in this limit. The main difference with (39) stems from the fact that there the spin degree of freedom is considered as an independent quantity, not enslaved by the dynamics as is the case in the relativistic problem.

The induced fermion number density and spin density may be obtained in a similar calculation as in the preceding section. From the effective action,

$$S_{\text{eff}} = -i\text{Tr}\ln\left(i\partial_0 - H_P + \mu + \frac{b}{2}\sigma^3\right), \tag{41}$$

we obtain

$$\mathcal{L}_{\text{eff}} = \frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\int\frac{dk_0}{2\pi i}\left[\ln\left(k_0 - E_{n,+} + \mu + \frac{b}{2}\right) + \ln\left(k_0 - E_{n,-} + \mu - \frac{b}{2}\right)\right]. \tag{42}$$

The only difference with the relativistic computation is that instead of integrals of the type (17), we now encounter integrals of the form

$$\int\frac{dk_0}{2\pi i}\,\frac{e^{ik_0\delta}}{k_0 + \xi + ik_0\delta} = \theta(\xi), \tag{43}$$

containing, as usual in nonrelativistic calculations [21], an additional convergence term $\exp(ik_0\delta)$. The resulting value of the induced fermion number density is

$$\rho = \frac{|eB|}{2\pi}(N_+ + N_-), \tag{44}$$

with $N_\pm$ the number of filled Landau levels for spin-$\uparrow$ and spin-$\downarrow$ electrons, and

$$N_\pm = \left[\frac{m\mu_\pm}{|eB|} + \frac{1}{2}\right] \tag{45}$$

and

$$\mu_\pm = \mu + \frac{eB}{m} S_\pm \tag{46}$$

their effective chemical potentials. The square brackets denote again the integer-part function. Implicit in this framework is the assumption that, just like in the relativistic case, the chemical potential lies between two Landau levels. The induced fermion number density (44) is related to a Chern-Simons term (23) in the effective action, with

$$\theta = \text{sgn}(eB)\frac{1}{2\pi}(N_+ + N_-). \tag{47}$$

Because of the presence of the sgn(eB) factor, which changes sign under a parity transformation, this Chern-Simons term is invariant under such transformations. The induced spin density turns out to be independent of $N_\pm$, viz.,

$$s = \frac{eB}{4\pi}. \tag{48}$$

This follows from the symmetry in the spectrum $E_{n+1,+} = E_{n,-}$ $(eB > 0)$ or $E_{n,+} = E_{n+1,-}$ $(eB < 0)$. The magnetic moment M is according to (29) obtained from (48) by multiplying s with twice the Bohr magneton μ_B. This leads to the textbook result for the magnetic spin susceptibity χ_P:

$$\chi_P = \frac{\partial M}{\partial B} = \frac{e^2}{4\pi m} = 2\mu_B^2\,\nu_{2\text{D}}(0), \tag{49}$$

with $\nu_{2\text{D}}(0) = m/(2\pi)$ the density of states per spin degree of freedom in two space dimensions.

At zero field, ρ reduces to the standard fermion number density in two space dimensions $\rho \to \mu m/\pi = k_F^2/(2\pi)$, where k_F denotes the Fermi momentum. A single fluxon carries according to (48) a spin $S_\otimes = \frac{1}{2}$ and, since, for small fields,

$$\rho \to \frac{\mu m}{\pi} + \frac{|eB|}{2\pi}, \tag{50}$$

also one unit of fermion charge. That is, in the nonrelativistic electron gas the fluxon may be thought of as a fermion in that it has both the spin and charge of a fermion. However, the close connection between spin of a fluxon and induced Chern-Simons term for arbitrary fields that we found in the relativistic case is lost. This can be traced back to the fact that in the nonrelativistic case the electron spin is an independent degree of freedom. In the next section we point out that the spin of the fluxon does not derive from the ordinary Chern-Simons term, but from a so-called mixed Chern-Simons term. Such a term is absent in the relativistic case.

To see how the spin contribution (49) to the magnetic susceptibility compares to the orbital contribution we evaluate the k_0 integral in the effective action (42) with $b = 0$ to obtain

$$\mathcal{L}_{\text{eff}} = \frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\sum_{\varsigma=\pm}(\mu - E_{n,\varsigma})\theta(\mu - E_{n,\varsigma}). \tag{51}$$

The summation over n is easily carried out with the result, for small fields,

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\sum_{\varsigma=\pm}\left[\mu_\varsigma^2 m - \frac{(eB)^2}{4m}\right]$$
$$= \frac{\mu^2 m}{2\pi} + \frac{(eB)^2}{8\pi m}[(2\sigma)^2 - 1], \quad (52)$$

where $\sigma = \frac{1}{2}$ and $\mu_\pm$ is given by (46). The first term in the right-hand side of (52), which is independent of the magnetic field, is the free-particle contribution

$$\frac{\mu^2 m}{2\pi} = -2\int\frac{d^2k}{(2\pi)^2}\left(\frac{k^2}{2m} - \mu\right)\theta\left(-\frac{k^2}{2m}\mu\right). \quad (53)$$

The second term yields the low-field susceptibility

$$\chi = (-1)^{2\sigma+1} 2\mu_B^2\, \nu_{2\text{D}}(0)\left[(2\sigma)^2 - 1)\right]. \quad (54)$$

Equation (54) shows that the ratio of orbital to spin contribution to χ is different from the three-dimensional case (36). Also, whereas a 3D electron gas is paramagnetic ($\chi > 0$) because of the dominance of the spin contribution, the 2D gas is not magnetic ($\chi = 0$) at small fields since the orbital and spin contributions to χ cancel.

IV. MIXED CHERN-SIMONS TERM

In this section we investigate the origin of the induced spin density (48) that we found in the nonrelativistic electron gas. To this end we slightly generalize the theory (37) and consider the Lagrangian

$$\mathcal{L} = \Psi^\dagger\left[i\partial_0 - eA_0 + \mu - \frac{1}{2m}(i\boldsymbol{\nabla} + e\boldsymbol{A})^2\right]\Psi$$
$$+\frac{e}{m}B^a\Psi^\dagger\frac{\sigma^a}{2}\Psi. \quad (55)$$

It differs from (37) in that the spin source term is omitted and in that the magnetic field in the Zeeman term is allowed to point in any direction in some internal space labeled by latin indices $a, b, c = 1, 2, 3$. As a result also the spin will have components in this space. It is convenient to consider a magnetic field whose direction in the internal space varies in space-time. We set

$$B^a(x) = Bn^a(x), \quad (56)$$

with n^a a unit vector in the internal space. The gauge potential A_μ appearing in the first term of (55) still gives $\epsilon_{ij}\partial_i A^j = B$. Equation (56) allows us to make the decomposition

$$\Psi(x) = S(x)\chi(x), \quad S^\dagger S = 1, \quad (57)$$

with $S(x)$ a local SU(2) matrix which satisfies

$$\boldsymbol{\sigma}\cdot\boldsymbol{n}(x) = S(x)\sigma^3 S^\dagger(x). \quad (58)$$

In terms of these new variables the Lagrangian (55) becomes

$$\mathcal{L} = \chi^\dagger\left[i\partial_0 - eA_0 - V_0 + \mu - \frac{1}{2m}(i\boldsymbol{\nabla} + e\boldsymbol{A} + \boldsymbol{V})^2\right]\chi$$
$$+\frac{eB}{2m}\chi^\dagger\sigma^3\chi, \quad (59)$$

where the 2×2 matrix $V_\mu = -iS^\dagger(\partial_\mu S)$ represents an element of the su(2) algebra, which can be written in terms of (twice) the generators σ^a as

$$V_\mu = V_\mu^a\sigma^a. \quad (60)$$

In this way the theory takes formally the form of a gauge theory with gauge potential V_μ^a. In terms of the new fields the spin density operator

$$j_0^a = \Psi^\dagger\frac{\sigma^a}{2}\Psi \quad (61)$$

becomes [22]

$$j_0^a = R_{ab}\chi^\dagger\frac{\sigma^b}{2}\chi = -\frac{1}{2}R_{ab}\frac{\partial\mathcal{L}}{\partial V_0^b}. \quad (62)$$

In deriving the first equation we employed the identity

$$S(\boldsymbol{\theta})\sigma^b S^\dagger(\boldsymbol{\theta}) = \sigma^a R_{ab}(\boldsymbol{\theta}), \quad (63)$$

which relates the SU(2) matrices in the $j = \frac{1}{2}$ representation, $S(\boldsymbol{\theta}) = \exp(\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma})$, to those in the adjoint representation ($j = 1$), $R(\boldsymbol{\theta}) = \exp(i\boldsymbol{\theta}\cdot\mathbf{J}^{\text{adj}})$. The matrix elements of the generators in the latter representation are $(J_a^{\text{adj}})_{bc} = -i\epsilon_{abc}$.

The projection of the spin density j_0^a onto the spin quantization axis, i.e., the direction n^a of the applied magnetic field [22],

$$\boldsymbol{n}\cdot\boldsymbol{j}_0 = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial V_0^3}, \quad (64)$$

only involves the spin gauge field V_μ^3. So when calculating the induced spin density $s = \langle\boldsymbol{n}\cdot\boldsymbol{j}_0\rangle$ we may set the fields V_μ^1 and V_μ^2 equal to zero and consider the simpler theory

$$\mathcal{L} = \sum_{\varsigma=\pm}\chi_\varsigma^\dagger\left[i\partial_0 - eA_0^\varsigma + \mu_\varsigma - \frac{1}{2m}(i\boldsymbol{\nabla} + e\boldsymbol{A}^\varsigma)^2\right]\chi_\varsigma, \quad (65)$$

where the effective chemical potentials for the spin-$\uparrow$ and spin-$\downarrow$ electrons are given in (46) and $eA_\mu^\pm = eA_\mu \pm V_\mu^3$. Both components $\chi_\uparrow$ and $\chi_\downarrow$ induce a Chern-Simons term, so that in total we have

$$\mathcal{L}_{\text{cs}} = \frac{e^2}{2}\epsilon^{\mu\nu\lambda}(\theta_+ A_\mu^+\partial_\nu A_\lambda^+ + \theta_- A_\mu^-\partial_\nu A_\lambda^-)$$
$$= \frac{\theta_+ + \theta_-}{2}\epsilon^{\mu\nu\lambda}(e^2 A_\mu\partial_\nu A_\lambda + V_\mu^3\partial_\nu V_\lambda^3)$$
$$+e(\theta_+ - \theta_-)\epsilon^{\mu\nu\lambda}V_\mu^3\partial_\nu A_\lambda, \quad (66)$$

where the last term involving two different vector potentials is a mixed Chern-Simons term. The coefficients are given by

$$\theta_\pm = \frac{1}{2\pi}\text{sgn}(eB)N_\pm, \tag{67}$$

assuming that $|eB| > \frac{1}{2}|\epsilon_{ij}\partial_i V_j^3|$, so that the sign of eB is not changed by spin gauge contributions. The integers $N_\pm$ are the number of filled Landau levels for spin-↑ and spin-↓ electrons given by (45). Since $N_+ - N_- = \text{sgn}(eB)$, we obtain for the induced spin density s precisely the result (48) we found, in the preceding section,

$$s = \langle \boldsymbol{n}\cdot\boldsymbol{j}_0\rangle = -\frac{1}{2}\left.\frac{\partial\mathcal{L}_{\text{eff}}}{\partial V_0^3}\right|_{V_\mu^3=0} = \frac{eB}{4\pi}. \tag{68}$$

The present derivation clearly shows that the induced spin in the nonrelativistic electron gas originates not from the standard Chern-Simons term (23), but from the mixed Chern-Simons term involving the electromagnetic and spin gauge potential.

The first term in (66) is a standard Chern-Simons term, the combination $\theta_+ + \theta_-$ precisely reproduces the result (47) and is related to the induced fermion number density (44).

Note added in proof. A. Polychronakos informed us that he has obtained a result [see Phys. Rev. Lett. **60**, 1920 (1988)] for the induced vacuum spin which differs by sgn(eBm) from that reported here.

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