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Exact operator quantization of a model of two-dimensional dilaton gravity

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Exact operator quantization is performed of a model of two-dimensional dilaton gravity in Lorentzian spacetime, classically equivalent to the one proposed by Callan, Giddings, Harvey, and Strominger, in the special case with 24 massless matter scalars. This is accomplished by developing a nonlinear and nonlocal quantum canonical transformation of basic interacting fields into a set of free fields, rigorously taking into account the spatially closed boundary condition. The quantized model enjoys conformal invariance and the entire set of physical states and operators are obtained in the BRST formalism. In addition, a rather detailed discussion of the nature of the basic issues for exact treatment of models of quantum gravity is provided.

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I. INTRODUCTION

The sentiment that we are on the verge of unveiling (at least a part of) the long-standing mystery of quantum gravity may well turn out to be too innocent, but the developments in the last few years in low-dimensional quantum gravity appear to contain encouraging evidence to make us feel tempted for such optimism. A powerful matrix model technique [1] opened up an avenue for exact nonperturbative analysis and the target space interpretation of the gauged Wess-Zumino-Witten models [2] revealed a way in which such an interesting and important physics as that of a black hole can be unraveled in string theory.

Encouraging signs are not confined to the realm of string theory. Notably after the work of Callan, Giddings, Harvey, and Strominger (CGHS) [3], it has been recognized that a class of field theoretical models of quantum gravity in two dimensions, characteristically containing a dilaton field, provides an excellent testing ground for a variety of fundamental issues in quantum gravity. Semiclassical analysis, which partially incorporates the back reaction of the metric field, has been performed by many authors and has demonstrated the usefulness of such models [3–9]. In particular, the phenomenon of evaporation of a black hole by the emission of Hawking radiation has been vigorously pursued with fair amount of success.

However, semiclassical analysis of course has its limitations. The type of analysis performed is valid only in the limit of large black hole mass and in the presence of a large number of massless matter fields, and the approximation breaks down for the most interesting phase which determines the ultimate fate of a black hole. Also, in such a framework, the difficult yet all important problem of the interpretation of the wave function, including the question of unitarity and loss of quantum coherence, cannot properly be addressed. It is therefore clear that a more powerful treatment, with full-fledged quantization of the gravitational degrees of freedom, is desired.

There have been a number of attempts to quantize the CGHS model beyond the semiclassical approximation, however, with moderate success [10–18]. One of the major difficulties is the choice of the proper functional measure which defines the quantum theory. Many of the procedures so far proposed lead, after a complicated nonlinear field transformation, to a free field theory, but the change of the measure due to such a transformation is, to say the least, treated in an unclear manner. One may take an attitude to regard the resultant free theory as defining a quantum model, but then the quantum mechanical relations between these free fields and the original interacting fields, for which we must make physical interpretations, will remain concealed. A slightly different choice of the measure which permits more rigorous treatment has also been proposed [17,18], but in this case there is a different problem; the quantum theory so obtained does not appear to yield to analysis beyond semiclassical approximation.

The purpose of the present article is to give an improvement on this situation by providing an interacting model for which the measure chosen is clear and at the same time exact operator quantization is possible. The model, to be defined precisely in the next section, can be regarded as a special case of CGHS model, namely the case where the number of massless matter scalars is exactly 24, with a certain choice of the measure. To make the model well defined we shall impose spatially closed boundary conditions and stay in Minkowski space throughout in order to be able to discuss spacetime physics, including such important concepts as causality, locality, etc. Exact operator quantization will then be accomplished by developing a nonlinear and nonlocal canonical transformation, valid quantum mechanically as well as classically, which maps the original interacting fields to a set of free fields. The model so quantized enjoys conformal invariance, and by utilizing it we shall provide a complete analysis of the physical states and operators in the Becchi-Rouet-Stora-Tyutin (BRST) framework. In this article, we will not be able to give a physical interpretation of our results as there are still many difficult problems to be overcome. These problems, which we believe must be faced in any serious attempt for exact treatment, will be explained in detail. In particular, the necessity of a careful examination of the choice of the inner product for the space of states is stressed. Our emphasis throughout will be the rigor of the analysis, which we believe is particularly important for quantum gravity, for various intuitions cultivated in our experience with ordinary field theories must be attentively scrutinized.

The rest of the paper is organized as follows. In Sec. II, we shall define the model and study its classical properties. A canonical transformation is introduced and proved in Sec. III and with its use the model is quantized in a rigorous manner. Conformal properties of various fields will also be clarified. Section IV is devoted to the analysis of the physical states and operators of the theory in the manner of BRST. Similarities to and differences from the case of noncritical closed string theory formulated in Euclidean space are clarified. In Sec. V, a rather detailed discussion of the nature of the remaining problems will be given. Finally two Appendixes are provided to supplement the technical details omitted in the text.

II. THE MODEL AND ITS CLASSICAL PROPERTIES

A. The model

The model we shall study in this article is classically identical to the one introduced by CGHS [3]. Its action is given by

$$S=\frac{1}{\gamma^2}\int d^2\xi\sqrt{-g}\left\{e^{-2\phi}[4g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+R_g-4\lambda^2]+\sum_{i=1}^{N}\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha f_i\partial_\beta f_i\right\}, \tag{2.1}$$

where ϕ is the dilaton field and f_i $(i=1,\ldots,N)$ are N massless scalar fields representing matter degrees of freedom. We shall stay in Minkowski space throughout and use the metric convention such that for flat space $\eta_{\alpha\beta}=\mathrm{diag}(1,-1)$. In order to define all the quantities unambiguously and to be able to perform integrations by parts which occur in various places, we shall take our universe to be spatially closed with period $2\pi L$. For this purpose, it is convenient to introduce the rescaled coordinates

$$x^\mu=(t,\sigma)=\xi^\mu/L\ , \tag{2.2}$$

and we require that all the fields appearing in the action be periodic in σ, i.e.,

$$F(t,\sigma+2\pi)=F(t,\sigma)\ . \tag{2.3}$$

When the action is rewritten in terms of x^μ, it retains its form except with the replacement $\lambda\to\mu\equiv\lambda L$. From now on, we will deal with such a "dimensionless" form and when necessary recover the correct dimensions by appropriately multiplying by the factors of $1/L$.

A choice of a classical action, of course, does not fix a quantum theory. We must specify the canonical variables and the form of the measure to be used for the functional integration. Except for the requirement of invariance under general coordinate transformations, there is no absolute maxim to be imposed by general principle and hence the choice is far from unique. In fact a number of choices for the measure have been proposed and analyzed with varying degrees of rigor and naturalness [10–18].

One attractive line of thought is to look for a choice such that the quantum theory so obtained consistently retains the on-shell conformal invariance present in the classical theory in the conformal gauge. Such an attempt was made, independently by de Alwis [10] and by Bilal and Callan [14], although the procedures employed were somewhat indirect. Recently a more transparent way of deriving a quantum theory with conformal invariance, which leads to essentially the same model as that in Refs. [10,14], was proposed by Hamada and Tsuchiya [18] (see, in particular, the Appendix). Their starting point is the action proposed by Russo and Tseytlin (RT) [19,20], which is classically equivalent to the CGHS action (2.1). It is obtained from (2.1) by the following transformation of fields:

$$\Phi\equiv e^{-2\phi}\ , \tag{2.4}$$

$$h_{\alpha\beta}\equiv e^{2\omega}g_{\alpha\beta}\ , \tag{2.5}$$

where

$$\omega=\tfrac{1}{2}(\ln\Phi-\Phi)\ . \tag{2.6}$$

Then the action becomes

$$S=\frac{1}{\gamma^2}\int d^2x\sqrt{-h}\left[h^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi+R_h\Phi-4\mu^{2e^{\Phi}}+\sum_i\tfrac{1}{2}h^{\alpha\beta}\partial_\alpha f_i\partial_\beta f_i\right], \tag{2.7}$$

where the curvature scalar R_h refers to the conformally transformed metric $h_{\alpha\beta}$. The authors of Ref. [18] take the functional measures for the fields $h_{\alpha\beta}$, Φ, and $\mathbf{f}$ to be those defined by the norms

$$\begin{aligned}\|\delta h\|^2&=\int d^2x\sqrt{-h}\,h^{\alpha\beta}h^{\gamma\delta}\delta h_{\alpha\gamma}h_{\beta\delta}\ ,\\ \|\delta\Phi\|^2&=\int d^2x\sqrt{-h}\,\delta\Phi\delta\Phi\ ,\\ \|\delta f_i\|^2&=\int d^2x\sqrt{-h}\,\delta f_i\delta f_i\quad(i=1,\ldots,N)\ .\end{aligned} \tag{2.8}$$

Then they separate $h_{\alpha\beta}$ into the Weyl factor ρ and the background metric $\hat{g}_{\alpha\beta}$ as $h_{\alpha\beta}=e^{2\rho}\hat{g}_{\alpha\beta}$ and rewrite the measure into the one with respect to $\hat{g}$ in the manner of David and Distler and Kawai [21,22]. The resulting action takes the form

$$\begin{aligned}S=\frac{1}{\gamma^2}\int d^2x\sqrt{-\hat{g}}\,[&\hat{g}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi+2\hat{g}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\rho\\ &+R_{\hat{g}}\Phi-4\mu^2e^{\Phi+2\rho}\\ &+\kappa(\hat{g}^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho+R_{\hat{g}}\rho)\\ &+\tfrac{1}{2}\hat{g}^{\alpha\beta}\partial_\alpha\mathbf{f}\cdot\partial_\beta\mathbf{f}]+S^{\mathrm{gh}}(\hat{g},b,c)\ ,\end{aligned} \tag{2.9}$$

where $\kappa=(N-24)/12$ and S^{gh} is the usual action for the reparametrization ghosts b and c. For $\kappa\neq0$, they further perform a simple redefinition of fields (with a unit Jacobian) and show that the result is essentially identical to the effective action obtained in [10] and [14].

Because of the transparency of its derivation, we shall take this action (2.9) seriously and analyze in this article the special case with $N=24$ in detail. Although the action still contains an exponential interaction, we shall see that it can be canonically mapped to a free theory and hence can be solved exactly.

In this sense the model is certainly mathematically consistent. We need, however, to mention two subtleties to be kept in mind. The first is the question of the range of Φ in the functional integration. Classical Φ is a non-negative quantity and as was emphasized in [18] there is no symmetry principle which allows one to extend this range naturally into the negative region. Ignoring this restriction may or may not be a serious problem and the answer should await a detailed analysis. (See, however, the discussion in [11].) The second concerns the physical interpretation of the model. Namely, the procedure outlined above can be interpreted in two ways. One point of view is to take the classical RT action seriously and regard $h_{\alpha\beta}$ as the genuine metric of the spacetime. Then the choice of the measure (2.8) seems natural. An alternative view is to regard the procedure as quantizing the original CGHS action with a definite but somewhat unusual measure. In this standpoint, one continues to interpret $g_{\alpha\beta}$ as the metric. Again the justification of one or the other can only be decided after a detailed study of the model.

These subtleties will have to be watched but we believe that the model has a big advantage in that one knows the precise setting and that it is still solvable.

B. Classical properties

Since the ghost part of the action can be handled in the usual way, we first look at the remainder, which we shall call the classical action $S^{\rm cl}$. Setting the reference metric $\hat{g}_{\alpha\beta}$ to be the flat metric $\eta_{\alpha\beta}$ and N to be 24, the classical action becomes

$$S^{\rm cl}=\frac{1}{\gamma^2}\int d^2x(\partial_\alpha\Phi\partial^\alpha\Phi+2\partial_\alpha\Phi\partial^\alpha\rho -4\mu^2e^{\Phi+2\rho}+\tfrac{1}{2}\partial_\alpha\mathbf{f}\cdot\partial^\alpha\mathbf{f}) . \tag{2.10}$$

Hereafter, we shall set $\gamma^2=1$ for simplicity. Variation with respect to Φ, ρ, and $\mathbf{f}$ gives the equations of motion

$$2\partial_+\partial_-\Phi+2\partial_+\partial_-\rho+\mu^2e^{\Phi+2\rho}=0 , \tag{2.11}$$

$$\partial_+\partial_-\Phi+\mu^2e^{\Phi+2\rho}=0 , \tag{2.12}$$

$$\partial_+\partial_-\mathbf{f}=0 . \tag{2.13}$$

Here and hereafter the light-cone coordinates are defined by

$$x^\pm=x^0\pm x^1=t\pm\sigma,\quad \partial_\pm=\tfrac{1}{2}(\partial_t\pm\partial_\sigma),\quad \Box=4\partial_+\partial_- . \tag{2.14}$$

General solutions for these equations of motion can easily be obtained: First, each matter scalar f_i is trivially a free field. Next, by eliminating the exponential term from Eqs. (2.11) and (2.12), one finds that $\Phi+2\rho\equiv\psi=\psi^++\psi^-$ is a free field. Next put this back into Eq. (2.12) and define the functions $A(x^+)$ and $B(x^-)$ by

$$\partial_+A(x^+)=\mu e^{\psi^+(x^+)} , \tag{2.15}$$

$$\partial_-B(x^-)=\mu e^{\psi^-(x^-)} . \tag{2.16}$$

Then it is easy to see that $\Phi+AB\equiv\chi$ is again a free field. Thus we can write

$$\Phi=\chi-AB , \tag{2.17}$$

$$\rho=\tfrac{1}{2}(\psi-\Phi) . \tag{2.18}$$

Since ψ and χ satisfy periodic boundary conditions, we can expand them into Fourier modes as follows.

$$\psi^\pm=\tfrac{1}{2}Q_\psi+\frac{P_\psi}{4\pi}x^\pm+\frac{i}{\sqrt{4\pi}}\sum_{n\neq0}\frac{1}{n}\alpha_n^\pm e^{-inx^\pm} , \tag{2.19}$$

$$\chi^\pm=\tfrac{1}{2}Q_\chi+\frac{P_\chi}{4\pi}x^\pm+\frac{i}{\sqrt{4\pi}}\sum_{n\neq0}\frac{1}{n}\beta_n^\pm e^{-inx^\pm} . \tag{2.20}$$

We now solve the equations for $A(x^+)$ and $B(x^-)$. Although the product AB must be a periodic function, $A(x^+)$ and $B(x^-)$ separately need not be periodic. In fact, since the left- and right-moving components $\psi^\pm$ each undergo a constant shift, the correct boundary conditions for $A(x^+)$ and $B(x^-)$ are of the form

$$A(x^++2\pi)=\alpha A(x^+) , \tag{2.21}$$

$$B(x^--2\pi)=\frac{1}{\alpha}B(x^-) . \tag{2.22}$$

From the mode expansion of ψ, the constant α is easily seen to be related to the momentum zero mode P_ψ by

$$\alpha=e^{P_\psi/2} . \tag{2.23}$$

Now we observe that the equations as well as the boundary conditions above for $A(x^+)$ and $B(x^-)$ are identical in form to those which appeared in the operator analysis of the Liouville theory [23,24]. Suppressing the dependence on t, the solutions can be expressed as

$$A(\sigma)=\mu C(\alpha)\int_0^{2\pi}d\sigma' E_\alpha(\sigma-\sigma')e^{\psi_+(\sigma')} , \tag{2.24}$$

$$B(\sigma)=\mu C(\alpha)\int_0^{2\pi}d\sigma'' E_{1/\alpha}(\sigma-\sigma'')e^{\psi_-(\sigma'')} , \tag{2.25}$$

where $C(\alpha)=1/(\sqrt{\alpha}-\sqrt{\alpha}^{-1})$ and the functions $E_\alpha(\sigma)$ and $E_{1/\alpha}(\sigma)$ are defined by

$$E_\alpha(\sigma)\equiv\exp(\tfrac{1}{2}\ln\alpha\epsilon(\sigma)) , \tag{2.26}$$

$$E_{1/\alpha}(\sigma)\equiv\exp(-\tfrac{1}{2}\ln\alpha\epsilon(\sigma)) . \tag{2.27}$$

$\epsilon(\sigma)$ is a stair-step function with the property

$$\epsilon(\sigma+2\pi)=2+\epsilon(\sigma) , \tag{2.28}$$

and it coincides with the usual ϵ function in the interval $[-2\pi,2\pi]$. It is useful to note that the derivatives of $E_\alpha(\sigma)$ and $E_{1/\alpha}(\sigma)$ are proportional to the periodic δ function, viz.,

$$\partial_\sigma E_\alpha(\sigma-\sigma')=\frac{1}{C(\alpha)}\delta(\sigma-\sigma') , \tag{2.29}$$

$$\partial_\sigma E_{1/\alpha}(\sigma-\sigma')=-\frac{1}{C(\alpha)}\delta(\sigma-\sigma') . \tag{2.30}$$

In addition to the equations of motion discussed above, there are constraint equations which follow from general covariance, namely those expressing the vanishing of the energy-momentum tensor $T_{\alpha\beta}$, which is obtained by varying the action (2.9) with respect to the reference metric $\hat{g}$. Classical parts of $T_{\alpha\beta}$, after setting $\hat{g}_{\alpha\beta}=\eta_{\alpha\beta}$, are given in the light-cone coordinates by

$$T_{\pm\pm}=(\partial_\pm\Phi)^2-\partial_\pm^2\Phi+2\partial_\pm\rho\partial_\pm\Phi+\tfrac{1}{2}(\partial_\pm \mathbf{f})^2 , \tag{2.31}$$

$$T_{+-}=\partial_+\partial_-\Phi+\mu^2 e^{\Phi+2\rho} . \tag{2.32}$$

From the previous equation of motion for ρ [Eq. (2.12)], T_{+-} is seen to vanish, showing the on-shell conformal invariance at the classical level. As for $T_{\pm\pm}$, use of Eqs. (2.17) and (2.18) readily yields

$$T_{\pm\pm}=\partial_\pm\chi\partial_\pm\psi-\partial_\pm^2\chi+\tfrac{1}{2}(\partial_\pm \mathbf{f})^2 . \tag{2.33}$$

Defining $\tilde{\phi}_1$ and $\tilde{\phi}_2$ by

$$\psi=\frac{1}{\sqrt{2}}(\tilde{\phi}_1+\tilde{\phi}_2) , \tag{2.34}$$

$$\chi=\frac{1}{\sqrt{2}}(\tilde{\phi}_1-\tilde{\phi}_2) , \tag{2.35}$$

it can be diagonalized as

$$T_{\pm\pm}=\tfrac{1}{2}(\partial_\pm\tilde{\phi}_1)^2-\frac{1}{\sqrt{2}}\partial_\pm^2\tilde{\phi}_1-\tfrac{1}{2}(\partial_\pm\tilde{\phi}_2)^2 +\frac{1}{\sqrt{2}}\partial_\pm^2\tilde{\phi}_2+\tfrac{1}{2}(\partial_\pm \mathbf{f})^2 . \tag{2.36}$$

Thus the vanishing of $T_{\pm\pm}$ simply relates the chiral components of the three types of free fields, χ, ψ, and $\mathbf{f}$. In order to enforce the proper boundary conditions, we find it advantageous to take the functions $A(x^+)$, $B(x^-)$, and χ as arbitrary and compute the corresponding ψ and $\mathbf{f}$.

$A(x^+)$ and $B(x^-)$ with the proper boundary conditions can be written as

$$A(x^+)=\mu e^{(P_\psi/4\pi)x^+}a(x^+) , \tag{2.37}$$

$$B(x^-)=\mu e^{(P_\psi/4\pi)x^-}b(x^-) , \tag{2.38}$$

where $a(x^+)$ and $b(x^-)$ are arbitrary periodic functions. Then by simple calculations, $\psi^\pm(x^\pm)$ and $(\partial_\pm \mathbf{f})^2$ can be expressed as

$$\psi_+(x^+)=\frac{P_\psi}{4\pi}x^+ +\ln\left[\frac{P_\psi}{4\pi}a(x^+)+\partial_+a(x^+)\right] , \tag{2.39}$$

$$\psi_-(x^-)=\frac{P_\psi}{4\pi}x^- +\ln\left[\frac{P_\psi}{4\pi}b(x^-)+\partial_+b(x^-)\right] , \tag{2.40}$$

$$\tfrac{1}{2}(\partial_+f)^2=-\left[\frac{P_\psi}{4\pi}+\frac{\frac{P_\psi}{4\pi}\partial_+a+\partial_+^2a}{\frac{P_\psi}{4\pi}a+\partial_+a}\right]\partial_+\chi+\partial_+^2\chi , \tag{2.41}$$

$$\tfrac{1}{2}(\partial_-f)^2=-\left[\frac{P_\psi}{4\pi}+\frac{\frac{P_\psi}{4\pi}\partial_-b+\partial_-^2b}{\frac{P_\psi}{4\pi}a+\partial_-b}\right]\partial_-\chi+\partial_-^2\chi . \tag{2.42}$$

In terms of these quantities, the original metric $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta}=e^{\psi}(\chi-AB)^{-1}\eta_{\alpha\beta} =e^{\psi}[\chi-\mu^2e^{(P_\psi/2\pi)t}a(x^+)b(x^-)]^{-1}\eta_{\alpha\beta} . \tag{2.43}$$

This is the *finite-universe* version of the general solution obtained by CGHS [3].

Let us give a simple example which describes a black hole metric in the limit that the size of the Universe L tends to infinity. It is given by the choice

$$\chi=c=\text{const},\quad \partial_\pm \mathbf{f}=0 , \tag{2.44}$$

$$a(x^+)=\sin x^+,\quad b(x^-)=\sin x^- . \tag{2.45}$$

The corresponding $\psi^\pm$ are given by

$$\psi_\pm=\frac{P_\psi}{4\pi}x^\pm+\ln\left[\frac{P_\psi}{4\pi}\sin x^\pm+\cos x^\pm\right] . \tag{2.46}$$

Notice that in the limit $L\to\infty$, $\psi^\pm$ vanish. Recalling that $\mu=\lambda L$, the metric becomes, in this limit,

$$\lim_{L\to\infty} g_{\alpha\beta}=\lim_{L\to\infty}[e^{\psi}(c-\mu^2e^{(P_\psi/2\pi)t}\sin x^+\sin x^-)^{-1}\eta_{\alpha\beta}] =(c-\lambda^2\xi^+\xi^-)^{-1}\eta_{\alpha\beta} , \tag{2.47}$$

which describes a black hole. We must note here that in fact infinitely many other configurations lead to the same $L\to\infty$ limit. Specifically, any choices of $a(x^+)$ and $b(x^-)$ for which $La(x^+)$ and $Lb(x^-)$ tend, respectively, to ξ_+ and ξ_- are indistinguishable in the limit of a large universe. We will have more to say on this point in the final section.

III. QUANTIZATION OF THE MODEL

A. Canonical transformation and operator quantization

In the previous section, we found that the general solution to the classical equations of motion can be described by three types of free fields, ψ, χ, and $\mathbf{f}$. Quantization of the matter sector is trivial since f_i's are canonical right from the beginning. For the dilaton-Liouville sector, however, it is not yet obvious what combinations of ψ and χ are to be quantized as *canonical* free fields for the following reasons: First, any function of free fields is again a free field and depending on the choice an additional functional Jacobian may arise. Moreover, the form of the energy-momentum tensor $T_{\pm\pm}$ alone is in general not sufficient to settle the question. After all, the fact that they take "free-field form," as in (2.31), does not even guarantee that the fields appearing in them are free fields.

Nevertheless, the expression obtained in (2.36) is suggestive. It appears to imply that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ should be regarded as canonical free fields and that, while the former being a normal field, the latter should be treated as a "negative metric" field. (This should not be taken as dictating the way the inner product should be defined. It is a separate problem to be discussed in the final section.) In this subsection, we shall prove rigorously that this expectation is indeed correct. Namely, it will be shown that the transformation of fields $(\Phi,\rho)\to(\tilde{\phi}_1,\tilde{\phi}_2)$ is canonical quantum mechanically as well as classically.

Let us begin with classical analysis. First, for later convenience, we shall rescale the fields $\tilde{\phi}_1$ and $\tilde{\phi}_2$ to define canonically normalized fields ϕ_1 and ϕ_2 and expand them into Fourier modes as follows:

$$\phi_i \equiv \sqrt{4\pi}\tilde{\phi}_i \ , \tag{3.1}$$

$$\phi_i(x^+,x^-)=\phi_i^+(x^+)+\phi_i^-(x^-) \ , \tag{3.2}$$

$$\phi_i^{\pm}(x^\pm)=\frac{q^i}{2}+p^i x^\pm + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^{(i,\pm)}e^{-inx^\pm} \ . \tag{3.3}$$

We take the basic Poisson brackets to be

$$i\{\alpha_m^{(1,\pm)},\alpha_n^{(1,\pm)}\}=m\delta_{m+n,0} \ , \tag{3.4}$$

$$\{q^1,p^1\}=1 \ , \tag{3.5}$$

$$i\{\alpha_m^{(2,\pm)},\alpha_n^{(2,\pm)}\}=-m\delta_{m+n,0} \ , \tag{3.6}$$

$$\{q^2,p^2\}=-1 \ , \tag{3.7}$$

$$\text{rest}=0 \ . \tag{3.8}$$

After some calculations, this leads to

$$\{\phi_1^+(x^+),\phi_1^+(y^+)\}=\tfrac{1}{2}(x^+-y^+)-\pi\epsilon(x^+-y^+) \ , \tag{3.9}$$

$$\{\phi_1^-(x^-),\phi_1^-(y^-)\}=\tfrac{1}{2}(x^--y^-)-\pi\epsilon(x^--y^-) \ , \tag{3.10}$$

$$\{\phi_1^+(x^+),\phi_1^-(y^-)\}=\tfrac{1}{2}(y^--x^+) \ , \tag{3.11}$$

$$\{\phi_1^-(x^-),\phi_1^+(y^+)\}=\tfrac{1}{2}(y^+-x^-) \ , \tag{3.12}$$

$$\{\phi_1(x),\phi_1(y)\}=-\pi[\epsilon(x^+-y^+)+\epsilon(x^--y^-)] \ , \tag{3.13}$$

and similar expressions with all the signs reversed for ϕ_2. (The ϵ function here is the stair-step function defined in the previous section.)

In terms of ϕ_i, the fields ψ and χ now take the form

$$\psi=\frac{1}{\sqrt{8\pi}}(\phi_1+\phi_2) \ , \tag{3.14}$$

$$\chi=\frac{1}{\sqrt{8\pi}}(\phi_1-\phi_2) \ , \tag{3.15}$$

and their Fourier modes, defined in Eqs. (2.19) and (2.20), are easily seen to satisfy the Poisson brackets relations

$$i\{\alpha_m^\pm,\beta_n^\pm\}=i\{\beta_m^\pm,\alpha_n^\pm\}=m\delta_{m+n,0} \ , \tag{3.16}$$

$$\{Q_\psi,P_\chi\}=1 \ , \tag{3.17}$$

$$\{Q_\chi,P_\psi\}=1 \ , \tag{3.18}$$

$$\text{rest}=0 \ . \tag{3.19}$$

Then with the use of previous formulas for ϕ_i, the following brackets are readily obtained:

$$\{\chi(x),\psi^+(y^+)\}=\frac{1}{8\pi}(x^+-x^-)-\tfrac{1}{4}\epsilon(x^+-y^+) \ , \tag{3.20}$$

$$\{\chi(x),\psi^-(y^-)\}=-\frac{1}{8\pi}(x^+-x^-)-\tfrac{1}{4}\epsilon(x^--y^-) \ , \tag{3.21}$$

$$\{\psi(x),\psi(y)\}=\{\chi(x),\chi(y)\}=0 \ , \tag{3.22}$$

$$\{\psi(x),\chi(y)\}=\{\chi(x),\psi(y)\} = -\tfrac{1}{4}[\epsilon(x^+-y^+)+\epsilon(x^--y^-)] \ . \tag{3.23}$$

We are now ready to prove the canonical nature of the transformation. The Poisson brackets we wish to reproduce *at equal time* are

$$\{\Phi(x),\Pi_\Phi(y)\}_{\mathrm{ET}}=\delta(\sigma_x-\sigma_y) \ ,$$

$$\{\rho(x),\Pi_\rho(y)\}_{\mathrm{ET}}=\delta(\sigma_x-\sigma_y) \ , \tag{3.24}$$

$$\text{rest}=0 \ ,$$

where the momentum fields are given by

$$\Pi_\Phi=2(\dot{\Phi}+\dot{\rho}) \ , \tag{3.25}$$

$$\Pi_\rho=2\dot{\Phi} \ . \tag{3.26}$$

Thanks to the fact that the modes of ψ have vanishing Poisson brackets with each other, the only nontrivial brackets to be compared are $\{\chi(x),AB(y)\}$ and the time derivatives thereof at equal time. Calculations are somewhat tedious and very similar to the ones needed for the self-interacting Liouville theory [24]. In particular one needs to be careful about the presence of the momentum zero mode in the functions $E_\alpha(\sigma)$ and $E_{1/\alpha}(\sigma)$ inside AB. Useful formulas are listed in Appendix A.

With the help of these formulas, it is straightforward to get the brackets

$$\{\Phi(x),\Phi(y)\}_{\mathrm{ET}} = -\{\chi(x),AB(y)\}_{\mathrm{ET}}-\{AB(x),\chi(y)\}_{\mathrm{ET}}=0 \ , \tag{3.27}$$

$$\{\dot{\Phi}(x),\Phi(y)\}_{\mathrm{ET}}=0 \ , \tag{3.28}$$

$$\{\dot{\Phi}(x),\dot{\Phi}(y)\}_{\mathrm{ET}}=0 \ , \tag{3.29}$$

$$\{\rho(x),\rho(y)\}_{\mathrm{ET}} = -\tfrac{1}{4}[\{\chi(x),\psi(y)\}_{\mathrm{ET}}+\{\psi(x),\chi(y)\}_{\mathrm{ET}}]=0 \ . \tag{3.30}$$

$$\{\dot{\rho}(x),\rho(y)\}_{\mathrm{ET}}=\tfrac{1}{2}\delta(\sigma_x-\sigma_y) \ , \tag{3.31}$$

$$\{\dot{\rho}(x),\dot{\rho}(y)\}_{\mathrm{ET}}=0 \ , \tag{3.32}$$

$$\{\Phi(x),\rho(y)\}_{\mathrm{ET}}=\tfrac{1}{2}\{\chi(x),\psi(y)\}_{\mathrm{ET}}=0 \ , \tag{3.33}$$

$$\{\dot{\Phi}(x),\rho(y)\}_{\mathrm{ET}}=-\tfrac{1}{2}\delta(\sigma_x-\sigma_y) \ , \tag{3.34}$$

$$\{\Phi(x),\dot{\rho}(y)\}_{\rm ET}=\tfrac{1}{2}\delta(\sigma_x-\sigma_y)\ , \quad (3.35)$$

$$\{\dot{\Phi}(x),\dot{\rho}(y)\}_{\rm ET}=0\ . \quad (3.36)$$

From these bracket relations, it is evident that our transformation correctly reproduces the canonical Poisson bracket relations (3.24).

Now we can quantize the theory by the replacement

$$\{\phi_i,\phi_j\}_{\rm ET}\to(-i)[\phi_i,\phi_j]\ . \quad (3.37)$$

To prove the canonical nature of the transformations quantum mechanically, we must define the composite operator AB. Here we have a situation far simpler than the corresponding case for the Liouville theory [23,24]: All the modes of ψ commute with each other, and AB is well defined without the need of normal ordering. Thus all the Poisson brackets relations previously obtained can be directly converted into quantum commutation relations, and the quantum canonicity follows immediately from the classical one.

B. Conformal properties

As was shown in Sec. II B, the off-diagonal part of the energy-momentum tensor T_{+-} vanishes due to the equations of motion and the model has invariance under the left- and right-conformal transformations. Upon quantization the left-Virasoro generators for the dilaton-Liouville and the matter sectors take the form

$$L_n^{dL}=L_n^1+L_n^2\ , \quad (3.38)$$

$$L_n^1=\tfrac{1}{2}\sum_m :\alpha^1_{n-m}\alpha^1_m:+iQn\alpha^1_n\ , \quad (3.39)$$

$$L_n^2=-\tfrac{1}{2}\sum_m :\alpha^2_{n-m}\alpha^2_m:-iQn\alpha^2_n\ , \quad (3.40)$$

$$L_n^f=\tfrac{1}{2}\sum_m :\boldsymbol{\alpha}^f_{n-m}\cdot\boldsymbol{\alpha}^f_m:\ , \quad (3.41)$$

where the background charge Q is given by $Q=\sqrt{2\pi}$. Paying attention to the negative metric nature of α_n^2's, we readily obtain

$$[L_m^1,L_n^1]=(m-n)L^1_{m+n}+\frac{1+12Q^2}{12}(m^3-m)\delta_{m+n,0}+Q^2m\delta_{m+n,0}\ , \quad (3.42)$$

$$[L_m^2,L_n^2]=(m-n)L^2_{m+n}+\frac{1-12Q^2}{12}(m^3-m)\delta_{m+n,0}-Q^2m\delta_{m+n,0}\ , \quad (3.43)$$

$$[L_m^{dL},L_n^{dL}]=(m-n)L^{dL}_{m+n}+\frac{2}{12}(m^3-m)\delta_{m+n,0}\ , \quad (3.44)$$

$$[L_m^f,L_n^f]=(m-n)L^f_{m+n}+\frac{N}{12}(m^3-m)\delta_{m+n,0}\ . \quad (3.45)$$

Thus, both L_n^{dL} and L_n^f satisfy the standard form of the Virasoro algebra with the central charge 2 and N, respectively, and together with the ghost contribution the total conformal anomaly vanishes for $N=24$.

Let us now discuss the conformal properties of the basic fields and the operators involving their exponentials. First, for the positive metric field ϕ_1, we easily find

$$[L_m^1,\phi_1(x)]=e^{imx^+}\left[\frac{1}{i}\partial_+\phi_1+Qm\right]\ . \quad (3.46)$$

As for the negative metric field ϕ_2, the sign of L_m, as well as those of the commutation relations, are reversed, and the net result is identical with the positive metric case. Combining them we immediately get

$$[L_m,\psi(x)]=e^{imx^+}\left[\frac{1}{i}\partial_+\psi+\frac{Q}{\sqrt{2\pi}}m\right]\ , \quad (3.47)$$

$$[L_m,\chi(x)]=e^{imx^+}\frac{1}{i}\partial_+\chi\ . \quad (3.48)$$

Note that χ field transforms as a genuine primary field with conformal dimension 0.

Next consider the simple exponential operator of the form $e^{\lambda\phi}$ where ϕ is either ϕ_1 or ϕ_2. For the "cylinder" type coordinates under use for our spatially closed Lorentzian universe, the proper normal ordering is the *symmetric* normal ordering defined by

$$:e^{\lambda\phi(x)}:=e^{\lambda q/2}e^{p(x^++x^-)}e^{\lambda q/2}\times e^{\lambda\phi_c^+(x^+)}e^{\lambda\phi_a^+(x^+)}e^{\lambda\phi_c^-(x^-)}e^{\lambda\phi_a^-(x^-)}\ , \quad (3.49)$$

where $\phi_a^\pm$ and $\phi_c^\pm$ refer to the annihilation and creation part of the nonzero modes, respectively. If we adopt the usual Hermiticity assignment for the modes this operator is manifestly Hermitian for real λ. More importantly, it becomes a primary field only with this normal ordering. As we cannot directly make use of the Euclidean operator product technique, the calculation of the commutator $[L_n,e^{\lambda\phi}]$ is slightly tedious. However the results are standard: For ϕ_1 and ϕ_2 we obtain

$$[L_n^1,e^{\lambda\phi_1(x)}]=e^{inx^+}\left[\frac{1}{i}\partial_++n\left[-\frac{\lambda^2}{2}+Q\lambda\right]\right]e^{\lambda\phi_1(x)}\ , \quad (3.50)$$

$$\dim e^{\lambda\phi_1}=-\frac{\lambda^2}{2}+Q\lambda\ , \quad (3.51)$$

$$[L_n^2,e^{\lambda\phi_2(x)}]=e^{inx^+}\left[\frac{1}{i}\partial_++n\left[\frac{\lambda^2}{2}+Q\lambda\right]\right]e^{\lambda\phi_2(x)}\ , \quad (3.52)$$

$$\dim e^{\lambda\phi_2}=\frac{\lambda^2}{2}+Q\lambda\ . \quad (3.53)$$

This immediately implies that $e^{\lambda\psi}$ and $e^{\lambda\chi}$ are primary fields with conformal dimensions:

$$\dim e^{\lambda\psi}=\frac{Q}{\sqrt{2\pi}}\lambda=\lambda\ , \quad (3.54)$$

$$\dim e^{\lambda\chi}=0\ \ (\text{independent of }\lambda)\ . \quad (3.55)$$

Finally we need to clarify the conformal property of the composite operator $AB(x)$. As remarked in Sec. II B, this operator consists only of modes of ψ, and there is no necessity of normal ordering. The calculation of the commutator with the Virasoro generator is rather tedious due to the nonlocal nature of the operator. However, if we formally adopt the symmetric normal ordering it becomes almost identical to the corresponding case encountered in the interacting Liouville theory [24] and we can easily transcribe their procedure. The final result is

$$[L_n^{dL}, AB(x)] = e^{inx^+}\frac{1}{i}\frac{\partial}{\partial x^+}[AB(x)] , \tag{3.56}$$

showing that AB is a primary field with dimension zero just like the field χ. It means that our fundamental field $\Phi = \chi - AB$ as a whole behaves as a dimension zero primary field, a gratifying result.

IV. BRST ANALYSIS OF PHYSICAL STATES AND OPERATORS

Having quantized the model in a rigorous manner, we now construct the physical states and the operators of the theory in the BRST formalism. Because of the conformal symmetry, the analysis will be quite similar to the one for the string theory. In fact, technically, our model has features which are hybrid of the critical and noncritical string theories: It can be regarded as a critical bosonic string theory since the total central charge is made up of the contributions from 26 free bosons, with matter fields playing the role of the 24 transverse coordinates. On the other hand, the presence of the background charges for the remaining two coordinates leads to structures reminiscent of the $c=1$ noncritical string theory.

Thus in the following, we shall be able to make use of the analysis performed on the noncritical string theory [25,26], albeit with some modifications. These modifications are due (i) to the fact that we stay in Minkowski space, (ii) to the extra presence of the matter fields, and (iii) to the special structure of the background charge terms for the dilaton-Liouville sector. Our analysis is closely analogous to those performed for noninteracting Liouville gravity [27] and for a free field model of dilaton gravity [16] both defined in Euclidean space, but we believe it is useful to give some details and clarify the difference between Minkowski and Euclidean formulations. This will also serve to make this article sufficiently self-contained.

A. Analysis of physical states

1. Preliminary

In this section, we shall deal explicitly with the left-moving sector only and ϕ_i will mean the chiral components with the following mode expansion and the commutation relations:

$$\phi_i(x^+) = \frac{1}{2}q_i + p_i x^+ + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-inx^+} , \tag{4.1}$$

$$\partial_+\phi_i = \sum_{n\in Z}\alpha_n^i e^{-inx^+} \quad (\alpha_0^i \equiv p^i) , \tag{4.2}$$

$$[q_1, p_1] = i, \quad [q_2, p_2] = -i , \tag{4.3}$$

$$[q_1, p_2] = [q_2, p_1] = 0 , \tag{4.4}$$

$$[\alpha_m^1, \alpha_n^1] = -[\alpha_m^2, \alpha_n^2] = m\delta_{m+n,0}, \quad [\alpha_m^1, \alpha_n^2] = 0 . \tag{4.5}$$

Since the Virasoro generators L_n^{dL} and L_n^f given in Eqs. (3.38)–(3.41) both satisfy the standard form of the Virasoro algebra with central charges 2 and 24, respectively, the nilpotent BRST operator, which we shall denote by d, is immediately obtained as

$$d = \sum c_{-n}(L_n^{dL} + L_n^f) - \tfrac{1}{2}\sum :(m-n)c_{-m}c_{-n}b_{m+n}: , \tag{4.6}$$

where sl(2) invariant normal-ordering for the ghosts is assumed. The physical ghost vacuum is defined as usual by $|0\rangle_\downarrow = c_1|0\rangle_{\text{inv}}$.

Hereafter we shall follow the work of Bouwknegt, McCarthy, and Pilch (BMP) [26]. Let us recall their general strategy. First the BRST operator d is decomposed with respect to the ghost zero mode in the form $d = c_0L_0 - Mb_0 + \hat{d}$, where L_0 is the total Virasoro operator including the ghosts. From the well-known relation $L_0 = \{b_0, d\}$ one can deduce by a standard argument that the nontrivial d cohomology must be in the sector satisfying

$$L_0\psi = 0 . \tag{4.7}$$

On the subspace $\mathcal{F}_0$ defined by

$$\mathcal{F}_0 = \{\psi | L_0\psi = 0, \quad b_0\psi = 0\} , \tag{4.8}$$

$\hat{d}$ becomes nilpotent and the d cohomology (absolute cohomology) is reduced to the $\hat{d}$ cohomology (relative cohomology).

The next step is to further decompose the $\hat{d}$ operator with respect to a grading called the *degree* to be assigned to the mode operators. For this purpose, define the following light-cone-like combinations of modes for the dilaton-Liouville sector:

$$q^\pm = \frac{1}{\sqrt{2}}(q_1 \pm q_2) ,$$

$$p^\pm = \frac{1}{\sqrt{2}}(p_1 \pm p_2) , \tag{4.9}$$

$$[q^\pm, p^\pm] = i ,$$

$$\alpha_m^\pm = \frac{1}{\sqrt{2}}(\alpha_m^1 \pm \alpha_m^2), \quad [\alpha_m^\pm, \alpha_n^\mp] = m\delta_{m+n,0} , \tag{4.10}$$

and assign the degree as

$$\deg(\alpha_n^+) = \deg(c_n) = 1, \quad \deg(\alpha_n^-) = \deg(b_n) = -1 . \tag{4.11}$$

The rest of the mode operators, *including the matter part,* are defined to carry degree 0. Then $\hat{d}$ is decomposed according to the degree as

$$\hat{d} = \hat{d}_0 + \hat{d}_1 + \hat{d}_2 , \tag{4.12}$$

$$\hat{d}_0 = \sum_{n\neq 0} P^+(n)c_{-n}\alpha_n^- , \tag{4.13}$$

$$\hat{d}_1 = \sum_{n.z.m.} :c_{-n}[\alpha^+_{-m}\alpha^-_{m+n} + \tfrac{1}{2}(m-n)c_{-m}b_{m+n} + L^f_n]: \,, \tag{4.14}$$

$$\hat{d}_2 = \sum_{n\neq 0} P^-(n)c_{-n}\alpha^+_n \,, \tag{4.15}$$

where $P^{\pm}(n)$ are given by

$$P^+(n) = \frac{1}{\sqrt{2}}(p_1 + p_2 + 2iQn) \,, \tag{4.16}$$

$$P^-(n) = p^- \,. \tag{4.17}$$

Notice that, because of the special structure of the background charges, $P^-(n)$ is independent of n, and, further, that the n dependence of $P^+(n)$ is shifted by 1 unit compared with the Euclidean case. It is easy to check that $\hat{d}_0^2 = \hat{d}_2^2 = 0$ holds identically and furthermore, $[\hat{d}_0, L_0] = \{\hat{d}_0, b_0\} = 0$, and similarly for $\hat{d}_2$. This allows us to consider $\hat{d}_0$ and $\hat{d}_2$ cohomologies actually on the entire Fock space $\mathcal{F}$.

Let us summarize, in advance, the rest of the procedure: Depending on the conditions on $P^{\pm}(n)$, we study the $\hat{d}_0$ or $\hat{d}_2$ cohomology on the Fock space $\mathcal{F}$ and then restrict it to the relative cohomology space $\mathcal{F}_0$. Then we shall show that for each case such a cohomology is in one to one correspondence with the relative $\hat{d}$ cohomology and explicitly construct the representatives for the $\hat{d}$ cohomology. The final step consists of the construction of d cohomology from the $\hat{d}$ cohomology. It turns out that except for the single case where the relative cohomology contains $b_{-1}|\mathbf{P}\rangle_\downarrow$, the absolute cohomology can be obtained in the simple way: If ψ is an element of $\hat{d}$ cohomology, then ψ and $c_0\psi$ represent the possible d cohomology. Therefore in the following, we shall consider explicitly the $\hat{d}$ cohomology only and will comment on the d cohomology when we deal with the special state mentioned above.

2. $P^+(n)\neq 0$ ($\forall n \in \mathbf{Z}$, $n\neq 0$) case

Let us begin with the case in which $P^+(n)\neq 0$ for all nonzero integer n. In this case $\hat{d}_0$ cohomology is useful since one can define the operator

$$K^+ \equiv \sum_{n\neq 0} \frac{1}{P^+(n)}\alpha^+_{-n}b_n \,, \tag{4.18}$$

which satisfies

$$\{\hat{d}_0, K^+\} = \sum_{n\neq 0} :(nc_{-n}b_n + \alpha^+_{-n}\alpha^-_n): + 1 \tag{4.19}$$

$$= \hat{N}^{\mathrm{DLG}} \,, \tag{4.20}$$

where $\hat{N}^{\mathrm{DLG}}$ is the level counting operator for the dilaton-Liouville-ghost (DLG) sector. By a standard argument, nontrivial $\hat{d}_0$ cohomology can only be in the sector where $\hat{N}^{\mathrm{DLG}}$ vanishes. We must also satisfy the $L_0\psi = 0$ condition, which in this case reads

$$p^+p^- + \tfrac{1}{2}\mathbf{p}_f^2 + \hat{N}^f - 1 = 0 \,, \tag{4.21}$$

where $\hat{N}^f$ counts the level for the matter sector. The states satisfying these conditions are indeed $\hat{d}_0$ nontrivial. To see this, look for a state Ω such that $\hat{d}_0\Omega$ gives a state without any nonzero mode excitations in the DLG sector. Because of the form of $\hat{d}_0$, Ω must be of the form $\sum_{m\neq 0}(A_m\alpha^+_{-m}b_m)\Omega_0$, where Ω_0 is a state without DLG excitations, but obviously such an expression vanishes.

It is evident from Eq. (4.21) that for $p^+p^- > 0$ only a zero-mode excitation is allowed in the matter sector. On the other hand, for $p^+p^- \leq 0$ there can be matter excitations at nonzero Virasoro levels. As will be discussed in Sec. V, this case will be of great importance especially when one considers $L \to \infty$ limit.

We now describe the construction of $\hat{d}$ cohomology including the latter case. Essentially we follow the procedure described in BMP, but due to the presence of the matter degrees of freedom, a part of the arguments will have to be modified.

Let ψ_0 be a $\hat{d}_0$ nontrivial state of the form $F_{-N}|\mathbf{P}\rangle_\downarrow$ where F_{-N} is a matter operator at level N and $\mathbf{P} = (p^+, p^-, \mathbf{p}_f)$. We have $\hat{d}_0\psi_0 = 0$ and in addition $\hat{d}_2\psi_0 = 0$ as well. Therefore, $\hat{d}\psi_0 = \hat{d}_1\psi_0$ and according to the general argument of BMP (see the Appendix of [26]) this state must be $\hat{d}_0$ exact. Thus we look for ψ_1 of degree 1 such that $\hat{d}_1\psi_0 = -\hat{d}_0\psi_1$. Let us apply the operator K^+ introduced previously. Then one gets

$$\begin{aligned} K^+\hat{d}_1\psi_0 &= -K^+\hat{d}_0\psi_1 \\ &= [-\{K^+, \hat{d}_0\} + \hat{d}_0K^+]\psi_1 \\ &= -\hat{N}^{\mathrm{DLG}}\psi_1 + \hat{d}_0K^+\psi_1 \,. \end{aligned} \tag{4.22}$$

It is instructive to write down the explicit form of $\hat{d}_1\psi_0$. It is given by

$$\hat{d}_1\psi_0 = \sum_{n\geq 1} c_{-n}L^f_nF_{-N}|\mathbf{P}\rangle_\downarrow \,. \tag{4.23}$$

Evidently, because of the presence of the matter, this is *not* an eigenstate of $\hat{N}^{\mathrm{DLG}}$ unlike the case treated in BMP. Similarly, $K^+\hat{d}_1\psi_0$ is also not an eigenstate. Nevertheless, *for each term making up such states,* $\hat{N}^{\mathrm{DLG}}$ does have positive integral value and this will be enough to consider the inverse operator $\hat{N}^{-1}_{\mathrm{DLG}}$ on such states. Since the rest of the argument, which is also modified from that in BMP, is somewhat technical, we shall relegate it to Appendix B. When all the dust settled, the result turned out to be formally identical to the one obtained in BMP. Namely, a representative ψ of the $\hat{d}$ cohomology corresponding to a $\hat{d}_0$ cohomology represented by $\psi_0 = F_{-N}|\mathbf{P}\rangle_\downarrow$ is given by

$$\psi = \sum_{n=0}^{\infty}(-1)^n(T^+)^n\psi_0 \,, \tag{4.24}$$

where the operator T^+ is defined by

$$T^+ \equiv \hat{N}^{-1}_{\mathrm{DLG}}K^+\hat{d}_1 \,. \tag{4.25}$$

Let us give some examples. For $F_{-N} = \alpha^i_{-1}$ and α^i_{-2}, application of the formula above gives the following physical states $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$|\psi_1\rangle=\left[\alpha^i_{-1}-\frac{p^i_f}{P^+(1)}\alpha^+_{-1}\right]|\mathbf{P}\rangle_\downarrow\ , \tag{4.26}$$

$$|\psi_2\rangle=\left[\alpha^i_{-2}-\frac{2}{P^+(1)}\alpha^i_{-1}\alpha^+_{-1}-\frac{p^i_f}{P^+(2)}\alpha^+_{-2}+\frac{p^i_f}{P^+(1)}\left\{\frac{1}{P^+(1)}+\frac{1}{P^+(2)}\right\}(\alpha^+_{-1})^2\right]|\mathbf{P}\rangle_\downarrow\ . \tag{4.27}$$

If we were dealing with a string theory, this would describe the transversality condition. However, in the dilation gravity context, it means that whenever matter fields are present they necessarily induce excitations in the dilaton-Liouville sector.

3. $p^-\neq 0$ case

In this case, the relevant tool is the $\hat{d}_2$ cohomology. The procedure is entirely similar to the previous case, except for the use of $K^-\equiv(1/p^-)\sum_{n\neq 0}\alpha^-_{-n}b_n$ in place of K^+. $L_0\psi=0$ condition is the same as given in (4.21) and we obtain the representative for $\hat{d}$ cohomology corresponding to Eqs. (4.24) and (4.25) as

$$\psi=\sum_{n=0}^{\infty}(-1)^n(T^-)^n\psi_0\ , \tag{4.28}$$

$$T^-\equiv\hat{N}^{-1}_{\mathrm{DLG}}K^-\hat{d}_1\ . \tag{4.29}$$

4. $P^+(r)=0$, $p^-=0$ case

Now we consider the remaining case where $P^+(r)=0$ for some integer r and at the same time $p^-=0$. Physical states for this case have come to be called "discrete states." We look at the $\hat{d}_0$ cohomology and define the operator K^+_r analogously to K^+ except without the term involving $P^+(r)$. That is,

$$K^+_r\equiv\sum_{n\neq 0,r}\frac{1}{P^+(n)}\alpha^+_{-n}b_n\ . \tag{4.30}$$

This operator satisfies the relation $\{\hat{d}_0,K^+_r\}=\hat{N}^{\mathrm{DLG}}_r$, where $\hat{N}^{\mathrm{DLG}}_r$ is the level-counting operator for the DLG sector this time excluding the rth level, and it must vanish in order for $\hat{d}_0$ cohomology to be nontrivial. $L_0\psi=0$ condition now reads

$$\tfrac{1}{2}\mathbf{p}^2_f+\hat{N}^f+\hat{N}_{\mathrm{DLG}}-1=0\ . \tag{4.31}$$

Since $\mathbf{p}^2_f$ and $\hat{N}^f$ are nonnegative, we see that $\hat{N}^{\mathrm{DLG}}$ can either be 1 or 0. Together with the vanishing of $\hat{N}^{\mathrm{DLG}}_r$ this means that $r=\pm 1$ and only a level 1 excitation is possibly allowed in the DLG sector. We now study these cases separately.

$r=1$ case: When there is an excitation at level 1 in the DLG sector, $\mathbf{p}_f=0$ and we find two types of $\hat{d}_0$ cohomology represented by the following states:

$$(A_+\alpha^+_{-1}+\mathbf{A}\cdot\boldsymbol{\alpha}^f_{-1})|\mathbf{P}\rangle_\downarrow\ \text{(ghost number 0)}\ , \tag{4.32}$$

$$c_{-1}|\mathbf{P}\rangle_\downarrow\ \text{(ghost number 1)}\ , \tag{4.33}$$

where A_+ and $\mathbf{A}$ are arbitrary coefficients. These states are easily checked to be $\hat{d}$ nontrivial as well. For the latter state, it is instructive to compare with the usual bosonic string case with $Q=0$. In that case all the momenta vanish and we have $c_{-1}|0\rangle$. This state, however, is d exact. Indeed

$$dc_0b_{-1}|0\rangle_\downarrow=b_0Mc_0b_{-1}|0\rangle_\downarrow=c_{-1}|0\rangle\ . \tag{4.34}$$

This phenomenon is allowed because BMP theorem 4.2 breaks down. Namely, in this particular configuration of the momenta, there exists another $\hat{d}$-nontrivial state $b_{-1}|0\rangle$ at ghost number -1 and the argument of theorem 4.2 which assumes the nonexistence of cohomologies separated by 2 units of ghost number is no longer valid.

Now for $\hat{N}^{\mathrm{DLG}}=0$, we simply have $|\mathbf{P}\rangle_\downarrow$ as representing nontrivial $\hat{d}_0$ and $\hat{d}$ cohomology, where $\mathbf{p}^2_f=2$.

$r=-1$ case: Going through an analysis similar to the previous case using K^+_{-1}, we find the following two nontrivial $\hat{d}_0$ cohomology representatives:

$$(A_-\alpha^-_{-1}+\mathbf{A}\cdot\boldsymbol{\alpha}^f_{-1})|\mathbf{P}\rangle_\downarrow\ , \tag{4.35}$$

$$b_{-1}|\mathbf{P}\rangle_\downarrow\ . \tag{4.36}$$

It is easy to check that these states represent $\hat{d}$ cohomologies as well.

Finally we must make a comment on the absolute cohomology which arises from $\psi\equiv b_{-1}|\mathbf{P}\rangle_\downarrow$. According to the general argument of BMP, we have, in addition to ψ itself, the second member $c_0\psi+\chi$, where χ is determined by the equation $M\psi=\hat{d}\chi$. Explicitly, this reads

$$M\psi=2c_{-1}c_1b_{-1}|\mathbf{P}\rangle_\downarrow=2c_{-1}|\mathbf{P}\rangle_\downarrow=\hat{d}\frac{2}{P^+(1)}\alpha^+_{-1}|\mathbf{P}\rangle_\downarrow\ . \tag{4.37}$$

Thus the second member of the absolute cohomology takes the form

$$\left[c_0b_{-1}+\frac{2}{P^+(1)}\alpha^+_{-1}\right]|\mathbf{P}\rangle_\downarrow\ . \tag{4.38}$$

As mentioned previously, this is the only exception in which the second member of absolute cohomology is not given simply by appending the c_0 operator to a representative of $\hat{d}$ cohomology.

B. Physical operators

Having obtained all the BRST nontrivial states, we now wish to discuss the corresponding physical operators. It is well known that in the *Euclidean plane coordinate* formulation a state $|\psi\rangle$ corresponding to an operator $\Psi(z,\bar{z})$ is given by

$$\lim_{\substack{z\to 0\\ \bar{z}\to 0}}\Psi(z,\bar{z})|0\rangle=|\psi\rangle\ , \tag{4.39}$$

where $|0\rangle$ is the sl(2) invariant vacuum. However, in the *Minkowski cylinder coordinate* we cannot directly use this procedure. Thus we shall first develop a useful machinery which allows us to convert between these two types of formulations and then try to make use of the simple correspondence (4.39).

Consider in Minkowski space the left conformal transformation of a scalar field ϕ with a background charge Q. It is given by

$$[T_\epsilon,\phi(x^+)]=\frac{1}{i}[\epsilon(x^+)\partial_+\phi(x^+)+Q\partial_+\epsilon(x^+)] , \quad (4.40)$$

where

$$T_\epsilon=\sum\epsilon_{-n}L_n^M , \quad (4.41)$$

$$\epsilon(x^+)\equiv\sum\epsilon_{-n}e^{inx^+} , \quad (4.42)$$

$$[L_n^M,\phi(x^+)]=e^{inx^+}\left[\frac{1}{i}\partial_+\phi+Qn\right] . \quad (4.43)$$

This implies that for a finite transformation (including the right transformation) given by $y^+=y^+(x^+)$, $y^-=y^-(x^-)$, the field ϕ transforms as

$$\phi(y)=\phi(x)-Q\ln\left[\frac{dy^+}{dx^+}\right]-Q\ln\left[\frac{dy^-}{dx^-}\right] . \quad (4.44)$$

Applying this result to the case of interest, namely,

$$y^+=z=e^{ix^+} , \quad (4.45)$$

$$y^-=\bar{z}=e^{ix^-} , \quad (4.46)$$

we immediately get

$$\phi^{\rm EP}(z,\bar{z})=\phi^M(x^+,x^-)-iQ(x^++x^-)-2Q\ln(i) , \quad (4.47)$$

where we have supplemented the superscript EP and M to distinguish the Euclidean-plane and Minkowski fields. By writing out the Fourier components, this means the following identifications:

$$q^{\rm EP}=q^M-iQ\pi , \quad (4.48)$$

$$p^{\rm EP}=p^M-iQ , \quad (4.49)$$

$$\alpha_n^{\rm EP}=\alpha_n^M . \quad (4.50)$$

An alternative more useful way of effecting this transformation is to agree to use *the same mode operators* and regard it as a similarity transformation. Specifically,

$$\mathcal{U}\phi^{\rm EP}\mathcal{U}^{-1}=\phi^M , \quad (4.51)$$

$$\mathcal{U}=e^{Qq}e^{Q\pi p} . \quad (4.52)$$

It is not difficult to check explicitly that this operation correctly converts the Virasoro generators. Namely,

$$L_n^{\rm EP}=\tfrac{1}{2}:\sum_m\alpha_{n-m}\alpha_m:+iQ(n+1)\alpha_n , \quad (4.53)$$

$$\mathcal{U}L_n^{\rm EP}\mathcal{U}^{-1}=L_n^M , \quad (4.54)$$

$$L_n^M=\tfrac{1}{2}:\sum_m\alpha_{n-m}\alpha_m:+iQn\alpha_n+\frac{Q^2}{2}\delta_{n,0} . \quad (4.55)$$

It is obvious from this relation that $L_n^{\rm EP}$ and L_n^M both satisfy the standard form of the Virasoro algebra.

Now the operator $\Psi(x^+,x^-)_M$ corresponding to a given state $|\psi\rangle_M$ in Minkowski space can be found through the following procedure: First get the corresponding Euclidean state $|\psi\rangle_E$ by $|\psi\rangle_E=\mathcal{U}^{-1}|\psi\rangle_M$. Then by the usual correspondence find the Euclidean operator $\Psi(z,\bar{z})_E$. Finally, it is converted to the Minkowski operator by $\Psi(x^+,x^-)_M=\mathcal{U}\Psi(z,\bar{z})_E\mathcal{U}^{-1}$. Since the conversion is done by a similarity transformation, conformal and BRST properties of the states and the fields are guaranteed to be preserved.

In particular, all the physical operators corresponding to the physical states obtained in the previous subsection can easily be constructed. One thing we must be careful about in this procedure, however, is that the normal ordering is properly defined for the independent fields ϕ_i and not for ψ and χ. Let us give an example at level 1 to illustrate this point. Consider a state of the form $(A_+\alpha_{-1}^++\mathbf{A}\cdot\boldsymbol{\alpha}_{-1}^f)|\mathbf{P}\rangle_\downarrow$, which is a physical state for $p_1=p_2=-iQ$, $\mathbf{p}_f=0$. The corresponding operator invariant under the left BRST transformation is given by

$$c[A_+\{(\partial_+:e^{2Q\phi_1}:):e^{-2Q\phi_2}:-:e^{2Q\phi_1}:(\partial_+:e^{-2Q\phi_2}:)\} + \mathbf{A}\cdot\partial_+\mathbf{f}:e^{2Q\phi_1}::e^{-2Q\phi_2}:] , \quad (4.56)$$

where the symmetric normal ordering must be adopted.

V. DISCUSSIONS

Starting from a definite model of dilaton gravity given by the action Eq. (2.9), we have succeeded in its operator quantization and analyzed the physical states and operators of the theory in the BRST framework. Our emphasis throughout is the rigor of the analysis, trying to avoid as much as possible any explicit or implicit assumptions and preconceptions, which often obscure the validity of the results obtained. This is particularly important for quantum gravity since it possesses many features quite distinct from ordinary quantum field theory.

The work we have performed in this article constitutes the first stage of our intended investigation. The task of extracting physical consequences of the model remains to be undertaken. We now wish to list and analyze in some detail the nature of the problems that lie ahead, which would be invariably encountered in any attempt for exact treatment of models of quantum gravity.

With the knowledge of the physical states and operators in hand, the obvious next step is to calculate the "matrix elements" and interpret them physically. There are a number of closely intertwined problems, both technical and conceptual, to be solved at this stage. The issue centers around the interpretation of the "wave function," namely, the difficulty of interpreting it as a probability amplitude. Perhaps the best attitude toward this problem is to try to stick rigorously to the principle of quantum mechanics and see what that leads to.

In our setting, the first specific problem to be solved is what inner product to be introduced in the space of states. The canonical commutation relation alone does not tell us the answer since we can have infinitely many different representations of the Heisenberg algebra. In an ordinary quantum field theory, this problem is settled by demanding appropriate Hermiticity property for the basic operators so that the physical observables have real eigenvalues and that the Hamiltonian is bounded from below. Clearly, this criterion cannot be applied directly in quantum gravity. Let us nevertheless try to see how far we can go along this line.

First we show that possible Hermiticity assignments for the modes are severely restricted once we adopt the usual assignments for the ghosts b and c, namely

$$c_n^\dagger = c_{-n}, \quad b_n^\dagger = b_{-n} . \tag{5.1}$$

The argument goes as follows. Consider a physical matrix element of a BRST invariant operator $\mathcal{O}$:

$$(|\text{phys}\rangle, \mathcal{O}|\text{phys}'\rangle) . \tag{5.2}$$

Since this should be independent of the choice of the representative of the cohomology, we must have

$$\begin{aligned} 0 &= (|\text{phys}\rangle, \mathcal{O}d|*\rangle) \\ &= (d^\dagger|\text{phys}\rangle, \mathcal{O}|*\rangle) , \end{aligned} \tag{5.3}$$

where d is the BRST operator. This means that $d^\dagger$ must always annihilate physical states and hence we must require $d^\dagger = \text{const} \times d$. But since the part of d consisting of ghosts alone is Hermitian, the constant above can only be unity. This in turn dictates the Hermiticity property of the Virasoro generator to be

$$L_n^\dagger = L_{-n} . \tag{5.4}$$

In general this does not fix the Hermiticity property of α_n completely, but for the dilaton-Liouville sector it does. The key is the presence of the background charge term $iQn(a_n^1 - \alpha_n^2)$ in L_n^{dL}, which is linear in the oscillator. Since Q for our model is real, this leads to $(\alpha_n^i)^\dagger = \alpha_{-n}^i$ for $n \neq 0$. Once the assignment for the nonzero mode is fixed as above, the zero mode α_0 (and its conjugate) should be taken to be Hermitian since $(\alpha_0 \alpha_n)^\dagger = \alpha_0 \alpha_{-n}$ must hold.

This, however, still does not settle the question of the inner product. As was first discussed in detail in [28] and subsequently applied to the case of gravity in [29], there are essentially two different realizations of the Heisenberg algebra with the Hermiticity assignments deduced above. The one that was argued to be relevant to quantum gravity is associated with the inner product with *indefinite norm,* for which Hermitian operators may have imaginary eigenvalues. We believe that this is a point of utmost importance which captures the very characteristic of gravity distinct from usual field theories.

In fact we can already see its relevance in our analysis of physical states. First, we have seen that for the discrete states with $P^+(r)=0$, the value of the zero mode p^+ is imaginary. Furthermore, for states involving matter excitations, it may be more relevant. Recall the $L_0\psi=0$ condition (4.21):

$$\tfrac{1}{2}(p_1^2 - p_2^2) + \tfrac{1}{2}\mathbf{p}_f^2 + \hat{N}^f - 1 = 0 . \tag{5.5}$$

As will be discussed shortly, states containing matter fields with finite momentum in the limit $L \to \infty$ must have matter excitations at arbitrary high Virasoro levels. From the condition above, this is possible only if $p_1^2 - p_2^2$ can take arbitrary large negative values. There are several ways of realizing this condition, including the one with imaginary values for p_i's. After all, the equation above should be expressing the energy balance between the matter and the dilaton-Liouville sectors, characteristic of a theory of gravity. Thus at the least, we must carefully examine the appropriate choice of the inner product in order to compute the matrix elements and to make physical interpretation of them, including the question of "unitarity" of the theory.

Deeply linked with the problem above is the question of how to extract physics. To see what sort of averaged spacetime configuration is associated with a given physical state, we must evaluate expectation values of some appropriate operators in such a state. BRST invariant operators are certainly preferred, but it is not an easy task to find such operators of direct physical significance. Another possibility is to devise a way to completely fix the gauge freedom within the conformal gauge and try to evaluate an operator, such as the metric, which allows for more direct interpretation. Preliminary investigation indicates that this attempt also requires some ingenuity to be successful.

At this juncture, let us make a brief remark on an important point which should be kept in mind when one tries to deal with a "quantum black hole" in *two dimensions.* It has to do with the notion of asymptotic flatness and that of the "mass" of the black hole. In semiclassical treatment, one first identifies a classical black hole configuration and then finds the Hawking radiation emitted into the asymptotically flat region of such a background, with subsequent modification of the metric due to the radiation itself, i.e., the back reaction. This sequential procedure can no longer be applied in an exact treatment such as the one we have been pursuing. The metric, the dilaton, and the matter degrees of freedom are inherently intertwined and especially in one spatial dimension, where the radiation does not disperse, there cannot be an asymptotically flat region. This is already indicated in the form of a physical state with matter excitation discussed in the previous section. Such a state is necessarily accompanied by excitations in the dilaton-Liouville sector. This would make the identification of the black hole "mass" even more difficult.

Finally, let us discuss the question of the large L limit. The problems described so far can be posed already for the case of the universe with finite size L. If, however, one wishes to study what happens in the $L \to \infty$ limit, one must face a further technical problem. As was remarked at the beginning of Sec. IV, the structure of the physical states of our model is very similar to that of a critical closed string theory. However, its interpretation is quite different. The p's should be interpreted only as a set of variables describing the zero modes of the theory and not as physical momenta. The actual momenta are discretized in the units of $1/L$ and are associated with higher Virasoro levels. Consequently, in the limit of large L, states carrying finite physical momenta correspond in string language to excited states at arbitrary high Virasoro levels. Moreover, as we have seen already in the analysis of classical solutions, large degeneracy is inevitable in this "infrared" limit. Bumpy structures which may be seen in a finite universe can be artifacts to be flattened away as L is taken to infinity. Although we believe that a lot of physics should be extractable for a finite universe, we must keep this difficulty in mind and try to develop some clever means to deal with it.

Although undoubtedly quite challenging as they are, the problems described above are all extremely intriguing and worth pursuing. They are presently under investigation and we hope to be able to report our progress elsewhere.

While preparing the manuscript, we received a preprint [30] which deals with a model similar to ours with 24 matter scalars. The treatment of the model, especially the boundary condition, is however different from ours.

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APPENDIX A

In this appendix, we list some useful Poisson brackets needed for the calculation of $\{\chi(x), AB(y)\}$ discussed in Sec. III A.

As remarked in the text, one needs to carefully take into account the fact that the quantity α, which specifies the boundary conditions for $A(x^+)$ and $B(x^-)$, is a function of P_ψ. From its form $\alpha=\exp(P_\psi/2)$, we derive

$$\{\chi(x),\alpha\}=\{Q_\chi,e^{P_\psi/2}\}=\tfrac{1}{2}\alpha\ ,$$

$$\{\chi(x),C(\alpha)\}=\tfrac{1}{2}\alpha\frac{\partial}{\partial\alpha}C(\alpha)=-\tfrac{1}{4}C(\alpha)^2\left[\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}\right]\ ,$$

$$\{\chi(x),E_\alpha(\sigma_y-\sigma')\}=\tfrac{1}{4}\epsilon(\sigma_y-\sigma')E_\alpha(\sigma_y-\sigma')\ ,$$

$$\{\chi(x),E_{1/\alpha}(\sigma_y-\sigma'')\}=-\tfrac{1}{4}\epsilon(\sigma_y-\sigma'')E_{1/\alpha}(\sigma_y-\sigma'')\ .$$

Using these formulas as well as Eqs. (3.20)–(3.23) in the text, we obtain the following basic Poisson brackets evaluated at equal time:

$$\{\chi(\sigma_x),A(\sigma_y)\}_{\rm ET}=\frac{\sigma_x}{4\pi}A(\sigma_y)-\tfrac{1}{2}C(\alpha)A(\sigma_x)E_{1/\alpha}(\sigma_x-\sigma_y)-\tfrac{1}{4}A(\sigma_y)\epsilon(\sigma_x-\sigma_y)\ ,$$

$$\{\chi(\sigma_x),\dot A(\sigma_y)\}_{\rm ET}=\frac{\sigma_x}{4\pi}\dot A(\sigma_y)-\tfrac{1}{4}\dot A(\sigma_y)\epsilon(\sigma_x-\sigma_y)\ ,$$

$$\{\dot\chi(\sigma_x),A(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}C(\alpha)\dot A(\sigma_x)E_{1/\alpha}(\sigma_x-\sigma_y)\ ,$$

$$\{\dot\chi(\sigma_x),\dot A(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}\dot A(\sigma_x)\delta(\sigma_x-\sigma_y)\ ,$$

$$\{\chi(\sigma_x),B(\sigma_y)\}_{\rm ET}=-\frac{\sigma_x}{4\pi}B(\sigma_y)-\tfrac{1}{2}C(\alpha)E_{1/\alpha}(\sigma_y-\sigma_x)B(\sigma_x)+\tfrac{1}{4}B(\sigma_y)\epsilon(\sigma_x-\sigma_y)\ ,$$

$$\{\chi(\sigma_x),\dot B(\sigma_y)\}_{\rm ET}=-\frac{\sigma_x}{4\pi}\dot B(\sigma_y)+\tfrac{1}{4}\dot B(\sigma_y)\epsilon(\sigma_x-\sigma_y)\ ,$$

$$\{\dot\chi(\sigma_x),B(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}C(\alpha)E_{1/\alpha}(\sigma_y-\sigma_x)\dot B(\sigma_x)\ ,$$

$$\{\dot\chi(\sigma_x),\dot B(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}\dot B(\sigma_x)\delta(\sigma_x-\sigma_y)\ .$$

Combining this equations it is now straightforward to get

$$\{\chi(\sigma_x),AB(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}C(\alpha)[A(\sigma_x)E_{1/\alpha}(\sigma_x-\sigma_y)B(\sigma_y)+A(\sigma_y)E_{1/\alpha}(\sigma_y-\sigma_x)B(\sigma_x)]\ ,$$

$$\{\dot\chi(\sigma_x),AB(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}C(\alpha)[\dot A(\sigma_x)E_{1/\alpha}(\sigma_x-\sigma_y)B(\sigma_y)+A(\sigma_y)E_{1/\alpha}(\sigma_y-\sigma_x)\dot B(\sigma_x)]\ ,$$

$$\{\chi(\sigma_x),\partial_t(AB)(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}C(\alpha)[A(\sigma_x)E_{1/\alpha}(\sigma_x-\sigma_y)\dot B(\sigma_y)+\dot A(\sigma_y)E_{1/\alpha}(\sigma_y-\sigma_x)B(\sigma_x)]\ ,$$

$$\{\dot\chi(\sigma_x),\partial_t(AB)(\sigma_y)\}_{\rm ET}=-\tfrac{1}{2}\delta(\sigma_x-\sigma_y)\partial_t(AB)(\sigma_y)$$
$$-\tfrac{1}{2}C(\alpha)[\dot A(\sigma_x)E_{1/\alpha}(\sigma_x-\sigma_y)\dot B(\sigma_y)+\dot A(\sigma_y)E_{1/\alpha}(\sigma_y-\sigma_x)\dot B(\sigma_x)]\ .$$

Notice that these expressions are all symmetric with respect to the interchange $x \leftrightarrow y$. Therefore the combination

$$\{\chi(\sigma_x), AB(\sigma_y)\}_{\rm ET} - \{\chi(\sigma_y), AB(\sigma_x)\}_{\rm ET}$$

and the time derivatives thereof (at most one each for x and y) all vanish.

APPENDIX B

In this appendix, we shall supply the arguments needed to justify the result (4.24) and (4.25) in Sec. IV A of the text.

First we prove that the solution of (4.22) is given by

$$\psi_1 = -\hat{N}^{-1}_{\rm DLG} K^+ \hat{d}_1 \psi_0 \ .$$

For this to be correct, we must show that $K^+\psi_1=0$. Noting that K^+ is a level zero operator in the DLG sector and hence commutes with $\hat{N}^{-1}_{\rm DLG}$, we get

$$K^+\psi_1 = -K^+\hat{N}^{-1}_{\rm DLG} K^+ \hat{d}_1 \psi_0$$

$$= -\hat{N}^{-1}_{\rm DLG} (K^+)^2 \hat{d}_1 \psi_0 \ .$$

This indeed vanishes because $(K^+)^2=0$.

Next we prove the important property that $\hat{d}_2\psi_1=0$ holds. Since $\hat{d}_2$ and $\hat{N}^{-1}_{\rm DLG}$ again commute, we have

$$\begin{aligned}\hat{d}_2\psi_1 &= -\hat{d}_2\hat{N}^{-1}_{\rm DLG}K^+\hat{d}_1\psi_0\\ &= -\hat{N}^{-1}_{\rm DLG}\hat{d}_2K^+\hat{d}_1\psi_0\\ &= -\hat{N}^{-1}_{\rm DLG}\{\hat{d}_2,K^+\}\hat{d}_1\psi_0+\hat{N}^{-1}_{\rm DLG}K^+\hat{d}_2\hat{d}_1\psi_0\\ &= -\hat{N}^{-1}_{\rm DLG}\sum_{n\neq 0}\frac{p^-}{P^+(n)}\alpha^+_{-n}\alpha^+_n\hat{d}_1\psi_0\\ &\quad -\hat{N}^{-1}_{\rm DLG}K^+\hat{d}_1\hat{d}_2\psi_0=0 \ ,\end{aligned}$$

where we used the fact that $\hat{d}_1\psi_0$ does not contain any α^-_{-n} and hence gets annihilated by α^+_n.

We now go to the next step. Form $\psi_0+\psi_1$ and act $\hat{d}$ on it. Because $\hat{d}_2\psi_1=0$, we get $\hat{d}(\psi_0+\psi_1)=\hat{d}_1\psi_1$. Again from the general argument this state must be $\hat{d}_0$ exact and we look for a state ψ_2 of degree 2 such that $\hat{d}_1\psi_1=-\hat{d}_0\psi_2$. It is clear that we can repeat the previous procedure and find a solution

$$\psi_2 = -\hat{N}^{-1}_{\rm DLG}K^+\hat{d}_1\psi_1 \ .$$

In order for this process to terminate, we must show that there exists the maximum degree. This is easy to prove once we note that all the degrees are carried only by the oscillators in the DLG sector. (In the construction above, the degree is carried solely by α^{+}_{-n}'s.) Thus every time the degree is raised by 1, the Virasoro level of the DLG sector is increased at least by one unit. But since the total Virasoro level, including the matter sector, must stay constant, namely at N, it means that at every step, the level in the matter sector is decreased by at least one unit. Thus, after at most N steps, the process terminates. In this way we have the general solution (4.24) and (4.25) for the relative cohomology.

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