

Cosmological theory without singularities

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A theory of gravitation is constructed in which all homogeneous and isotropic solutions are nonsingular, and in which all curvature invariants are bounded. All solutions for which curvature invariants approach their limiting values approach de Sitter space. The action for this theory is obtained by a higher-derivative modification of Einstein's theory. We expect that our model can easily be generalized to solve the singularity problem also for anisotropic cosmologies.

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I. INTRODUCTION

One of the outstanding problems in the theory of gravitation (and more generally in the quest for a unified theory of all interactions) is the singularity problem. According to the Penrose-Hawking theorems [1], general relatively (GR) manifolds are, in general, geodesically incomplete, which is a sign that singularities in space-time occur.

Singularities are undesirable for a theory which claims to be complete since their existence implies that space-time cannot be continued past them. The space-time structure becomes unpredictable already at the classical level.

Two important examples of singularities in GR are the initial and final singularities in a closed universe and the singularity in the center of the black hole. In the former case, the singularity implies we cannot answer the question what will happen after the "big crunch" or (in the case of an expanding universe) what was before the "big bang."

The presence of singularities is an indication that GR is an incomplete theory. Wheeler even talks about a "crisis in physics" [2]. It is a widespread opinion that either quantum gravity or a more fundamental theory such as string theory will provide a cure for the "sickness" of GR. However, quantum gravity does not yet exist as a self-consistent nonperturbative theory. Neither does string theory exist as a unique theory capable of addressing the singularity problem of gravity in a definitive way, although interesting string-specific ideas have recently been put forward [3].

Because of the absence of a completely developed fun-

damental theory on the basis of which we could address the singularity problem, we will use a rather different approach. Any fundamental theory will, in the region of low curvature, give an effective action for a four-dimensional space-time metric $g_{\mu\nu}$ which to lowest order must agree with the Einstein action. We will try to construct (guess) an effective action for $g_{\mu\nu}$ which solves the singularity problem and which in the low-curvature limit reduces to the Einstein action. It is possible that in such a manner we will be able to discover important features of the future fundamental theory. We might also gain information which will help in finding this fundamental theory.

Before discussing the ideas behind our construction of the effective action for gravity, we return to the Penrose-Hawking theorems [1]. They do not give us any detailed information about the nature of the singularity. However, in the two examples discussed above, a collapsing universe and a black hole, we know that at the singularity some of the physically measurable curvature invariants such as R , $R_{\mu\nu}R^{\mu\nu}$, and $C^2 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverge (here R is the Ricci scalar, $R_{\mu\nu}$ the Ricci tensor, and $C_{\alpha\beta\gamma\delta}$ the Weyl tensor). It is reasonable to assume that the divergence of some curvature invariants at the singularity is a fairly general phenomenon. In fact, for singularities reached on timelike curves in a globally hyperbolic space-time it can be proved [4] that the Riemann tensor becomes infinite. Hence, as a first step we will find a mechanism to bound all the curvature invariants.

Limitation principles play a very important role in physics. Special relativity includes as one of its fundamental assumptions the principle that no particle velocity can exceed the speed of light. The cornerstone of quantum mechanics is the uncertainty principle which states that the second fundamental constant, Planck's constant \hbar , gives the minimal phase-space volume a particle can be localized in. The third fundamental constant, Newton's gravitational constant G , has not yet been used in any limitation principle.

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Thus it is natural to assume that there exists a fundamental length

$$l_{\text{Pl}} \sim (G\hbar c^{-3})^{1/2} \approx 10^{-33} \text{ cm}$$

in nature (determined by G) such that there is no curvature corresponding to scales $l < l_{\text{Pl}}$. There are strong indications that this will in fact arise in quantum gravity [5] or string theory [3]. From the existence of a fundamental length, it follows by simple dimensional considerations that all curvature invariants are limited:

$$\begin{aligned} |R| &\leq l_{\text{Pl}}^{-2}, \\ |R_{\mu\nu}R^{\mu\nu}| &\leq l_{\text{Pl}}^{-4}, \\ |C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}| &\leq l_{\text{Pl}}^{-8}, \dots \end{aligned} \quad (1.1)$$

To realize the idea of a fundamental length it is necessary to construct a theory in which all curvature invariants are bounded. Since there are an infinite number of curvature invariants and since bounds on low-order invariants do not necessarily imply bounds on higher-order invariants, it is a rather formidable task to construct such a theory.

Fortunately we can simplify the problem drastically by making use of the “limiting curvature hypothesis” (LCH) construction [6], according to which one looks for a theory in which (i) a finite number of invariants are bounded by an explicit construction (e.g., $|R| \leq l_{\text{Pl}}^{-2}$ and $|R_{\mu\nu}R^{\mu\nu}| \leq l_{\text{Pl}}^{-4}$), and (ii) when these invariants take on their limiting values, any solution of the field equations reduces to a definite nonsingular solution (e.g., de Sitter space). In this case it follows automatically that all curvature invariants are limited. Note, however, that it is necessary to demonstrate the absence of singular solutions for which the curvature invariants which were singled out in step 1 above do not approach their limiting values. Whether this is the case or not will depend on the specific model. Examples are discussed in Secs. III and IV.

The LCH contains the part of Penrose’s hypothesis [7], which states that the Weyl tensor C should vanish at the beginning of the Universe. This follows since by the LCH the Universe near the big bang is de Sitter and that $C=0$ for a de Sitter universe.

A theory in which the LCH is realized has some attractive features, both for cosmology [6] and black holes [8]. In cosmology, the present homogeneous expanding Universe would have started out with a de Sitter phase. In this case we would have some (maybe unusual) realization of the oscillating universe scenario. Entropy considerations tell us that only for a perfectly homogeneous and isotropic universe could we have perfect periodicity. In general, we must have a nontrivial realization. Including inhomogeneities, we might obtain a multiple-universe model in which one collapsing universe splits into several de Sitter bounces.

For black holes the LCH gives the attractive picture that inside of the horizon instead of a singularity at the center we would have a piece of a de Sitter universe which could be the source of other Friedmann (baby) universes (see Fig. 1). In this case the difficult question

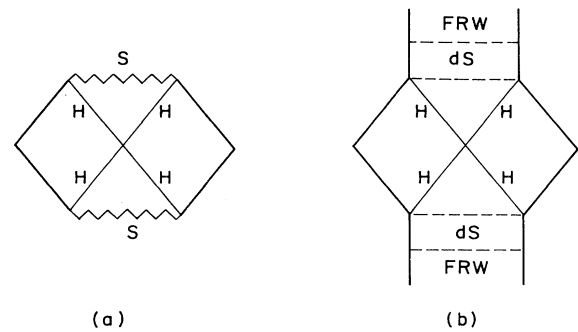


FIG. 1. Penrose diagram of an eternal black hole (a) in Einstein gravity and (b) in the nonsingular universe theory. The singularities (S) are replaced by de Sitter phases (dS) which couple to Friedmann universes (FRW). The horizons (H) are not affected.

[9] concerning information loss when matter falls into a black hole has a natural answer: The information which is lost to an observer external to the Schwarzschild horizon is stored in the baby universe. In addition, using this picture provides a good starting point to attack the issue of the final stage of an evaporating black hole, a problem which has recently been of high interest in the context of two-dimensional quantum gravity [10].

In this paper we construct an effective action for gravity in which all homogeneous and isotropic solutions are nonsingular and at high curvature approach de Sitter space. (A brief summary of our work was published in Ref. [11].) In order to implement the LCH, we proceed in analogy to a technique by which point-particle velocities can be limited, thus achieving the transition between Newtonian mechanics and point-particle motion in special relativity (SR) (see also Ref. [12]). An extension of our construction to inhomogeneous cosmologies and to black hole metrics will be presented separately [13].

In the following section, we present the general theory of how to implement the LCH. We obtain a fairly general effective action for gravity as a higher-derivative modification of the Einstein action, specialize to the case of an isotropic, homogeneous universe, and derive the resulting equations of motion.

In Sec. III we analyze a simple model which yields a nonsingular universe without limiting curvature. We discuss the effects of including spatial curvature (i.e., $k \neq 0$) and hydrodynamical matter. The analysis of the more complicated model with limiting curvature is given in Sec. IV. Section V contains conclusions and further discussion.

II. THEORY

In order to realize the LCH and hence to avoid singularities, it is necessary to abandon at least one of the key assumptions on which the Penrose-Hawking theorems are based. The two most important assumptions are (i) the energy-dominance condition, a simplified version of which appropriate for cosmology is $\epsilon > 0$ and $\epsilon + 3p \geq 0$, where ϵ and p are matter-energy density and pressure re-

spectively, and (ii) the Einstein equations are universally true.

There is no reason to believe that these assumptions will be valid at very high energies and curvatures. First of all, already in matter theories routinely studied by particle physicists, the energy-dominance condition is not always true. For example, the effective equation of state for a homogeneous, slowly varying scalar-field configuration with potential energy is $p \simeq -\epsilon$, thus violating the energy-dominance condition. This matter-evolution scenario is in fact the basis for the inflationary universe [14].

Note, however, that inflationary-universe models do not cure the problem of the final singularity. There may be nonsingular solutions for a collapsing universe filled with scalar-field matter, but they are of measure zero. Rather, in this case typical solutions have an effective equation of state $p = \epsilon$ (the kinetic term for the scalar field dominates), not $p = -\epsilon$, and hence have a final singularity. Our goal is to construct a theory in which *all* solutions are nonsingular.

Concerning the second key assumption of the Penrose-Hawking theorems, it is well known that Einstein theory can only be an effective theory of gravity at low curvatures. Perturbative quantum-gravity calculations [15], vacuum polarization effects of quantum matter fields in an external gravitational background [16], and also considerations based on string theory [17] all show that the effective equations for the gravitational field should be modified at higher curvatures. In a perturbative analysis, the modifications take the form of higher-derivative terms which are usually important only at very high (Planck) curvatures. Hence, provided the effective-action approach is valid at all at high curvatures, this effective action will certainly not be of pure Einstein form.

To summarize, there are two ways to modify the theory at high curvatures in order to avoid singularities: (i) Modify the matter action by including terms which violate the energy-dominance condition; (ii) modify the gravitational-field equations.

The first approach was explored in Ref. [18]. However, the weakness of this approach is the absence of a good physical motivation for the modification. In addition, it seems impossible to avoid singularities associated with purely gravitational modes which do not couple to matter.

The second approach is much better motivated since higher-derivative correction terms to the Einstein action are predicted by many theories [15–17]. Hence our starting point will be to look for an effective action for gravity of the form

$$S_g = -\frac{1}{16\pi G} \int F(R, R_{\mu\nu}R^{\mu\nu}, C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \dots) \sqrt{-g} d^4x + \text{nonlocal terms}, \quad (2.1)$$

where the ellipsis denotes the dependence of F on other curvature invariants. At low curvatures the leading term in F is simply R .

The action (2.1) can be viewed as the effective action of some fundamental theory such as quantum gravity or

string theory. In these theories we are at present unable to calculate the nonperturbative effective action. Hence, as mentioned in the Introduction, our approach will be to construct (guess) an effective action of the form (2.1) to obtain a theory in which *all* solutions are nonsingular.

To simplify the considerations we shall neglect nonlocal terms. In our approach this is justifiable since if we are able to solve the singularity problem in a purely local theory, we expect that the nonlocal terms (which are inevitable, for example, because of particle production) will not drastically change the asymptotic behavior of our theory because of its special properties (see Sec. V).

The key to the analysis is the assumption about the validity of the background-field approximation for the gravitational field up to high curvatures. Such an approximation will only be justified if the quantum fluctuations around this metric are sufficiently small. If the gravitational field is asymptotically free at high curvatures (see Sec. III), we can hope that this approach will be valid. As we shall see, there are features in our theory which indicate that this will really be the case.

For the moment we shall ignore matter (later we will show that the presence of matter does not change the solutions at high curvatures in an important way). Thus our starting point is the effective action

$$S_g = -\frac{1}{16\pi G} \int F(R, R_{\mu\nu}R^{\mu\nu}, C^2, \dots) \sqrt{-g} d^4x. \quad (2.2)$$

The usual Einstein theory in the absence of matter has only one solution, Minkowski space, for a homogeneous and isotropic universe. Any non-Einstein theory of gravity gives rise to fourth- (or higher-) order equations of motion and hence to a large number of cosmological solutions. In general, the singularity problems of such a theory are much worse than in Einstein gravity. A simple example is R^2 gravity,

$$F(R) = R + \alpha R^2, \quad (2.3)$$

which is conformally equivalent [19] to Einstein gravity plus scalar-field matter and which hence has many isotropic singular solutions (even without matter). Thus the theory we are looking for must be a very special higher-derivative gravity model.

We wish to construct an effective action for gravity in which all homogeneous and isotropic solutions are nonsingular and in which all curvature invariants are limited (in Sec. V we will indicate how to extend our analysis to anisotropic models [13]). To motivate our construction it is useful to keep in mind ways of writing the action for two well-known physical theories in which certain physical quantities are bounded: special relativity and the Born-Infeld theory of electromagnetism [20].

To impose bounds on physical quantities in an explicit manner, it is convenient to employ a Lagrange-multiplier technique proposed by Altshuler [12]. To explain how this technique works we first consider the simple example of point-particle motion. We start with the action for a nonrelativistic particle of mass m and world line $x(t)$. We demonstrate how to explicitly implement the limitation on the particle velocity and, in particular, how to ob-

tain the action for point-particle motion in special relativity. The nonrelativistic action with which we start is

$$S_{\text{old}} = m \int dt \frac{1}{2} \dot{x}^2. \tag{2.4}$$

In order to construct a new theory with bounded velocity, we introduce a ‘‘Lagrange-multiplier field’’ $\phi(t)$, which couples to some function of the quantity whose value we want to limit, and a potential $V(\phi)$ for this field:

$$S_{\text{new}} = m \int dt \left[\frac{1}{2} \dot{x}^2 + \phi \dot{x}^2 - V(\phi) \right]. \tag{2.5}$$

Let us stress that ϕ is not a dynamical field. Provided that $\partial V/\partial\phi$ is bounded, the constraint equation (i.e., the variational equation with respect to ϕ) ensures that \dot{x} is bounded. In order to obtain the correct Newtonian limit for small \dot{x} and small ϕ , $V(\phi)$ must be proportional to ϕ^2 as $|\phi| \rightarrow 0$. One of the simplest potentials which satisfies the above asymptotic conditions,

$$V(\phi) = \frac{2\phi^2}{1+2\phi}, \tag{2.6}$$

leads to special relativity. In fact, eliminating the Lagrange multiplier using the constraint equation and substituting the result into (2.5) yields (up to a constant term which does not affect the equations of motion) the relativistic point-particle action

$$S_{\text{new}} = m \int dt \sqrt{1 - \dot{x}^2}. \tag{2.7}$$

Let us return to the theory of gravitation. In the notation of the above example, the ‘‘old’’ theory will be given by the Einstein action. In order to implement the LCH we wish to impose restrictions on some curvature invariants I_1, I_2, \dots, I_n in an explicit manner. The general form of a higher-derivative local modification of the Einstein action involving the invariants I_1, \dots, I_n is

$$S_g = -\frac{1}{16\pi G} \int [R + F(I_1, I_2, \dots, I_n)] \sqrt{-g} d^4x, \tag{2.8}$$

where F is some function of the invariants I_1, \dots, I_n .

By introducing Lagrange-multiplier fields $\phi_1(t), \dots, \phi_n(t)$, the above action can be rewritten as

$$S_g = -\frac{1}{16\pi G} \int [R + \phi_1 f_1(I_1) + \dots + \phi_n f_n(I_n) + V(\phi_1 \dots \phi_n)] \sqrt{-g} d^4x, \tag{2.9}$$

where $f_i(I_i)$ are functions we can choose as we want. The actions (2.8) and (2.9) are equivalent provided that the potential $V(\phi_1, \dots, \phi_n)$ satisfies the partial differential equation

$$-\sum_{i=1}^n \phi_i \frac{\partial V}{\partial \phi_i} + V(\phi_1 \dots \phi_n) = F \left[f_1^{-1} \left[\frac{\partial V}{\partial \phi_1} \right], \dots, f_n^{-1} \left[\frac{\partial V}{\partial \phi_n} \right] \right]. \tag{2.10}$$

This follows immediately by using the constraint equations for (2.9):

$$f_i(I_i) = \frac{\partial V}{\partial \phi_i}, \quad i = 1, \dots, n. \tag{2.11}$$

We see from the constraint equations (2.11) that by appropriate choice of the functions f_i and V we can implement bounds on the invariants I_1, \dots, I_n . Variation of the action (2.9) with respect to $g_{\mu\nu}$ yields the field equations.

First, we try to construct the simplest theory in which the LCH is realized. At least for simple models (such as the isotropic universe), it is natural to choose as one of the invariants

$$I_1 = R - \sqrt{3}(4R_{\mu\nu}R^{\mu\nu} - R^2)^{1/2}, \tag{2.12}$$

since for a homogeneous, spatially flat universe it is equal to $12H^2$. This invariant will be used to limit the curvature by some (e.g., Planckian) value. The second invariant I_2 will take on such a form as to implement in the theory the condition that in the asymptotic regions all of the solutions evolve to de Sitter. The simplest way to do this is to pick I_2 such that $I_2=0$ only for de Sitter space (Minkowski space is included as a special case) and to make sure that

$$I_2 \rightarrow 0 \quad \text{as} \quad |\phi_2| \rightarrow \infty. \tag{2.13}$$

For homogeneous and isotropic space-times, it can be shown that

$$I_2 = 4R_{\mu\nu}R^{\mu\nu} - R^2 \tag{2.14}$$

is a good choice, since $I_2=0$ only for de Sitter space. Note that, in general, I_2 is positive semidefinite. However, for inhomogeneous and anisotropic space-times (e.g., when $C^2 \neq 0$), the above form of I_2 is insufficient to single out de Sitter space as an asymptotic solution. This is obvious from considering the Schwarzschild metric for which $I_2=0$. Hence, in the general case, we [13] should add to (2.14) terms which depend on C^2 and vanish for conformally flat space-times.

However, for a homogeneous and isotropic universe it is (as we will show) sufficient to consider the action in the general form

$$S_g = -\frac{1}{16\pi G} \int [R + \phi_1 f_1(I_1) + \phi_2 f_2(I_2) + V(\phi_1, \phi_2)] \sqrt{-g} d^4x. \tag{2.15}$$

The variational field equations which follow from (2.15) are

$$\begin{aligned}
R_{\beta}^{\alpha} - \frac{1}{2} \delta_{\beta}^{\alpha} R - \frac{1}{2} \delta_{\beta}^{\alpha} V = & \left[\phi_1 \frac{\partial f_1}{\partial I_1} \right]_{,\beta}^{\alpha} - \left[\phi_1 \frac{\partial f_1}{\partial I_1} \right]_{,\sigma}^{\sigma} \delta_{\beta}^{\alpha} - \phi_1 \frac{\partial f_1}{\partial I_1} R_{\beta}^{\alpha} + \frac{1}{2} \phi_1 f_1 \delta_{\beta}^{\alpha} - 4 \left[\phi_2 \frac{\partial f_2}{\partial I_2} R^{\sigma\tau} \right]_{,\sigma\tau} \delta_{\beta}^{\alpha} \\
& - 4 \left[\phi_2 \frac{\partial f_2}{\partial I_2} R_{\beta}^{\alpha} \right]_{,\sigma}^{\sigma} + 4 \left[\phi_2 \frac{\partial f_2}{\partial I_2} R_{\sigma}^{\alpha} \right]_{,\beta}^{\sigma} + 4 \left[\phi_2 \frac{\partial f_2}{\partial I_2} R_{\beta}^{\sigma} \right]_{,\sigma}^{\alpha} \\
& + 2 \delta_{\beta}^{\alpha} \left[\phi_2 \frac{\partial f_2}{\partial I_2} R \right]_{,\sigma}^{\sigma} - 2 \left[\phi_2 \frac{\partial f_2}{\partial I_2} R \right]_{,\beta}^{\alpha} - 8 \phi_2 \frac{\partial f_2}{\partial I_2} R^{\alpha\gamma} R_{\gamma\beta} + 2 \phi_2 \frac{\partial f_2}{\partial I_2} R R_{\beta}^{\alpha} + \frac{1}{2} \delta_{\beta}^{\alpha} \phi_2 f_2, \quad (2.16)
\end{aligned}$$

and the constraint equations are

$$f_1(I_1) = -\frac{\partial V}{\partial \phi_1}, \quad f_2(I_2) = -\frac{\partial V}{\partial \phi_2}. \quad (2.17)$$

We will simplify the theory further by assuming a factorizable potential

$$V(\phi_1, \phi_2) = V_1(\phi_1) + V_2(\phi_2). \quad (2.18)$$

The asymptotic conditions on the potentials V_1 and V_2 follow from demanding that the theory reduce to the Einstein theory at small curvatures and that the LCH be realized. The first condition yields

$$V_i(\phi_i) \sim \phi_i^2, \quad |\phi_i| \ll 1, \quad i=1,2. \quad (2.19)$$

In order to limit R explicitly, we can try a potential which to leading order takes the form

$$V_1(\phi_1) \sim \phi_1, \quad |\phi_1| \gg 1, \quad (2.20)$$

and to obtain de Sitter solutions in the asymptotic regions we need a potential which at large ϕ_2 increases less quickly than ϕ_2 . We assume an asymptotic form

$$V_2(\phi_2) \sim \text{const}, \quad |\phi_2| \gg 1. \quad (2.21)$$

In this case, provided $f_2(I_2) \rightarrow 0$ as $I_2 \rightarrow 0$, the constraint equation (2.17) implies that $I_2 \rightarrow 0$ as $|\phi_2| \rightarrow \infty$, and we have a chance of realizing the LCH, provided that the evolution of the scalar fields ϕ_1 and ϕ_2 is appropriate, a question which needs detailed investigation.

To conclude this section we will write down Eqs. (2.16) and (2.17) explicitly for a homogeneous and isotropic metric with scale factor $a(t)$ in the contracting phase (i.e., $H < 0$):

$$ds^2 = dt^2 - a^2(t) \left[\frac{1}{1-kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]. \quad (2.22)$$

We choose simple functions f_1 and f_2 :

$$f_1(I_1) = I_1, \quad f_2(I_2) = -\sqrt{I_2}. \quad (2.23)$$

Thus our final action takes the form

$$\begin{aligned}
S_g = & -\frac{1}{16\pi G} \int [(1+\phi_1)R \\
& - (\phi_2 + \sqrt{3}\phi_1) \sqrt{4R_{\mu\nu}R^{\mu\nu} - R^2} \\
& + V_1(\phi_1) + V_2(\phi_2)] \sqrt{-g} d^4x. \quad (2.24)
\end{aligned}$$

As is well known from the derivation of the Friedmann-Robertson-Walker equations in Einstein gravity, the only independent equation of motion is the 0-0 equation. In our case we have in addition the constraint equations (2.17). The full set of equations can be obtained by inserting the metric (2.22) into (2.16) and (2.17).

The resulting ϕ_1, ϕ_2 and 0-0 equations are

$$H^2 + \frac{k}{a^2} = \frac{1}{12} V_1', \quad (2.25)$$

$$\dot{H} - \frac{k}{a^2} = -\frac{1}{\sqrt{12}} V_2', \quad (2.26)$$

$$\begin{aligned}
& -\frac{1}{2}(V_1 + V_2) + 3H^2(1-2\phi_1) + 3\frac{k}{a^2}(4\phi_1 + 1) \\
& = \sqrt{3}H \left[\dot{\phi}_2 + 3H\phi_2 - \frac{k}{Ha^2}\phi_2 \right]. \quad (2.27)
\end{aligned}$$

Another way to obtain the same equations is to substitute the ansatz (2.22) with $g_{00} = N(t)^2$ into the action (2.24) and to vary it with respect to N , ϕ_1 , and ϕ_2 (see, e.g., Ref. [21]). Adding to the system matter with the action

$$S_m = \int L_m \sqrt{-g} d^4x, \quad (2.28)$$

where L_m is the matter Lagrangian, only leads to an addition term

$$\frac{8\pi}{3} G \rho_m \quad (2.29)$$

on the left-hand side of the 0-0 equation.

In the following sections we shall show that all solutions of the above equations are free of singularities.

III. NONSINGULAR UNIVERSE WITHOUT LIMITING CURVATURE

Since our goal is primarily to construct a nonsingular universe model and only secondarily to limit the curvature, we first consider a simple model in which the ϕ_1 field is absent. In this case it is easier to discuss our techniques of analysis.

We will show that for this model all solutions for a collapsing universe are nonsingular and asymptotically approach de Sitter solutions. However, there is no general (i.e., solution independent) bound on the effective cosmological constant of the de Sitter period.

In this section we set $\phi_2 \equiv \phi$ and $V_2 \equiv V$. The equations of motion are given by (2.26) and (2.27). Let us first consider a spatially flat ($k=0$) collapsing model without

matter. In this case the equations of motion are

$$\dot{H} = -\frac{1}{2\sqrt{3}} V', \quad (3.1)$$

$$\dot{\phi} = -3H\phi + \sqrt{3}H - \frac{1}{2\sqrt{3}H} V. \quad (3.2)$$

The phase space of this model is the two-dimensional (ϕ, H) plane. The phase-space trajectories can be understood by considering $dH/d\phi$ [determined immediately from (3.1) and (3.2)]:

$$\frac{dH}{d\phi} = -\frac{V'}{\sqrt{12}} \left[-3H\phi + \sqrt{3}H - \frac{1}{2\sqrt{3}H} V \right]^{-1}. \quad (3.3)$$

From (3.3) it follows that provided that $V(\phi)$ is bounded at large ϕ [as postulated in (2.21)], then as ϕ tends to infinity H approaches a finite value; i.e., for any solution, the effective cosmological constant in the large- ϕ region is bounded. In this case it follows from (3.2) that in a collapsing universe, for large ϕ ,

$$\phi(t) \sim e^{3|H|t}. \quad (3.4)$$

Our choice of invariant I_2 has led to the conclusion that the asymptotic de Sitter solutions are attractor solutions. This conclusion holds independent of the specific choice of the potential $V(\phi)$, as long as the asymptotic condition (2.21) is satisfied.

From (3.4) it follows that all solutions for a contracting universe are free of singularities. It takes infinite time to reach $\phi = \infty$.

To concretize the consideration we consider a simple potential which satisfies the asymptotic conditions (2.19) and (2.21):

$$V = \sqrt{12} H_0^2 \frac{\phi^2}{1 + \phi^2}, \quad (3.5)$$

where H_0 is a constant (in the model with limiting curvature discussed in Sec. IV, H_0 sets the scale of this limiting

curvature).

The phase-space trajectories $(\phi(t), H(t))$ in a collapsing universe are shown in Fig. 2. The numerical results were obtained using the specific potential (3.5). However, as discussed above, the main features of the diagram depend only on the asymptotic properties.

First, we note that there is only one singular point ($\dot{\phi} = \dot{H} = 0$) in the phase plane. This point is

$$(\phi, H) = (0, 0) \quad (3.6)$$

and corresponds to Minkowski space-time.

There are two classes of trajectories which are asymptotically de Sitter. Those starting at large positive values of ϕ go off to $\phi = \infty$, reaching their asymptotic value of H from above (i.e., $\dot{H} < 0$). Those starting with large negative values of ϕ tend to $\phi = -\infty$ with $\dot{H} > 0$.

For small values of H and ϕ we can use the asymptotic condition (2.19) on $V(\phi)$ to conclude that there are periodic solutions about Minkowski space. In this limit the basic equations (3.1) and (3.2) become

$$\dot{H} = -\frac{1}{2\sqrt{3}} \frac{\partial V}{\partial \phi} \simeq -2H_0^2 \phi, \quad (3.7)$$

$$\dot{\phi} \simeq \frac{1}{\sqrt{3}} \frac{3H^2 - \frac{1}{2}V}{H} \simeq H_0 \frac{\sqrt{3}(H/H_0)^2 - \phi^2}{H/H_0}, \quad (3.8)$$

where for $V(\phi)$ we have inserted the general asymptotic form

$$V(\phi) \simeq 2\sqrt{3} H_0^2 \phi^2, \quad (3.9)$$

valid for small ϕ . The numerical factor $2\sqrt{3}$ has been inserted to eliminate numerical constants in the following equations.

It is convenient to introduce a rescaled time

$$\tau \equiv H_0 t \quad (3.10)$$

and a dimensionless measure of H :

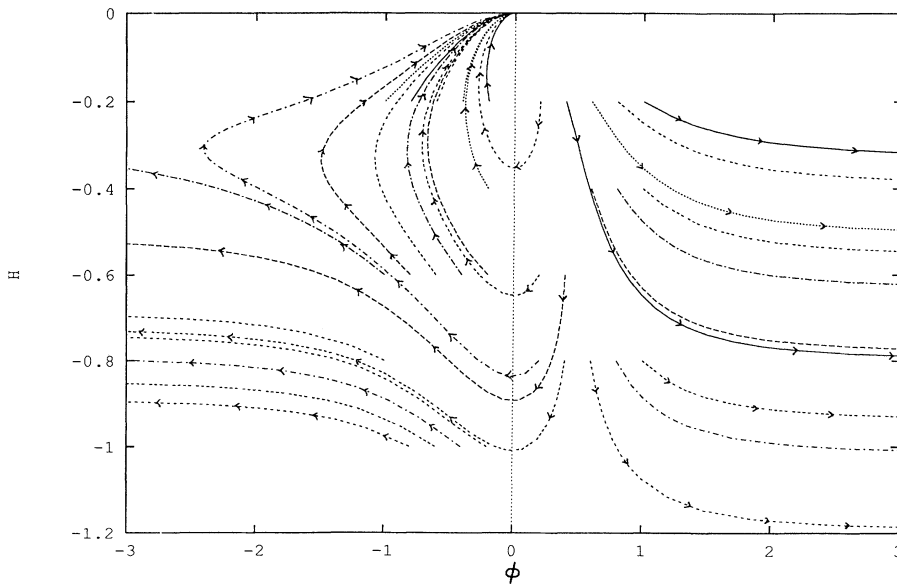


FIG. 2. Phase-space diagram (ϕ, H) (arrows indicating the direction of time evolution) for a spatially flat universe without limiting curvature and with no matter ($k = c = 0$). Generated using the potential (3.5).

$$y \equiv H/H_0. \quad (3.11)$$

With $d/d\tau$ denoted by a prime, Eqs. (3.7) and (3.8) become

$$y' = -2\phi, \quad \phi' = \frac{\sqrt{3}y^2 - \phi^2}{y}. \quad (3.12)$$

To see the oscillatory nature of the solutions we introduce radial and angular coordinates r and ψ :

$$\phi = r \sin\psi, \quad y = -3^{-1/4}r(1 - \cos\psi). \quad (3.13)$$

The resulting equations for r and ψ are

$$\psi' = \omega, \quad r' = 0, \quad (3.14)$$

where the frequency is $\omega = 2 \times 3^{1/4}$. The corresponding solutions oscillate with frequency given by H_0 (which we expect to be Planck scale) about Minkowski space.

Based on the preceding discussion of asymptotic solutions we see that there is a separatrix [22] in phase space dividing solutions which tend to $\phi = \infty$ from those which oscillate or tend to $\phi = -\infty$. We observe that for large $|H|$ the separatrix will asymptotically (and from the right-hand side on Fig. 2) approach the line of turning points given by $d\phi/dH = 0$. From (3.1) it follows that for large $|H|$ the turning points lie at

$$\phi \simeq \frac{1}{\sqrt{3}}. \quad (3.15)$$

For small values of ϕ and H , the separatrix is well to the right of the line of turning points given by

$$\phi \simeq 3^{1/4} \frac{|H|}{H_0}. \quad (3.16)$$

The above analysis of the phase-space trajectories is an indication that in our theory Minkowski space is stable toward homogeneous perturbations. As long as the initial values of $|H|$, $|\phi|$, and $\phi/|H|$ are small, a solution

starting close to Minkowski space will remain close for all times. The issue of stability of Minkowski space toward inhomogeneous perturbations is an important unsolved problem.

We stress again that all the general features of the phase-space analysis are true for any potential $V(\phi)$ which satisfies the required asymptotic conditions (2.19) and (2.21). However, the results depend crucially on the choice of the invariant I_2 .

Next, we include hydrodynamical matter with the energy density

$$\rho_m(t) = ca(t)^{-n}, \quad (3.17)$$

where $n = 3$ for dust and $n = 4$ for radiation. For the moment we keep to a collapsing spatially flat model. In this case Eq. (3.1) is unchanged, while Eq. (3.2) becomes

$$\dot{\phi} = -3H\phi + \sqrt{3}H - \frac{1}{2\sqrt{3}H}V - \frac{8\pi G}{\sqrt{3}H}ca(t)^{-n}. \quad (3.18)$$

With matter, phase space is three dimensional, the third dimension being $a(t)$. In Fig. 3 we show the projection of some of the trajectories onto the $(\phi(t), H(t))$ plane for potential $V(\phi)$ given by (3.5). All trajectories have $8\pi Gc = 1$ and $a(t_0) = 10$, t_0 being the initial time. The main impression is that the trajectories look very similar to those without matter in the asymptotic region. We shall now explain why this is the case.

First, we note that as $|\phi| \rightarrow \infty$, the solutions approach de Sitter space since $\dot{H} \rightarrow 0$. Hence

$$a(t) \simeq e^{-|H|(t-t_0)} a(t_0). \quad (3.19)$$

Next, we combine (3.1) and (3.18) to obtain, for $|\phi| \gg 1$,

$$\frac{dH}{d\phi} \simeq \frac{V'}{2\sqrt{3}} \left[3H\phi + \frac{8\pi Gc}{\sqrt{3}H}a(t)^{-n} \right]^{-1}. \quad (3.20)$$

Our model incorporates a very important feature: In the

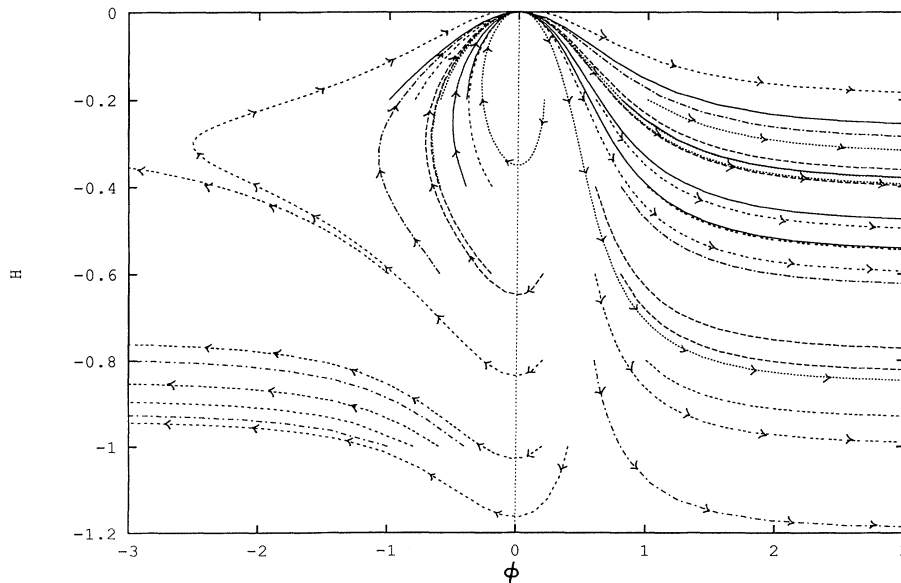


FIG. 3. Projection onto the (ϕ, H) plane of the three-dimensional phase-space diagram (ϕ, H, a) (arrows again indicating direction of time evolution) for a spatially flat universe but without limiting curvature but with matter ($k=0, c \neq 0$). Generated using the potential (3.5) with initial condition $a(t_0) = 10$.

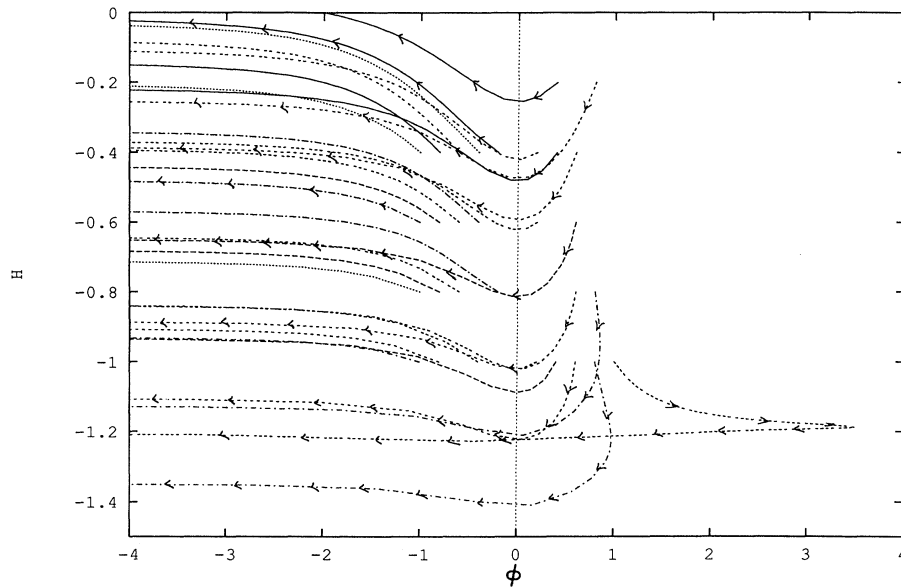


FIG. 4. Phase-space diagram as in Fig. 3, but with $a(t_0)=1$. Therefore the initial matter-energy density is larger than for the trajectories of Fig. 3.

asymptotic de Sitter region, matter does not have an important effect on the geometry. The effective gravitational constant which describes the influence of matter on the geometry goes to zero as space-time approaches de Sitter space. In this sense the model is asymptotically free.

Some understanding of asymptotic freedom can be obtained by solving the ϕ and H equations of motion (3.1) and (3.18) in the asymptotic region $|\phi| \gg 1$. Equation (3.18) becomes

$$\dot{\phi} \simeq 3|H|\phi + \frac{c}{|H|} a(t_0)^{-n} e^{n|H|(t-t_0)} \quad (3.21)$$

(where we have incorporated the factor $8\pi G/\sqrt{3}$ into the definition of c). From (3.21) it follows that $\phi(t)$ is a linear combination of the homogeneous solution (3.4) and (assuming that $H \simeq \text{const}$) the inhomogeneous contribution $\phi_I(t)$:

$$\phi_I(t) = \frac{c}{n|H|^2} a(t_0)^{-n} (e^{n|H|(t-t_0)} - 1). \quad (3.22)$$

For dust ($n=3$), both the homogeneous and inhomogeneous terms grow at the same rate, and the coefficient of the inhomogeneous term is smaller. Hence matter does not affect even the time dependence of the phase-space trajectories. For radiation ($n=4$), $\phi_I(t)$ grows faster than (3.4). At sufficiently late times, therefore, it will dominate. In this period, however, we can [for potential (3.5)] solve the H equation (3.1) to obtain

$$H(t) \simeq H(t_1) - \frac{2H_0^2}{3n|H|} \left[\frac{\sqrt{3}n|H|^2}{c} \right]^3 a(t_0)^{3n} e^{-3n|H|(t_1-t_2)} \quad (3.23)$$

(where t_1 is some time $\gg t_0$ well into the asymptotic re-

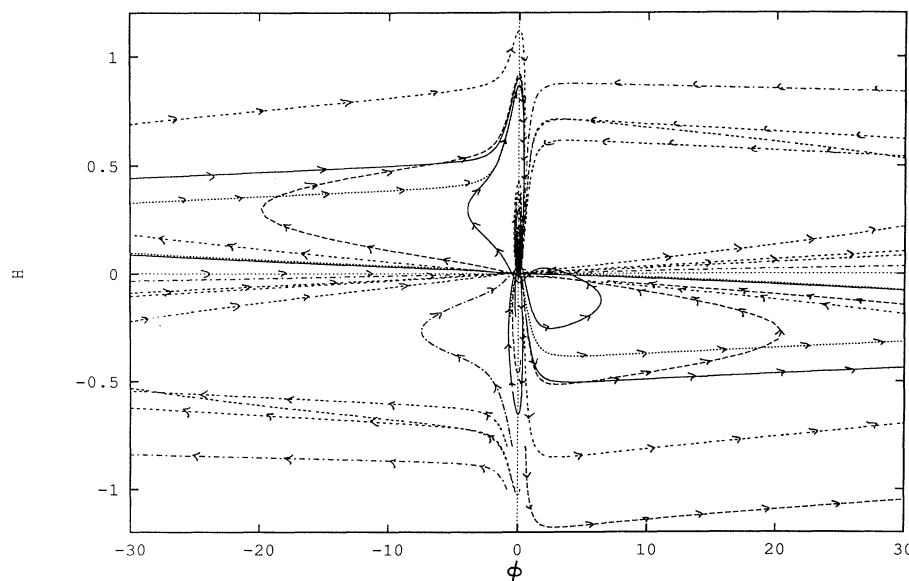


FIG. 5. Projection onto the (ϕ, H) plane of the three-dimensional phase-space diagram (ϕ, H, a) in a closed ($k=1$) universe without limiting curvature and in the absence of matter ($c=0$). The potential (3.5) was used and $a(t_0)=10$ was chosen as the initial condition.

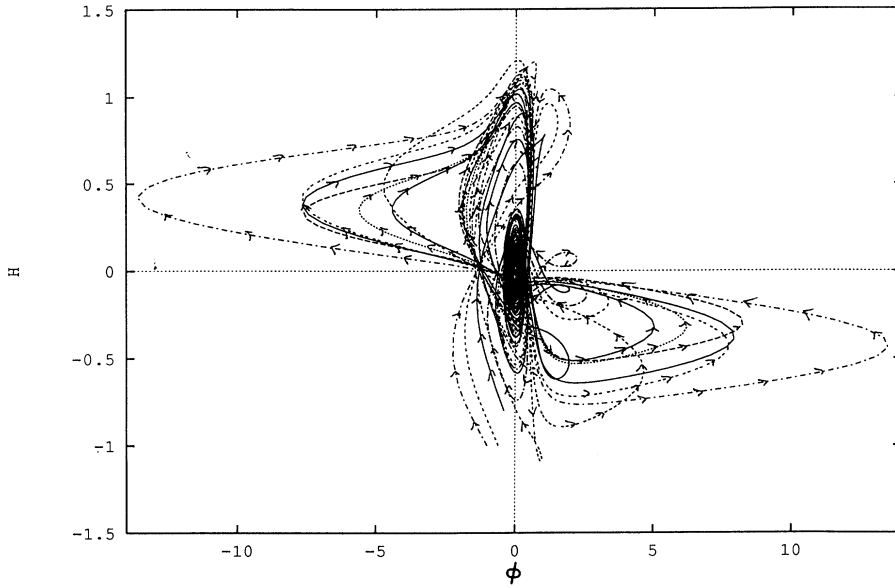


FIG. 6. Same as in Fig. 5, but with the initial condition $a(t_0)=1$. Note the different scales on the axes.

gion), which shows that the presence of matter does not affect the final value of the curvature when starting the evolution in the asymptotic region.

For small $|\phi|$ the presence of matter does have a significant effect on the phase-space trajectories. As $a(t_0)$ decreases (or, equivalently, c and thus the matter-energy density increase), the distortions of the trajectories increase, as can be seen by comparing Figs. 3 and 4. Figure 4 corresponds to a matter-energy density which is 10 times larger.

Finally, we consider the effects of spatial curvature. In this case Eq. (3.1) and (3.2) generalize to [see (2.26) and (2.27)]

$$\dot{H} = -\frac{1}{2\sqrt{3}}V' + \frac{k}{a^2}, \quad (3.24)$$

$$\begin{aligned} \dot{\phi} = & -3H\phi + \frac{k}{Ha^2}\phi - \frac{1}{2\sqrt{3}H}V + \sqrt{3}H \\ & + \sqrt{3}\frac{k}{Ha^2} - \frac{c}{Ha^n}, \end{aligned} \quad (3.25)$$

where the constant c is as in Eq. (3.21).

In the case of the potential (3.5) and for $c=0$, some resulting phase-space trajectories projected onto the (ϕ/H) plane are shown in Figs. 5 and 6. For the trajectories of Fig. 6, the initial value of $a(t)$ was chosen to be 10 times smaller than in Fig. 5. Hence the effects of curvature are more pronounced.

Consider a sample trajectory of Fig. 5. It starts out with large initial value of a . The trajectory tends toward $|\phi| \gg 1$ and $\dot{H} \rightarrow 0$, as in the case $k=0$. Since $a(t)$ is now decreasing almost exponentially, the role of curvature increases. At a critical value of ϕ , the value of \dot{H} becomes 0. This will occur when

$$\frac{1}{2\sqrt{3}}V'(\phi(t)) = \frac{k}{a^2(t)}. \quad (3.26)$$

Hence the smaller the initial value of $a(t)$, the earlier (3.26) will be satisfied (compare Figs. 5 and 6). At a simi-

lar time, the curvature terms also start to dominate in Eq. (3.24). Therefore, as is obvious from the k -dependent terms in (3.25), $\phi(t)$ will rapidly decrease, as will $|H(t)|$. At some finite and negative value of ϕ , $H(t)$ vanishes. Thereafter, the Universe reexpands. The evolution of this model for small $a(t)$ resembles a de Sitter bounce.

Note that all solutions are nonsingular. In particular, the solutions can be integrated through the point when $H=0$ [when terms on the right-hand side of (3.25) become infinite].

In conclusion, we have constructed a higher-derivative modification of Einstein's theory in which all homogeneous and isotropic solutions are nonsingular. Without curvature (i.e., for $k=0$), the solutions either are periodic about Minkowski space or else converge to a $k=0$ de Sitter solution. For $k \neq 0$ the solutions which do not remain close to Minkowski space go through a de Sitter bounce and are future extendable to $t = \infty$. In addition, we have shown that our model is asymptotically free in the sense that the effective coupling of matter to gravity goes to zero as the curvature increases.

IV. NONSINGULAR UNIVERSE WITH LIMITING CURVATURE

Now we turn to the discussion of the full model in which the LCH is implemented, the model given by the action (2.24), in which for a homogeneous and isotropic metric the equations of motion reduce to (2.25)–(2.27). We include hydrodynamical matter with the energy density given by (3.17).

In the general case ($k \neq 0$ and $c \neq 0$), the phase space of the model is three dimensional: $\phi_1(t)$, $\phi_2(t)$, and $a(t)$. For $k=0$ and $c=0$, the dependence on $a(t)$ drops out and the phase space can be reduced to the two-dimensional ϕ_1/ϕ_2 diagram. The first-order equations of motion in phase space are found by combining Eqs. (2.25)–(2.27). To derive the equation for $\phi_1(t)$ we

differentiate (2.25) with respect to t and use (2.26) to substitute for \dot{H} to obtain

$$\dot{\phi}_1 = -4\sqrt{3} \frac{HV'_2}{V'_1} \tag{4.1}$$

The equation of motion for ϕ_2 is (2.27):

$$\begin{aligned} \dot{\phi}_2 = & -3H\phi_2 + \frac{k}{Ha^2}\phi_2 \\ & + \frac{1}{\sqrt{3}H} \left[3H^2(1-2\phi_1) + 3\frac{k}{a^2}(4\phi_1-1) \right. \\ & \left. - \frac{1}{2}(V_1+V_2) - \frac{c}{a^n} \right], \end{aligned} \tag{4.2}$$

where H can be expressed in terms of ϕ_1 and a via (2.25). From (4.1), (4.2), and (2.25), it is obvious that for $k=c=0$ the $a(t)$ dependence disappears.

In the case $k=c=0$ we may use (2.25) to get

$$\frac{d\phi_2}{d\phi_1} = -\frac{V'_1}{4V'_2} \left[-\sqrt{3}\phi_2 + (1-2\phi_1) - \frac{2}{V'_1}(V_1+V_2) \right], \tag{4.3}$$

the key equation for the following phase-space analysis.

For all potentials $V_1(\phi_1)$ and $V_2(\phi_2)$ with the asymptotic behavior

$$V_i \propto \phi_i^2, \quad \phi_i \ll 1, \tag{4.4}$$

and

$$V_1 \propto \phi_1 - \ln\phi_1 + O\left[\frac{1}{\phi_1}\right], \quad \phi_1 \gg 1, \tag{4.5}$$

$$V_2 \propto \text{const} + O\left[\frac{1}{\phi_2}\right], \quad \phi_2 \gg 1, \tag{4.6}$$

the phase diagrams have the same feature as depicted schematically in Fig. 8 for spatially collapsing universes without matter. The numerical solutions depicted in Fig. 7 were obtained for the particular choice of potentials [which satisfy the asymptotic conditions of (4.4)–(4.6)]

$$V_1(\phi_1) = 12H_0^2 \frac{\phi_1^2}{1+\phi_1} \left[1 - \frac{\ln(1+\phi_1)}{1+\phi_1} \right], \tag{4.7}$$

$$V_2(\phi_2) = 2\sqrt{3} H_0^2 \frac{\phi_2^2}{1+\phi_2^2}. \tag{4.8}$$

The presence of the logarithmic term in (4.7) will be justified shortly.

We can identify four classes of trajectories. Note that by (2.26), $|\phi_2| \rightarrow \infty$ implies that the evolution approaches de Sitter space. The first class of trajectories start in the de Sitter phase at $\phi_2 \rightarrow -\infty$ and evolve to de Sitter at $\phi_2 \rightarrow \infty$. For small initial values of ϕ_1 , trajectories starting at $\phi_2 = -\infty$ reach a turning point and return to $\phi_2 = -\infty$. The third class of trajectories are periodic solutions about Minkowski space-time ($\phi_1 = \phi_2 = 0$). Finally, trajectories starting with small ϕ_1 and ϕ_1/ϕ_2 with ϕ_2 positive evolve toward de Sitter solutions at $\phi_2 = \infty$. There are two separatrices dividing phase space into regions corresponding to the four above classes (see Fig. 8).

Note that in order to prevent solutions starting with $\phi_1 \gg 1$ and $\phi_2 \simeq 0$ from escaping to $\phi_1 = \infty$ at $\phi_2 < 1$ in finite time (such solutions which violate the LCH and would lead to singularities in higher-order curvature invariants) it was necessary to add the logarithmic correction term to $V_1(\phi_1)$.

Phase space is the half plane $\phi_1 \geq 0$. Negative values of ϕ_1 are unphysical since by (2.25) and using the small ϕ_1 asymptotic form of $V_1(\phi_1)$, they would correspond to imaginary values for $H(t)$. This half plane can be divided

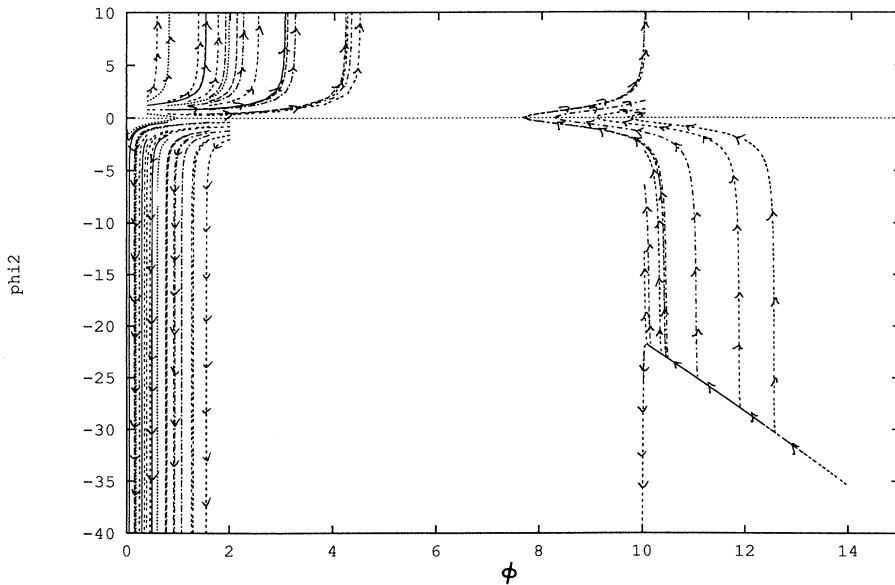


FIG. 7. Phase-space diagram for the spatially flat ($k=0$) universe with limiting curvature based on the potentials (4.7) and (4.8). There is no matter ($c=0$).

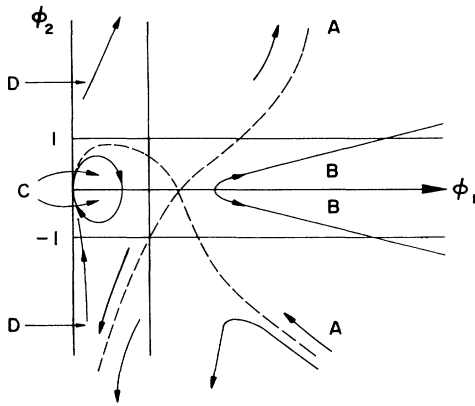


FIG. 8. Sketch of the generic phase-space diagram for a two-field model with $k=c=0$ and potentials satisfying the asymptotic conditions (4.4)–(4.6). Lines with arrows indicate phase-space trajectories, arrows pointing in the direction of increasing time. Separatrices are shown as dashed lines. With A , B , C , and D we denote the asymptotic region of phase space discussed in the text.

into four regions: in region A , $\phi_1 \gg 1$ and $|\phi_2| \gg 1$; in region B , $\phi_1 \gg 1$ and $|\phi_2| \ll 1$; in region C , $\phi_1 \ll 1$ and $|\phi_2| \ll 1$; and in region D , $\phi_1 \ll 1$ and $|\phi_2| \gg 1$. We will analyze the phase-space trajectories in each of the above regions, focusing on three features: the asymptotic expressions for $d\phi_2/d\phi_1$ (which give the tangent vectors to the trajectories), the separatrices, and the equations for the trajectories. To concretize the discussion we use the potentials (4.7) and (4.8). However, except in region B , the asymptotic solutions are independent of the specific choice of potentials.

In region A , Eq. (4.3) becomes

$$\frac{d\phi_2}{d\phi_1} \approx \frac{\sqrt{3}}{4} \frac{\phi_2^4}{\phi_1^2} \left[\sqrt{3} + 4 \frac{\phi_1}{\phi_2} \right]. \tag{4.9}$$

The direction of the tangent vectors is sketched in Fig. 8. Arrows indicate the direction of increasing time and are obtained by inspecting (4.1) and (4.2) directly. By inspecting the tangent vectors, it is clear that all solutions in the upper region $\phi_2 > 1$ quickly approach de Sitter space ($|\phi_2| \rightarrow \infty$ implies de Sitter space). In the lower region $\phi_2 < -1$ there are two domains separated by a separatrix which for $\phi_1 \gg 1$ and $|\phi_2| \gg 1$ is close to the line of turning points where $d\phi_2/d\phi_1 = 0$, its equation being given by

$$\frac{\phi_2}{\phi_1} = -\frac{4}{\sqrt{3}} \tag{4.10}$$

(see Fig. 8). To the right of the separatrix, trajectories correspond to solutions starting out in de Sitter phase. To the left of the line given by (4.10) we have $d\phi_2/d\phi_1 > 0$ and trajectories go off to de Sitter space at $\phi_2 \rightarrow -\infty$. In all cases, de Sitter space is reached at finite ϕ_1 values. This is seen by explicitly integrating (4.9). In the region where the first term on the right-hand side of (4.9) dominates, we have

$$\phi_1 \approx c - \frac{2}{3}\phi_2^{-3}, \tag{4.11}$$

while in the domain where the second term dominates the approximate solution is

$$\phi_1 \approx \frac{1}{\frac{3}{4}\phi_2^3 + c} \tag{4.12}$$

(c is a constant of integration). Note that all of the solutions starting in region A start in de Sitter space and end up in de Sitter space.

In region B the tangents in phase space are given by

$$\frac{d\phi_2}{d\phi_1} \approx \sqrt{3} \frac{1}{\phi_1 \phi_2}, \tag{4.13}$$

which integrates to

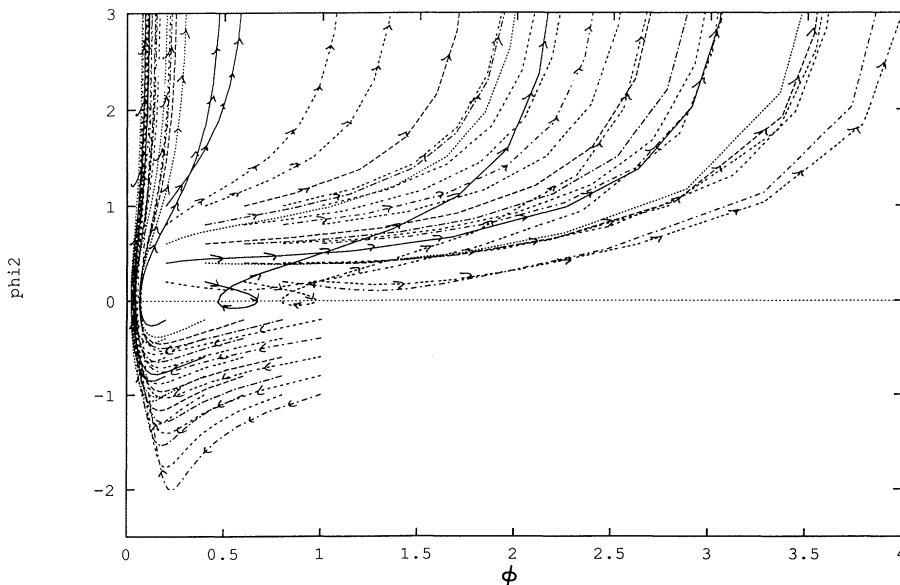


FIG. 9. Projection onto the (ϕ_1, ϕ_2) plane of the three-dimensional phase-space diagram (ϕ_1, ϕ_2, a) , for a two-field model which is spatially flat but contains matter. The potentials used are (4.7) and (4.8).

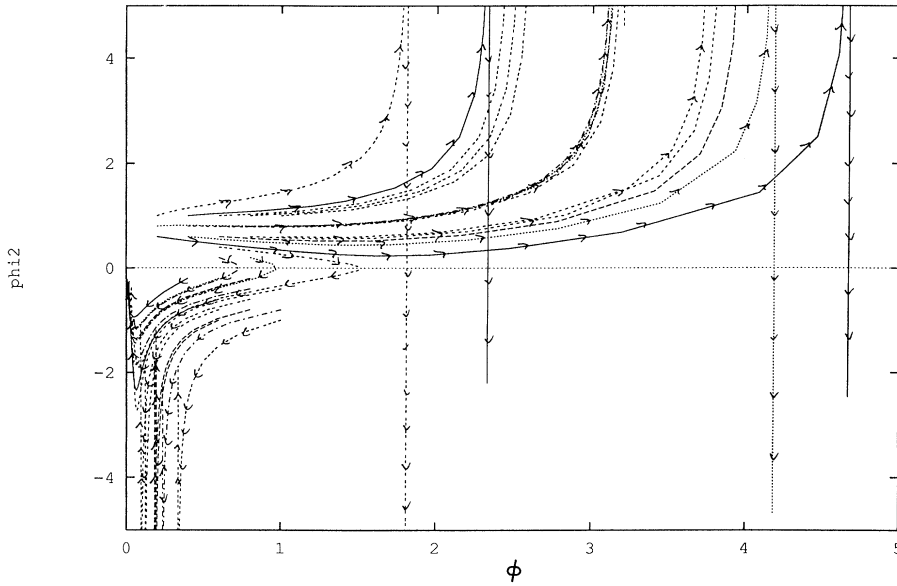


FIG. 10. Same for a two-field model without matter but including spatial curvature ($k \neq 0$).

$$\phi_1 = c \exp \left\{ \frac{1}{2\sqrt{3}} \phi_2^2 \right\}. \quad (4.14)$$

$$\phi_1 \approx -\frac{1}{\sqrt{3}} \phi_2^2 + c. \quad (4.17)$$

The tangent vectors are again sketched in Fig. 8. From (4.14) it follows that trajectories leave region *B* at a finite value of ϕ_1 . They enter region *A* and hence asymptotically approach de Sitter space.

In region *C*, Eq. (4.3) becomes

$$\frac{d\phi_2}{d\phi_1} \approx -\frac{\sqrt{3}}{2\phi_2} \left[1 - 3\phi_1 - \frac{1}{\sqrt{12}} \frac{\phi_2^2}{\phi_1} \right]. \quad (4.15)$$

The separatrix in the upper half planes is close to the line of turning points $d\phi_2/d\phi_1 = 0$ for large values of ϕ_i :

$$\phi_2 \approx \pm 12^{1/4} \phi_1^{1/2} (1 - 3\phi_1)^{1/2}. \quad (4.16)$$

Where the first term dominates, the trajectories obey

From the sketch of Fig. 8, it is clear that the trajectories which pass through $\phi_1 = \phi_2 = 0$ with $\dot{\phi}_2/\dot{\phi}_1(\phi_1 = \phi_2 = 0)$ not too large correspond to periodic motion about Minkowski space. This, as in the model of Sec. III, is an indication that Minkowski space is stable in our theory toward homogeneous perturbations.

Finally, in region *D* the equation for the tangent vector is

$$\frac{d\phi_2}{d\phi_1} \approx +\frac{\sqrt{3}}{2} \phi_2^3 \left[\sqrt{3}\phi_2 + \frac{1}{2\sqrt{3}\phi_1} \right]. \quad (4.18)$$

There is a separatrix which is (for large ϕ_i) approximately described by

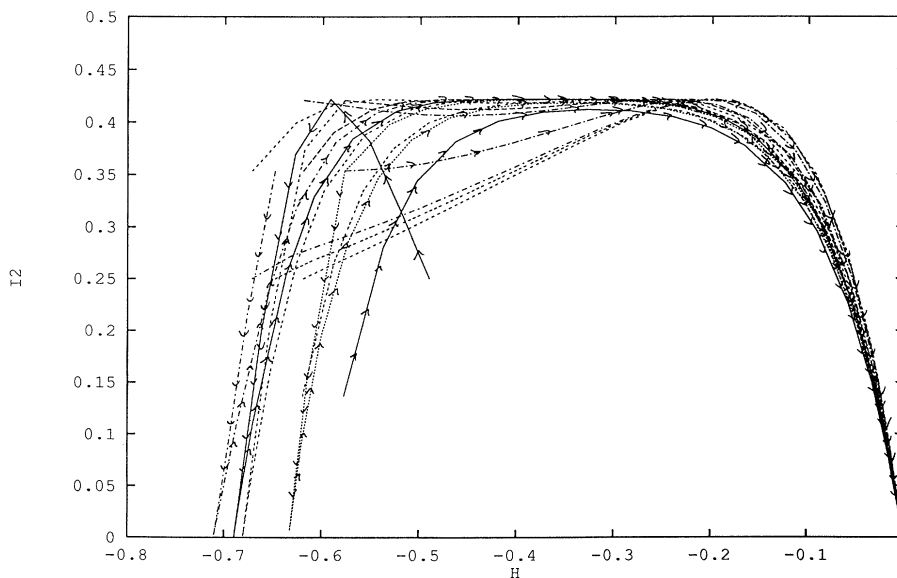


FIG. 11. Trajectories in the (I_2, H) plane for the same model as in Fig. 10.

$$\phi_2 \simeq -\frac{1}{6\phi_1}. \quad (4.19)$$

To the right of this line, the trajectories are given by

$$\phi_1 \simeq c - \frac{2}{9}\phi_2^{-3} \quad (4.20)$$

and to the left by

$$\phi_1 \simeq ce^{-2\phi_2^{-2}}. \quad (4.21)$$

In conclusion, all solutions are either periodic about Minkowski space or are asymptotically de Sitter. All solutions can be extended to $t = \pm\infty$, and hence there are no singularities.

As in the model of Sec. III, including matter does not effect the asymptotic solutions. The coupling between matter and gravity is asymptotically free also in the theory with action (2.24). However, including matter changes the nature of solutions starting near Minkowski space. These solutions now approach de Sitter space (see Fig. 9). This result is not surprising, since also in Einstein gravity Minkowski space is not a solution of the field equations in the presence of matter.

The projections of some phase-space trajectories onto the (ϕ_1, ϕ_2) plane in a model with $k \neq 0$ but $c = 0$ are shown in Fig. 10. As in the single-field model of Sec. III, the trajectories initially evolve as for $k = 0$ toward de Sitter space. Hence, for finite ϕ_1 , ϕ_2 becomes very large. Eventually, however, the curvature terms become important; ϕ_2 reaches a turning point and rapidly (within time period H_0^{-1}) relaxes to zero (for finite value of ϕ_1). As is obvious from Fig. 11, the rapid decrease in ϕ_2 corresponds to the de Sitter bounce during which H changes sign.

V. CONCLUSIONS AND DISCUSSION

We have constructed a theory of gravity in which all homogeneous and isotropic solutions (not only special solutions as in some other models [23]) are nonsingular, regardless of the matter content of the Universe. Our effective action for gravity contains higher-derivative terms which modify the Einstein action at high curvatures. Such terms are expected to be important near the Planck curvature in any fundamental theory such as quantum gravity or string theory.

Most higher-derivative gravity theories have much worse singularity properties than Einstein gravity. We use a particular construction based on implementing the "limiting curvature hypothesis" to obtain a class of models without singularities. We discussed two models, one in which all curvature invariants are bounded and all

solutions except those periodic about Minkowski space asymptotically approach de Sitter space (Sec. IV) and a simpler model without limiting curvature (Sec. III).

The theory presented in this paper is "asymptotically free" in the sense that the coupling of matter to gravity goes to zero as the curvature approaches its limiting value (similar features have been discussed by Linde [24] under the name "gravitational confinement").

When applied to an expanding universe, our theory implies that it started out in a de Sitter phase with scale factor $a(t) = e^{Ht}$ (for $k = 0$) or else (for $k = 1$) it emerged from a de Sitter bounce. In particular, there was a period of inflation driven by gravity. This is no surprise as it is well known [25] that higher-derivative gravity theories often produce inflation.

Note that the property of asymptotic freedom might also justify using the effective-action approach to gravity until the curvature reaches the Planck scale. Asymptotic freedom will also play an important role in controlling nonlocal terms. For example, nonlocal terms due to particle production may be expected to vanish in the asymptotic regions of phase space.

Our action is constructed by adding two Lagrange-multiplier terms (and their corresponding potentials) to the Einstein action. Each Lagrange multiplier is coupled to a curvature invariant. The role of the first Lagrange multiplier is to limit the curvature; the role of the second one (ϕ_2) is to force space-time to be de Sitter at large curvature. For a homogeneous and isotropic model, it was sufficient to couple ϕ_2 to the invariant $I_2 = 4R_{\mu\nu}R^{\mu\nu} - R^2$, since in this case $I_2 = 0$ singles out de Sitter space.

However, for an anisotropic cosmology, we must extend the invariant I_2 by including a term which effects the anisotropy. In a subsequent paper [13] (see also Ref. [26]) we will show that $I_2 = 4R_{\mu\nu}R^{\mu\nu} - R^2 + C^2$ is an appropriate invariant. This invariant also works for a spherically symmetric metric. Thus, in a model such as the one presented here, but with the new I_2 , we will be able to show that there will be no singularities inside the black hole horizon [13].

Open questions include the generalization of our model to general inhomogeneous metrics and a full stability analysis.

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