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Equations of state and transport equations in viscous cosmological models

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The aim of this paper is to compare the evolution of viscous cosmological models for three different constitutive equations for the viscous pressure σ, namely: Eckart theory ($\sigma=-\xi\nabla_\mu u^\mu$), Maxwell-Cattaneo theory ($\sigma+\tau\dot{\sigma}=-\xi\nabla_\mu u^\mu$), and generalized second-order Israel-Stewart theory ($\sigma+\tau\dot{\sigma}=-\xi\nabla_\mu u^\mu-\frac{1}{2}\xi\sigma T\rho u^\mu\nabla_\mu(v\tau/T\xi)$). In the last case, the nonlinear terms may be interpreted as a consequence of nonlinear equations of state for absolute temperature and thermodynamic pressure. We show that the nonlinear terms of the latter equation are compatible with a viscosity-driven inflationary expansion.

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I. INTRODUCTION

Viscous effects are the subject of growing attention in the analysis of cosmological models [1–4]. The simplest and most usual way to take them into account is to consider the well-known linear relation between bulk viscous pressure σ and the divergence of the barycentric velocity u^μ as

$$\sigma=-\xi\nabla_\mu u^\mu\ , \tag{1}$$

where ξ denotes the viscosity coefficient. This equation, which is also called Eckart's equation, is well known, but it, however, has two important shortcomings, namely (a) it predicts an infinite speed for the propagation of viscous pulses, and (b) it presents some pathological instabilities [5]. In order to overcome these problems, the relaxational equation

$$\tau\dot{\sigma}+\sigma=-\xi\nabla_\mu u^\mu\ , \tag{2}$$

where τ indicates the relaxation time, has been proposed instead. This relationship, also known as the Maxwell-Cattaneo equation, predicts a finite speed of propagation for viscous pulses, given by

$$v=\left[\frac{\xi}{\rho\tau}\right]^{1/2}c\ , \tag{3}$$

with ρ the energy density. When $\tau\to 0$, v diverges. Some authors [6,7,22] have applied Eq. (2) for the analysis of cosmological models.

Recently, Hiscock and Salmonson [8] (HS, hereafter) have used a generalization of Eq. (2) of the form

$$\sigma+\tau\dot{\sigma}=-\xi\nabla_\mu u^\mu-\frac{1}{2}\xi\sigma T\rho u^\mu\nabla_\mu\left[\frac{\tau}{T\rho\xi}\right]\ . \tag{4}$$

This equation incorporates nonlinear terms which are not present in (2). The motivation for these terms must be looked for in the thermodynamic formalism of extended irreversible thermodynamics [9–16]. In HS the results obtained with (4) are contrasted with those obtained with (2). These authors showed that important differences arise, namely, at late times the behavior of the cosmological scale factor $a(t)$ in the various theories is remarkably different. In the Eckart cosmology, which uses for σ Eq. (1), the scale factor behaves as $a(t)\approx t^{7/9}$. When the viscous pressure is described by the purely relaxational equation (2), one has an exponential or inflationary behavior $a(t)\approx\exp(kt)$. Finally, when the nonlinear terms are included, as in Eq. (4), the scale factor $a(t)$ behaves asymptotically as $a(t)\approx t^{2/3}$. HS conclude that the viscosity-driven inflation is a spurious effect due to the neglect of nonlinear terms in the transport equation (4).

In their analysis, HS used the equations of state obtained from the Boltzmann equation. Therefore, their conclusions may follow either from the inclusion of nonlinear terms in (4) or from the special form adopted for the equation of state for p, T, ξ or τ. Since HS raised an important point in the description of viscous phenomena, it is worthwhile to examine to what extent their conclusions are influenced by the choice of the equations of state.

The purpose of our paper is twofold: (a) to give a different, but complementary, interpretation of the nonlinear terms appearing in (4), by relating them to the nonequilibrium equations of state of extended irreversible thermodynamics, and (b) to use a different set of equations of state than that of HS. Our equations of state are commonly used in standard cosmological literature, so that it is worthwhile to check how the equations of state for p, T, and ξ/τ may affect the behavior of the cosmological model. We shall not discuss the relative merits of the different sets of equations of state in the description of the Universe; instead, our attention will be focused on their mathematical consequences for the evolution of their respective cosmological models. In Sec. II we deal with the purely thermodynamical aspect, whereas Sec. III is devoted to its cosmological consequence.

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II. EXTENDED THERMODYNAMICS: EQUATIONS OF STATE AND TRANSPORT EQUATIONS

The fundamental variables for describing relativistic dissipative fluids are the energy-stress tensor $T^{\mu\nu}$ and the particle number four-vector N^{μ}, which may be written as [1–3,9–12]

$$T^{\mu\nu}=\rho u^{\mu}u^{\nu}+(p+\sigma)\Delta^{\mu\nu}+q^{\mu}u^{\nu}+q^{\nu}u^{\mu}+\tau^{\mu\nu} \tag{5}$$

and

$$N^{\mu}=nu^{\mu} , \tag{6}$$

where n is the number density, u^{μ} the four-velocity of the fluid, p the equilibrium pressure, $\Delta^{\mu\nu}$ the projection tensor orthogonal to u^{μ}, σ the bulk-viscous stress, q^{μ} the heat flux, and $\tau^{\mu\nu}$ the shear-viscous stress tensor. In order that the stress tensor $T^{\mu\nu}$ is unique, the heat flux and the shear-viscous stress have to satisfy the constraints

$$\nabla^{\mu\nu}u_{\nu}=q^{\mu}u_{\mu}=\tau^{\mu\nu}u_{\nu}=\tau^{u}_{\mu}=\tau^{\mu\nu}-\tau^{\nu\mu}=0 . \tag{7}$$

The nonequilibrium entropy flow is given in extended irreversible thermodynamics by the Israel-Stewart expression [10,11,16]

$$s^{\nu}=snu^{\mu}+\frac{q^{\mu}}{T}-\frac{1}{2}[\beta_0\sigma^2+\beta_1 q^{\nu}q_{\nu}+\beta_2\tau^{\mu\nu}\tau_{\mu\nu}]\frac{u^{\mu}}{T}+\frac{\alpha_0\sigma q^{\mu}}{T}+\frac{\alpha_1\tau^{\mu}_{\nu}q^{\nu}}{T} , \tag{8}$$

where T is the absolute temperature and s the entropy per particle. The three parameters β_i model the deviation of the entropy density with respect to the local equilibrium entropy due to the nonvanishing values of the flows. The two other coefficients α_i describe couplings between the heat flux and the viscous stress deviation from equilibrium. If the α_i and β_i are taken to be identically zero, we find Eckart's theory [17], which is the usual local-equilibrium description of nonequilibrium systems.

According to the second law of thermodynamics the divergence of the entropy four-vector is positive [10,16,17]:

$$\nabla_{\mu}s^{\mu}\geq 0 . \tag{9}$$

The conservation of the energy and momentum take the form [5,6]

$$\nabla_{\mu}T^{\mu\nu}=0 , \tag{10}$$

$$\nabla_{\mu}N^{\nu}=0 . \tag{11}$$

According to (8)–(11) the entropy production has the form [8,10,16]

$$\begin{aligned}T\nabla_{\mu}s^{\mu}=&-\sigma[\nabla_{\mu}u^{\mu}+\beta_0 u^{\mu}\nabla_{\mu}\sigma-\alpha_0\nabla_{\mu}q^{\mu}+\tfrac{1}{2}T\sigma\nabla_{\mu}((\beta_0/T)u^{\mu})]\\&-q_{\mu}[\Delta^{\mu\nu}\{(1/T)\nabla_{\nu}T+u^{\gamma}\nabla_{\gamma}u_{\nu}+\beta_1 u^{\gamma}\nabla_{\gamma}q_{\nu}-\alpha_0\nabla_{\nu}\sigma-\alpha_1\nabla_{\gamma}\tau^{\gamma}_{\nu}+\tfrac{1}{2}Tq_{\nu}\nabla_{\gamma}((\beta_1/T)u^{\gamma})\\&-\sigma T\nabla_{\nu}(\alpha_0/T)-T\tau^{\gamma}_{\nu}\nabla_{\gamma}(\alpha_1/T)\}]-\tau_{\mu\nu}\langle\nabla^{\mu}u^{\nu}+\beta_2 u^{\gamma}\nabla_{\gamma}\tau^{\mu\nu}-\alpha_1\nabla^{\mu}q^{\nu}+\tfrac{1}{2}T\tau^{\mu\nu}\nabla_{\gamma}(\beta_2/T)u^{\gamma}\rangle .\end{aligned} \tag{12}$$

Slightly more general but more cumbersome expressions than (12) may be found in [8,10]. It is usual to express (12) in the short form

$$T\nabla_{\mu}s^{\mu}=-\sigma X^{\upsilon}_0-q_{\mu}X^{\mu}_1-\tau_{\mu\nu}X^{\mu\nu}_2 .$$

Note that the time derivatives of the fluxes σ, q_{μ}, and $\tau_{\mu\nu}$ appear in X^{ν}_0, X^{μ}_1, and $X^{\mu\nu}_2$, respectively. Then, in order to obtain evolution equations for the fluxes one assumes that X^{υ}_0, X^{μ}_1, and $X^{\mu\nu}_2$ are respectively proportional to σ, q^{μ}, and $\tau^{\mu\nu}$. In this way, and after conveniently identifying the proportionality coefficients, one may write (12) as [9–11,16]

$$T\nabla_{\mu}s^{\mu}=\frac{\sigma^2}{\xi}+\frac{q_{\mu}q^{\mu}}{\kappa T}+\frac{\tau_{\mu\nu}\tau^{\mu\nu}}{2\eta} , \tag{13}$$

where κ,η are the coefficients of thermal conductivity and shear viscosity respectively. The evolution equations for σ, q^{μ}, and $\tau^{\mu\nu}$ read

$$\sigma=-\xi[\nabla_{\mu}u^{\mu}+\beta_0 u^{\mu}\nabla_{\mu}\sigma-\alpha_0\nabla_{\mu}q^{\mu}+\tfrac{1}{2}T\sigma\nabla_{\mu}((\beta_0/T)u^{\mu})], \tag{14}$$

$$\begin{aligned}q^{\mu}=&-\kappa T\Delta^{\mu\nu}[(1/T)\nabla_{\nu}T+u^{\gamma}\nabla_{\gamma}u_{\nu}+\beta_1 u^{\gamma}\nabla_{\gamma}q_{\nu}-\alpha_0\nabla_{\nu}\sigma-\alpha_1\nabla_{\gamma}\tau^{\gamma}_{\nu}\tfrac{1}{2}Tq_{\nu}\nabla_{\gamma}((\beta_1/T)u^{\gamma})\\&-\sigma T\nabla_{\nu}(\alpha_0/T)-T\tau^{\gamma}_{\nu}\nabla_{\gamma}(\alpha_1/T)] ,\end{aligned} \tag{15}$$

$$\tau^{\mu\nu}=-2\eta\langle\nabla^{\mu}u^{\nu}+\beta_2 u^{\gamma}\nabla_{\gamma}\tau^{\mu\nu}-\alpha_1\nabla^{\mu}q^{\nu}+\tfrac{1}{2}T\tau^{\mu\nu}\nabla_{\gamma}((\beta_2/T)u^{\gamma})\rangle . \tag{16}$$

To return to the Eckart theory, which has just five variables (three components of the velocity and two thermodynamic variables), one only has to set $\alpha_i=\beta_i=0$. Equation (14) (with vanishing heat flux) is the equation used by HS as the basis of their model, namely, Eq. (4).

Here, we want to stress a complementary point of view about the nonlinear terms appearing in (14) or in (4): we wish to relate them to the nonequilibrium equations of state for the temperature and pressure. The meaning of these generalized equations is an active topic in nonequilibrium thermodynamics [18–20], so that it is useful to examine this connection. For the sake of simplicity, we will do that for vanishing heat flux and shear-viscous stress.

If the heat flux and the shear-viscous stress vanish, the entropy per unit mass of the deviation from equilibrium has the form

$$s = s_{eq} - \frac{1}{2}\frac{\tau}{T\xi\rho}\sigma^2 . \tag{17}$$

Here, the comoving reference is assumed, and we have identified $\beta_0 \equiv \tau/\rho\xi$, which is well known in extended irreversible thermodynamics [16].

The generalized Gibbs equation corresponding to (17) may be written as

$$ds = \theta^{-1} dw + \theta^{-1}\pi\, dv - \frac{\tau}{T\xi\rho}\sigma\, d\sigma , \tag{18}$$

where $v \equiv (1/\rho)$ is the volume per unit of mass energy and w the internal energy per unit volume, and where, in analogy with the standard Eckart theory, the generalized absolute temperature θ and generalized pressure π have been defined as [16–20]

$$\theta^{-1} = \left[\frac{\partial s}{\partial w}\right]_{v,\sigma} = T^{-1} - \frac{1}{2}\sigma^2 \frac{\partial}{\partial w}\left[\frac{v\tau}{T\xi}\right] , \tag{19}$$

$$\theta^{-1}\pi = \left[\frac{\partial s}{\partial v}\right]_{w,\sigma} = T^{-1}p - \frac{1}{2}\sigma^2 \frac{\partial}{\partial v}\left[\frac{v\tau}{T\xi}\right] . \tag{20}$$

We may obtain explicitly the expression for the entropy production by writing the time derivative of s as

$$\rho\dot{s} = \theta^{-1}\rho\dot{w} + \theta^{-1}\pi\rho\dot{v} - \frac{\tau}{T\xi}\sigma\dot{\sigma} . \tag{21}$$

Taking into account (19) and (20) we get

$$\rho\dot{s} = T^{-1}\rho\dot{w} + T^{-1}p\rho\dot{v} - \frac{1}{2}\sigma^2\rho u^{\mu}\nabla_{\mu}\left[\frac{\tau}{T\xi\rho}\right] - \frac{\tau}{T\xi}\sigma\dot{\sigma} . \tag{22}$$

The first two terms are those of the standard theory, and are given by the well-known expression $-T^{-1}\sigma\nabla_{\mu}u^{\mu}$, therefore we may write

$$\rho\dot{s} = -T^{-1}\sigma\left[\nabla_{\mu}u^{\mu} + \frac{\tau}{\xi}\dot{\sigma} + \frac{1}{2}T\sigma\rho u^{\mu}\nabla_{\mu}\left[\frac{\tau}{T\xi\rho}\right]\right] . \tag{23}$$

We now assume that σ is proportional to its conjugate thermodynamic force in (2) and obtain

$$\sigma = -\xi\left[\nabla_{\mu}u^{\mu} + \frac{\tau}{\xi}\dot{\sigma} + \frac{1}{2}T\sigma\rho u^{\mu}\nabla_{\mu}\left[\frac{\tau}{T\xi\rho}\right]\right] . \tag{24}$$

If one takes into account just the first term on the right-hand side, the standard Eq. (1) follows. If the two first terms are considered, one obtains (2). And if the whole right-hand side is taken into account, we have essentially the equation of HS.

The difference with these authors is simply that they arrived at (24) [or (4)] without explicitly writing the Gibbs equation, but by directly differentiating Eq. (14). Their approach is of course correct. However, since the meaning of generalized temperature and thermodynamic pressure in nonequilibrium situations is one of the basic thermodynamic problems, we think that it is of interest to outline that the nonlinear term in (24) comes, precisely, from the generalized temperature θ and the generalized pressure π, when the Gibbs equation corresponding to the generalized entropies is taken into account. Then, we see that the nonclassical terms in θ and π appearing in (19) and (20) lead indeed to definite physical consequences.

III. COSMOLOGICAL APPLICATION

In this section we apply Eq. (24) to the description of a cosmological model along the lines of Ref. [8]. In order to solve the dynamical equations, we need equations of state for T, p and also for the new parameter τ/ξ of the extended theory. HS have used the equations of state arising from the Boltzmann equation, namely,

$$p = nm\beta^{-1} , \tag{25}$$

$$T = \frac{mc^2}{k\beta} , \tag{26}$$

$$\xi = \frac{mc\eta^2\Omega^2}{I} , \tag{27}$$

$$\rho = A_0[\beta^{-1}K_1(\beta) + 3\beta^{-2}K_2(\beta)] , \tag{28}$$

where β, m, A_0, I, and $K_i(\beta)$ are respectively the inverse temperature, particle mass, a constant, collision integral, and Bessel's functions, and where $\Omega = 3\gamma[1+1/\eta\beta]-5$, $\eta = K_3(\beta)/K_2(\beta)$, and γ is the solution of

$$\frac{\gamma}{\gamma-1} = \beta^2(1+5\eta/\beta-\eta^2) .$$

Instead of (25)–(28) we propose to use the much simpler set of equations

$$p = \lambda\rho , \tag{29}$$

$$\xi = \alpha\rho^m , \tag{30}$$

$$\frac{\xi}{\tau} = \rho . \tag{31}$$

The first equation, with λ is a constant ($\lambda = 1/3$ for thermal radiation and $\lambda = 0$ for dust), is standard in cosmological models, whereas the third one was proposed by Belinskii *et al.* [21,22] as a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light [see Eq. (3)]. The equations of state (29)–(31) are simpler and more usual in cosmology than those of HS. We insist again that we are not referring here to the problem of which equations are most suited to the description of the real world; we only wish to emphasize the decisive influence of the equations of state concerning the possibility of viscous inflation. It is then worthwhile to analyze to what extent the results of these authors are sensitive to the choice of the equations of state.

We will only consider the case of flat spatial sections, $k=0$, in order to simplify the equations. For a Friedmann-Robertson-Walker universe, the Einstein equations can be written as

$$\rho=\left[\frac{3}{8\pi G}\right]H^2 , \tag{32}$$

$$\dot{H}+\tfrac{3}{2}(\lambda+1)H^2=-4\pi G\sigma , \tag{33}$$

with $H\equiv\dot{a}/a$ the Hubble factor and $a(t)$ the scale factor of the universe.

According to (33), the constitutive equation for σ should play an important role in the evolution equation for the universe. When the Eckart equation (1) is used one obtains

$$\dot{H}+\tfrac{3}{2}(\lambda+1)H^2-\tfrac{9}{2}\beta_m^{-1}H^{2m+1}=0 , \tag{34}$$

with $\beta_m=\alpha^{-1}(3/8\pi G)^{1-m}$, and where we have employed (28)–(30). If instead the relaxational equation (2) is used, one has

$$\ddot{H}+\dot{H}[3(1+\lambda)H+\beta_m H^{2-2m}]+H^3[\tfrac{3}{2}(1+\lambda)\beta_m H^{1-2m}-\tfrac{9}{2}]=0 . \tag{35}$$

Note that both equations admit as a solution an inflationary expansion given by

$$H_0=[(\beta_m/3)(1+\lambda)]^{1/(2m-1)} . \tag{36}$$

Now, we want to ascertain the effect of the nonlinear terms included in (22). Taking into account that $\xi/\tau=\rho$ and that $\nabla_\mu u^\mu=3H$, the equation (24) for the viscous pressure takes the form

$$\sigma=-\xi\left[3H+\frac{1}{\rho}\dot{\sigma}+\frac{1}{2}T\sigma\rho u^\mu\nabla_\mu\left[\frac{1}{T\rho^2}\right]\right] . \tag{37}$$

To relate T with ρ we assume $T=b\rho^n$ with b a positive constant. For instance, for radiation, $n=\frac{1}{4}$. When this expression is used, (37) takes the form

$$\sigma=-\xi\left[3H+\frac{1}{\rho}\dot{\sigma}-\frac{n+2}{2}\sigma\frac{\dot{\rho}}{\rho^2}\right] . \tag{38}$$

Combination of (38) with (33) and (32) yields

$$\ddot{H}+3(1+\lambda)H\dot{H}=\frac{9}{2}H^3+\left[\dot{H}+\frac{3}{2}(1+\lambda)H^2\right]\times\left[(n+2)\frac{\dot{H}}{H}-\beta_m H^{2-2m}\right] . \tag{39}$$

Note that this equation also admits an inflationary solution with $\ddot{H}=\dot{H}=0$, and H_0 given again by (36).

IV. CONCLUDING REMARKS

We have related the nonlinear terms of the transport equation used by HS to the generalized nonequilibrium equations of state for pressure and temperature. This connection is of interest from a thermodynamic point of view, for it shows that the nonclassical equations, instead of being an unphysical artifact, may have physical effects. In fact it is known that the generalized equation of state for absolute temperature in rigid heat conductors leads to different speed for thermal pulses in a nonequilibrium steady state along or opposed to the direction of the mean heat flux [23,24].

When the equations of state (29)–(31) are used together with the transport equation (24), one obtains Eq. (36) for the evolution of the cosmological model. This equation may have a solution of the kind $H=$const, i.e., an inflationary expansion of viscous origin. This is not the case in the model of HS, who argue that the viscosity-driven inflationary behavior may be a pathological one resulting from the use of truncated transport equations, as those of Eckart [Eq. (1)] or Maxwell-Cattaneo (2). Here it is seen that the conclusion of HS depends on the equations of state used in their work (those corresponding to a Boltzmann gas), so that it is not a general conclusion. Of course, which equations of state and which transport equations are the most suitable for the Universe is an open problem.

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