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New types of inflationary universe

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We investigate a range of scalar-field potentials which give rise to slow-roll inflation. The behavior of the scale factor of the Universe is derived in each case. Exact solutions are found which illustrate new types of inflationary behavior that arise when the potential is asymptotically of the form $V=V_0\phi^N\exp(A\phi^M)$. The gravitational wave and density perturbation spectral indices arising in these models are derived and discussed. We give a detailed discussion of the interrelationships between our exact solutions and approximate solutions derived in the slow-roll approximation.

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I. INTRODUCTION

The concept of chaotic inflation has motivated the study of a wide variety of scalar-field potentials as possible sources of the energy-momentum tensor of matter during the early stages of the Universe. At first, the alternatives were studied solely with a view to discovering the most natural manifestation of the inflationary universe concept, demonstrating its ubiquity and consequences [1]. Recent observations by the Cosmic Background Explorer (COBE) satellite [2] have led to detailed studies of the density and gravitational wave fluctuations that would be produced during any period of inflation [3]. The fact that the contributions of the density and gravitational wave fluctuations differ significantly in different inflationary universe models has motivated a detailed study of all the alternatives. The original de Sitter style of inflation proposed by Guth has been extensively studied, together with power-law inflationary models [4], since these can also arise from scalar-tensor gravity theories [5]. The author found a class of "intermediate" inflationary universe models [6] (in which the scale factor increases as the exponential of a fractional power of the time) which have turned out to have very interesting properties with regard to the balance of density and gravitational waves produced during inflation [7]. This collection of different inflationary universe models leads one to delineate the whole range of inflationary behaviors that are possible for the set of scalar-field potentials which allow slow rolling of the scalar field. In this paper we shall show how the situation can be classified in a straightforward way in terms of the behavior of the scalar-field potential. A number of new types of inflation will be found and are illustrated by a set of new exact solutions for the zero-curvature Friedmann universe containing a scalar-field source with potential $V(\phi)$.

In Sec. II we give a general discussion of the evolution of scalar fields in the flat Friedmann universe and derive conditions on the potential under which inflation will not occur and persist as $t\to\infty$. In Sec. III two broad classes of new exact inflationary universe solutions are derived for potentials which combine products of power-law with exponential behavior for ϕ. In Sec. IV the density inhomogeneity and gravitational wave production during these new varieties of inflation are computed and discussed. In Sec. V we relate our exact solutions and their asymptotic behaviors to the more general behavior of inflationary universes in the slow-roll regime.

II. SCALAR FIELDS IN FRIEDMANN UNIVERSES

We shall be interested in the evolution of a scalar field ϕ with potential $V(\phi)$ in a zero-curvature Friedmann universe with expansion scale factor $a(t)$. We define the Hubble expansion parameter as

$$H=\dot{a}/a\ . \tag{1}$$

The Einstein equations with $8\pi G=c=1$ reduce to the set

$$3H^2=\tfrac{1}{2}\dot{\phi}^2+V\ , \tag{2}$$

$$2\dot{H}=-\dot{\phi}^2\ , \tag{3}$$

$$\ddot{\phi}+3H\dot{\phi}+V'(\phi)=0\ . \tag{4}$$

Any one of these three equations may be derived from the remaining two. For our purposes, Eqs. (2) and (3) will prove most useful.

Various attempts have been made to prove "no-hair" theorems for Eqs. (2)–(4) and their anisotropic generalizations; this has led to some confusion. If a potential possesses a minimum into which the ϕ field evolves, then one cannot expect to prove a no-hair theorem even if inflation does occur. For example, if $V=V_0\phi^2$, then for a suitable choice of V_0 and of the initial value of $\dot{\phi}$, a period of de Sitter inflation will occur; however, there can be no asymptotic approach of the space-time to the de Sitter metric because the period of inflation will end when the ϕ field begins oscillating about the minimum at $\phi=0$. Heusler [8] has investigated the averaged behavior of the ϕ field in potentials with a minimum, but his subsequent use of those results to draw conclusions about the late-time isotropization of homogeneous models is not relevant to the issue of whether inflation can isotropize the expansion. No inflationary model, in which inflation ends, can ensure isotropization of the Universe from arbitrary initial conditions as $t\to\infty$. It does not need to. It merely offers a way of ensuring that the level of anisotropy is very small at some finite time ($\sim 10^{10}$ yr) after the inflation ended. This situation means that while one can prove a no-hair theorem for power-law inflation driven by specific choices of scalar-field potential (e.g., exponential [9]) or by specifying the equation of state to be a perfect fluid with density ρ and pressure p obeying $\rho+3p<0$ [4,10], there can be no proof of a no-hair theorem for power-law inflation which demands only that $\rho+3p<0$. For this would permit the inclusion of scalar fields whose potentials possess minima in which the field ultimately oscillates at late time after inflation has occurred.

If $V=\lambda\phi^{2N}$, where N is a positive integer, then the ϕ field will evolve into a regime in which it performs oscillations about the potential minimum with a period much shorter than the expansion time scale of the background universe. In this regime we can neglect H to a first approximation in (2) to obtain [11]

$$\dot{\phi}^2 = A^{2N} - 2\lambda\phi^N , \tag{5}$$

where the integration constant A^{2N} gives the amplitude of the oscillations. For a particular choice V, this equation may be integrated to obtain the first approximation to $\phi(t)$. Denote this "adiabatic" solution by ϕ_a. Now allow A to become a slowly varying function of t, so that

$$\dot{\phi} = \dot{\phi}_a + \phi\dot{A}/A . \tag{6}$$

Substituting into (4) with the assumption of a slow time variation of A that (so $\ddot{A} \ll \dot{A}H$), this gives

$$(N+1)\dot{A} = -3AH ; \tag{7}$$

hence,

$$A \propto a^{-3/(N+1)} . \tag{8}$$

This means that the total-energy density of the scalar field on the right-hand side of (2) evolves as [11]

$$\rho_\phi \propto a^{-6N/(N+1)} . \tag{9}$$

Hence its averaged behavior around the minimum of the potential mimics that of a perfect fluid with an equation of state $p=(\gamma-1)\rho$ in which

$$\gamma = 2N(1+N)^{-1} . \tag{10}$$

We see that $1 \le \gamma \le 2$ and the behavior is noninflationary (inflation requires $0<\gamma<\frac{2}{3}$). If $k=0$, the expansion asymptotes to $a \propto t^{2/3\gamma}$; if $k<0$, then it approaches $a \propto t$ [12].

Now consider potentials without minima in which inflation occurs by slow rolling. One can regard these as descriptions of the slow-roll portion of a more complicated potential possessing one or more minima. It is useful to begin by proving something about the scalar-field potentials which do not lead to inflation. If the potential never comes to dominate the solution of (2) and (3) as $t\to\infty$, then (with the exception of the case where $V\propto\exp\{-\lambda\phi\}$ and $\lambda^2>2$) the kinetic energy of ϕ will dominate the evolution and the solution will approach the one with $V=0$; that is, the free-field solution

$$H=\frac{1}{3t}, \quad \phi=(\tfrac{2}{3})^{1/2}\ln t \tag{11}$$

will be a stable attractor as $t\to\infty$. If this solution is stable as $t\to\infty$ when $V\neq 0$, then it indicates that inflation will not continue indefinitely (we are ignoring the reheating and decay of the ϕ field here). Consider a small perturbation of (11) by small quantities ϵ and h, so that

$$H=(1+h)/3t, \quad \phi=(1+\epsilon)(\tfrac{2}{3})^{1/2}\ln t . \tag{12}$$

Keeping only terms linear in h and ϵ, Eqs. (2), (3), and (12) give

$$h(t)=h_0t^{-1}+3t^{-1}\int t^2V(\phi_*)dt , \tag{13}$$

where ϕ_* is given by ϕ in (11). For any particular choice of V, we can use (13) to determine the stability of the noninflationary behavior. A solution of (13) which decays with time is only a sufficient condition for inflation not to occur. It is not a necessary condition because another term (for example, the curvature) might come to dominate. As expected, Eq. (13) confirms that inflation arises when $V=V_0=$ const. Consider now the more general potential

$$V(\phi)=V_0\phi^N\exp[-\lambda\phi^M], \quad M,N \text{ const} . \tag{14}$$

Using (13), we see that when $N=0$ and $M>1$ the noninflationary behavior $a\propto t^{1/3}$ remains stable (another proof of this will be given in Sec. V). This will also be the case for $M>1$ when $N\neq 0$. When $M=0$, $V\propto\phi^N$, and if there is no minimum, then there is an asymptotic approach to the intermediate inflationary models found by the author [6,7], in which

$$a(t)\propto\exp[At^f] , \tag{15}$$

$$\phi\propto t^{f/2} , \tag{16}$$

$$V(\phi)=\tfrac{1}{2}A^2f^2\phi^{-\beta}(2A\beta)^{\beta/2}(1-\beta^2\phi^{-2}) , \tag{17a}$$

where

$$\beta\equiv 4(f^{-1}-1) . \tag{17b}$$

The range of general behaviors of $V(\phi)$ remaining to be investigated is when $N\neq 0$ and $0<M<1$.

III. EXACT SOLUTIONS

We first search for solutions of (2) and (3) with

$$\phi=A(\ln t-B)^n, \quad A,B,n \text{ const} . \tag{18}$$

Hence, from (3), we have

$$\dot{H}=-\tfrac{1}{2}A^2n^2t^{-2}(\ln t-B)^{2n-2} , \tag{19}$$

and introducing $\theta=\ln t-B$, we have

$$H=-\tfrac{1}{2}A^2n^2\int\theta^{2n-2}e^{-\theta}d\theta , \tag{20}$$

and so

$$H=\frac{1}{2}A^2n^2t^{-1}\left\{\theta^{2n-2}+\frac{d(\theta^{2n-2})}{d\theta}+\cdots+\frac{d^{2n-2}(\theta^{2n-2})}{d\theta^{2n-2}}\right\}$$
$$\equiv\tfrac{1}{2}A^2n^2t^{-1}\Sigma_n(\theta)\ . \qquad (21)$$

The corresponding potential is determined from (2) and is

$$V(\phi)=\tfrac{3}{4}A^4n^4\exp[-2B]\exp[-2(\phi/A)^{1/n}]\Sigma_n{}^2(\theta)\ , \qquad (22)$$

where $\theta=(\phi/A)^{1/n}$. The expansion scale factor is given by

$$a(t)\propto\exp\left[\tfrac{1}{2}A^2n^2\int\Sigma_n(\theta)d\theta\right]\ , \qquad (23)$$

which, since $\theta=\ln t-B$, has the form of a finite series:

$$a\propto\exp\left[\frac{1}{2}A^2n^2\left[\frac{\theta^{2n-1}}{2n-1}+\theta^{2n-2}+(2n-2)\theta^{2n-3}+\cdots+\frac{d^{2n-1}}{d\theta^{2n-1}}(\theta^{2n-2})\right]\right]\ . \qquad (24)$$

Thus (18), (22), and either (23) or (24) give the complete exact solution.

A number of particular cases in which $(2n-2)$ is an integer are instructive.

(i) $n=1$. This gives the well-known case of power-law inflation with

$$H=\tfrac{1}{2}A^2t^{-1}\ , \qquad (25)$$

$$a(t)\propto t^{A^2/2}\ , \qquad (26)$$

$$V(\phi)=\tfrac{1}{4}A^2(3A^2-2)\exp[-2B-2\phi A^{-1}]\ . \qquad (27)$$

The choice $3A^2=2$ gives the free-field solution (11).

(ii) $n=\frac{3}{2}$. This gives a new type of inflationary model with

$$H=9A^2t^{-1}(\ln t-B+1)/8 \qquad (28)$$

and an expansion scale factor

$$a(t)\propto t^{9A^2(1-B)/8}t^{(9A^2\ln t)/16}\ . \qquad (29)$$

It arises from the potential

$$V=\exp[-2B-2(\phi/A)^{2/3}]\left[\frac{243}{64}A^4\{1+(\phi/A)^{2/3}\}^2-\frac{9A^2}{8}(\phi/A)^{2/3}\right]\ . \qquad (30)$$

Thus we see that the constant B permits an overall rescaling of the potential. At large ϕ the scalar field is evolving in a potential of the asymptotic form $V\propto\phi^{4/3}\exp[-2(\phi/A)^{2/3}]$. When $B=1$ there is a simpler exact solution with

$$a(t)\propto t^{(9A^2\ln t)/16}\ . \qquad (31)$$

(iii) $n=2$. The expansion rate now has the form

$$H=2A^2t^{-1}[(\ln t-B)^2+2(\ln t-B)+2]\ ; \qquad (32)$$

hence,

$$a(t)\propto\exp\{2A^2[\tfrac{1}{3}\ln^3t+(1-B)\ln^2t+(B^2-2B+2)\ln t]\}\ , \qquad (33)$$

with the potential given by

$$V=12A^4\exp\{-2B-2(\phi/A)^{1/2}\}\times[(\phi/A)+2(\phi/A)^{1/2}+2]^2-2A\phi\exp\{-2B-2(\phi/A)^{1/2}\}\ . \qquad (34)$$

The asymptotic forms of the general solutions (18)–(24) for large ϕ and large t, for general $n>1$, describe slow-rolling inflation in potentials that combine power-law and fractional exponential functions of ϕ. For $n>1$, they are

$$\phi\simeq A(\ln t)^n\ , \qquad (35)$$

$$H\simeq\tfrac{1}{2}A^2n^2t^{-1}(\ln t)^{2n-2}\ , \qquad (36)$$

$$a(t)\propto\exp\{\tfrac{1}{2}A^2n^2(2n-1)^{-1}(\ln t)^{2n-1}\}\ , \qquad (37)$$

with a potential of the form

$$V(\phi)\simeq\tfrac{3}{4}A^4n^4(\phi/A)^{4(n-1)/n}\exp[-2(\phi/A)^{1/n}]\ . \qquad (38)$$

A second broad class of new inflationary models can be found with the general behavior

$$\phi=A(t^\lambda+B)^n\ , \qquad (39)$$

where $A>0$, B, n, and λ are constants. The Hubble rate is given by

$$H=-\tfrac{1}{2}\lambda^2A^2n^2\int t^{2(\lambda-1)}(B+t^\lambda)^{2n-2}dt\ . \qquad (40)$$

Writing

$$x=t^\lambda\ , \qquad (41)$$

exact solutions are obtained in the cases where $(2n-2)\equiv m$ is a positive integer. We find, in this case, that H is given by a finite binomial series:

$$H=-\frac{1}{2}\lambda^2A^2n^2\left[\frac{x^{m+2-\lambda^{-1}}}{m+2-\lambda^{-1}}+\cdots+\frac{\begin{bmatrix}m\\k\end{bmatrix}B^kx^{m+k-\lambda^{-1}}}{m+k-\lambda^{-1}}+\cdots+\frac{B^mx^{1-\lambda^{-1}}}{2-\lambda^{-1}}\right]\ . \qquad (42)$$

Hence the expansion scale factor integrates to the form

$$a(t)=\exp\left\{\frac{1}{2}A^2n^2\left[\frac{t^{2n\lambda}}{2n(\lambda^{-1}-2n)}+\cdots+\frac{\begin{bmatrix}2n-2\\k\end{bmatrix}B^k t^{\lambda(2n-k)}}{(2n-k)(\lambda^{-1}-2n+k)}+\cdots+\frac{B^{2n-2}t^{2\lambda}}{2(\lambda^{-1}-2)}\right]\right\}. \quad (43)$$

When $B=0$ we recover the previously studied case of "intermediate" inflation [6,7] if $\lambda^{-1}-2n>0$, that is, with $n\lambda<\frac{1}{2}$ when n and λ are positive. This is also the asymptotic behavior of the general solution (43) a large t. The form of the potential that gives the evolution (43) is

$$V(\phi)=-\frac{1}{2}\lambda^2A^2n^2\left[\frac{\phi}{A}\right]^{2(n-1)/n}\left[\left[\frac{\phi}{A}\right]^{1/n}-B\right]^{(2\lambda-2)/\lambda}$$
$$+\frac{3}{4}A^4\lambda^2n^4\left[\frac{\left[\left[\frac{\phi}{A}\right]^{1/n}-B\right]^{(2n-\lambda^{-1})}}{2n-\lambda^{-1}}+\cdots+B^{2n-2}\frac{\left[\left[\frac{\phi}{A}\right]^{1/n}-B\right]^{2-\lambda^{-1}}}{2-\lambda^{-1}}\right]^2. \quad (44)$$

The asymptotic form of this function is an inverse power law in ϕ as $\phi\to\infty$, just as one expects from (15)–(17), since the behavior of the scale factor is that of intermediate inflation. Since the general solution (39), (43), and (44) is algebraically cumbersome, it is instructive to display the explicit form of the scale factor in two particular cases ($\lambda^{-1}>2n$).

(i) $n=\frac{3}{2}$:

$$a(t)\propto\exp\left[-\frac{9A^2}{8}\left\{\frac{t^{3\lambda}}{3(3-\lambda^{-1})}+\frac{Bt^{2\lambda}}{2(2-\lambda^{-1})}\right\}\right]. \quad (45)$$

(ii) $n=2$:

$$a(t)\propto\exp\left[-2A^2\left\{\frac{4^{4\lambda}}{4(4-\lambda^{-1})}+\frac{2Bt^{3\lambda}}{3(3-\lambda^{-1})}+l\frac{B^2t^{2\lambda}}{2(2-\lambda^{-1})}\right\}\right]. \quad (46)$$

IV. SCALAR AND TENSOR FLUCTUATIONS

As we remarked in the Introduction, one of the most interesting features of inflationary universe models is the spectral slope of scalar and tensor fluctuations and their contribution to the measured COBE signal. In this spirit we can draw some conclusions about the density and gravitational wave perturbations arising in the slow-roll regime, at large ϕ, of the two new classes of inflationary expansion given in Sec. III above. The second class of solutions, [Eqs. (39)–(44)] approaches the intermediate inflationary model (15)–(17) as $\phi\to\infty$. This model was discussed in detail by Barrow and Liddle [7], and their conclusions will apply to (39)–(44) with only small modifications. The first class of new solutions, [Eqs. (18)–(24)] with asymptotic slow-roll behavior given by (35)–(38) is not contained in previous studies and requires a separate analysis. Following the analysis of the intermediate inflationary models in [7], we introduce the slow-roll parameters ϵ and η:

$$\epsilon\equiv\frac{3\dot\phi^2/2}{V+\frac{1}{2}\dot\phi^2}=\frac{2H'^2}{H^2}, \quad (47)$$

$$\eta\equiv\frac{-\ddot\phi}{H\dot\phi}=\frac{2H''}{H}, \quad (48)$$

where $H'\equiv dH/d\phi$. From (18)–(24) we have

$$\frac{H'}{H}\sim\frac{2(n-1)}{n\phi},\quad\frac{H''}{H}\sim\frac{2(n-1)(n-2)}{n^2\phi^2}. \quad (49)$$

Hence, asymptotically, as $\phi\to\infty$,

$$\eta\sim\frac{4(n-1)(n-2)}{n^2\phi^2} \quad (50)$$

and

$$\epsilon\sim\frac{8(n-1)^2}{n^2\phi^2}. \quad (51)$$

The scalar and gravitational wave spectral indices n_s and n_g are given to first order in the slow-roll parameters ϵ and η as [7]

$$n_s=1-4\epsilon+2\eta, \quad (52)$$

$$n_g=-2\epsilon, \quad (53)$$

where ϵ and η are to be evaluated when the wave number of interest leaves the horizon during inflation. Hence, from (50)–(51), we obtain

$$n_s-1\sim 8(1-n)(3n-2)n^{-2}\phi^{-2}<0, \quad (54)$$

$$n_g\sim-16(n-1)^2n^{-2}\phi^{-2}. \quad (55)$$

Since the inflationary solution which possesses the slow-roll form (35)–(38) requires n to be an integer and $n>1$, we always have $n_s<1$, and so a COBE-normalized spectrum has reduced small-scale power compared to a similarly normalized scale-invariant spectrum. The relative contribution made by the tensor and scalar modes to the microwave background signal on a scale corresponding to the lth multipole of the spherical harmonic expansion of the temperature anisotropy is [7]

$$R_l(\text{tensor})/(\text{scalar}) \sim 12.4\epsilon \sim 99.2(n-1)^2 n^{-2} \phi^{-2} \tag{56}$$

or, equivalently, eliminating ϕ using (54),

$$R_l \sim 12.4(1-n)(n_s-1)(3n-2)^{-1} . \tag{57}$$

We see that for large ϕ the gravitational wave contribution is small (recall that $3n-2$ cannot be smaller than 1 in this solution).

V. DISCUSSION

By combining the results of this paper with our knowledge of other inflationary models, we can produce a tabulation of the results of scalar-field evolution down a potential $V(\phi)$, which is shown in Table I.

We note that the class of potentials shown in the last row, with the form $\phi^P \exp[-\lambda\phi^Q]$, only includes those with $P=4(1-Q)$. An analysis of (2)–(4) in the slow-roll approximation highlights why these special cases arise and why the range of M and N values are so restricted in the exact solutions. In the slow-roll approximation, the Friedmann equations (2)–(4) reduce to

$$3H^2 = V , \tag{58}$$

$$3H\dot{\phi} = -V' . \tag{59}$$

If we choose a potential of the form

$$V(\phi) = V_0 \phi^P \exp[-\lambda\phi^Q] , \quad V_0, P, Q > 0, \ \lambda > 0, \ \text{const} , \tag{60}$$

then, since $V' \sim -\lambda\phi^{P+Q-1} V_0 \exp[-\lambda\phi^Q]$ as $\phi \to \infty$, (58) and (59) lead to

$$\int \phi^{1-Q-P/2} \exp[\tfrac{1}{2}\lambda\phi^Q] d\phi = \lambda(V_0/3)^{1/2}(t+t_0) , \tag{61}$$

where t_0 is an integration constant. The integral in (61) exists in a simple closed form when $1-Q-\frac{1}{2}P = Q-1$, that is, for

$$P = 4(1-Q) , \tag{62}$$

in which case (61) integrates to give

$$2\exp[\tfrac{1}{2}\lambda\phi^Q] = \lambda^2 Q (V_0/3)^{1/2}(t+t_0) . \tag{63}$$

At large t, we obtain $\phi^Q \sim (2/\lambda)\ln t$. Using (58) and (61), we get the evolution of the scale factor as

$$a(t) \propto \exp\{Q(2-Q)^{-1}(\ln t)^{(2-Q)/Q}\} . \tag{64}$$

We recognize (63) and (64) as the asymptotic forms (35)–(38) for our first class of exact solutions which generate the entry in the last row of Table I after we substitute $N^{-1}=Q$. We note the restriction $N > \frac{1}{2}$ on the entry in Table I corresponds to the requirement that $Q<2$ in (64), so that the Universe expands rather than contracts as $t \to \infty$.

Finally, we indicate what occurs in the slow-roll approximation when (62) does not hold. If we put $v = \exp(\frac{1}{2}\lambda x^q)$, then (61) becomes

$$(2/\lambda)^{q+1} \int (\ln v)^q dv = \lambda(V_0/3)^{1/2}(t+t_0) , \tag{65}$$

where

$$q \equiv (4-4Q-P)/2Q . \tag{66}$$

When q is a positive integer, we can express the integral in (65) as a finite series; thus,

$$\left[\frac{2}{\lambda}\right]^{q+1} \exp\{\tfrac{1}{2}\lambda\phi^Q\} \sum_{k=0}^{k=q} (-1)^k \begin{bmatrix} q \\ k \end{bmatrix} k!(\tfrac{1}{2}\lambda\phi^Q)^{q-k} = \lambda(V_0/3)^{1/2}(t+t_0) . \tag{67}$$

The case considered above, in which condition (62) holds, corresponds to the simplest situation with $q=0$, and so the right-hand side of (67) is $2\lambda^{-1}\exp\{\frac{1}{2}\lambda\phi^Q\}$. This enables ϕ to be expressed as an explicit function of t and so $H(t)$, and hence $a(t)$, can be found from (58). But when $q \neq 0$, although t can be expressed as an explicit function of ϕ, this function cannot be inverted. Consequently, we cannot find $H(t)$ and $a(t)$ explicitly in the cases where q is a positive integer. The following conclusions can be drawn. Since $2Q(q+2)=4-N$ and M and q are both positive, our remarks are confined to the case with $N<4$.

For a given q, the dominant term on the left-hand side of (67) for large ϕ will be contributed by the $k=0$ term in the series. Hence, asymptotically,

$$2\lambda^{-1}\exp\{\tfrac{1}{2}\lambda\phi^Q\}\phi^{Qq} \sim \lambda(V_0/3)^{1/2}(t+t_0) , \tag{68}$$

with q give by (66). We see that $Q>0$ ensures that $\phi \to \infty$ as $t \to \infty$. Using (58), we see that

$$H = 2\lambda^{-2}(t+t_0)^{-1}\phi^{(2P+qQ)/2} . \tag{69}$$

If $P=q=0$, then we get power-law inflation with $V \propto \exp\{-\lambda\phi\}$ as long as $\lambda^2<2$. Since $\frac{1}{2}P+qQ \geq 0$, the inclusion of the last term in (69) serves to increase the inflation over and above any which obtains in the power-law case. However, we note that for $P \neq 0 \neq Q$ the $t(\phi)$ relation [Eq. (68)] cannot be analytically inverted to yield $\phi(t)$, and so we cannot obtain $H(t)$ explicitly from (58).

TABLE I. Results of scalar field evolution down a potential $V(\phi)$.

Potential $V(\phi)$	Scale factor $a(t)$	Inflation
V_0=const	$\exp\{t(V_0/3)^{1/2}\}$	yes
$\exp(-\lambda\phi)$	t^{2/λ^2}	if $\lambda^2<2$
$\phi^N\exp(-\lambda\phi^M)$, $M>2$	$t^{1/3}$	no
ϕ^{2M}, $M\in\mathbb{Z}^+$	$t^{4M/3(M+1)}$	no (oscillates)
ϕ^{-N}, $N>0$	$\exp\{At^{4/(4+N)}\}$	yes
$\phi^{4(N-1)/N}\exp[-2\phi^{1/N}]$	$\exp\{\frac{1}{2}N^2(2N-1)^{-1}(\ln t)^{2N-1}\}$	if $N>\frac{1}{2}$

We can also examine the $P \neq 0 \neq Q$ cases in terms of the slow-roll parameters ϵ and η defined by Eqs. (47) and (48). For a potential of the form (60), we have

$$\epsilon = \tfrac{1}{2}P^2\phi^{-2} + \tfrac{1}{2}\lambda^2 Q^2 \phi^{2Q-2} - \lambda P Q \phi^{Q-2} , \tag{70}$$

$$\eta = -\tfrac{1}{2}P^2\phi^{-2} + \tfrac{1}{2}\lambda^2 Q^2 \phi^{2Q-2} - \lambda Q \phi^{Q-2}(P+Q-1) . \tag{71}$$

The slow-roll approximation corresponds to $\epsilon < 1$ and $\eta < 1$. Inflation will end when $\epsilon = 1$. The solution (68) shows that $t \to \infty$ corresponds to $\phi \to \infty$, and in this limit (70) and (71) give $\epsilon \to 0$ and $\eta \to 0$ as required for the existence of slow-roll inflation as long as $Q < 1$. This confirms our earlier remarks following Eq. (14).

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