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Helicity-coupling amplitudes in tensor formalism

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The decay of a particle with spin to two other particles with spin is best described in the helicity formalism. It is the purpose of this paper to show that the helicity-coupling amplitudes, which appear in the angular distributions, can be expressed in terms of the covariant amplitudes in the tensor formalism. This allows for a systematic derivation for the energy dependence of the helicity-coupling amplitudes within the framework of the tensor formalism. The concept of pure intrinsic spin has been developed in the tensor formalism, for decays involving two spins in the final state, in order to bring the formalism to a form comparable to the standard ℓS-coupling scheme. Among several examples worked out in this paper are those involving spin-0, spin-1, or spin-2 states in the initial and the final systems and, in particular, a spin-1 state coupling to two spin-1 final states. The latter example is then specialized to the J/ψ radiative decays into a pseudovector ($J^{PC} = 1^{++}$) or into a vector ($J^{PC} = 1^{-+}$) which is exotic, i.e., not a quarkonium. Orbital angular momenta up to $\ell = 4$ have been included in the examples; in particular, the decay $\pi_2(1670) \to f_2(1270) + \pi$, which involves three ℓ's (ℓ =0, 2, and 4), is shown to exhibit a complicated energy-dependent form for the helicity-coupling amplitude.

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I. INTRODUCTION

The purpose of this paper is to demonstrate a new approach to the description of the decay of particles with spin in which the covariant tensor formalism is incorporated inherently into the helicity formalism.

The helicity formalism was originally developed by Jacob and Wick [1] in their seminal paper in 1959. For a covariant description of particles with spin, one may consult the paper of Rarita and Schwinger [2] and for the projection operators of arbitrary spin those of Fronsdal [3] and Behrends and Fronsdal [4]. The description of the tensor wave functions for particles with spin used in this paper follows the method originally proposed by Auvil and Brehm [5]. The reader may consult the paper by Chung [6] for a general exposition of the spin formalisms.

The description of particle decays into two other particles is most compactly given within the helicity formalism in which the rotation functions $D^J_{mm'}$ are used for the angular dependence. One deals with, in general, short-lived states in both the initial and final states, i.e., with the masses which are described by Breit-Wigner shapes. One therefore needs to keep careful track of the masses, energies and the momenta of all the states involved. The covariant tensor formalism gives naturally this "energy" dependence satisfying the requirement of Lorentz invariance.

It is shown that the helicity-coupling amplitudes may be expressed compactly in terms of the covariant decay amplitudes, provided the four-vectors and the z components of spin for both the initial and the final states are all defined along the direction of the decay products in the parent rest frame. In this way the tensor formalism, which gives all the energy dependence in the problem, can be incorporated into the helicity formalism for the decay process. Different "strengths" of the two different approaches, those of helicity and tensor formalisms, can therefore be combined into a single, coherent formulation. The philosophy one has adopted here is that a general decay amplitude can be expressed as a product of a complex constant, to be determined by experiments, and the covariant amplitude resulting from the Rarita-Schwinger formalism—this is a model, to be sure, but a model that incorporates Lorentz invariance as its sole tenet.

The helicity-coupling amplitudes have a well-known expansion in terms of the ℓS-coupling amplitudes. In a traditional approach, each ℓS-coupling amplitude is given an r^ℓ dependence where r is the *relative* momentum of the two decay products in the parent rest frame. The formulation proposed in this paper leads to a modification in which an ℓS-coupling amplitude is a polynomial in E/m of order up to the spin for either or both decay products, where m is the mass of a final particle and E its energy in the parent rest frame. It should be noted that the factor E/m tends to infinity as $E \to \infty$ or $m \to 0$. However, the helicity-coupling amplitude with its helicity equal to the spin does not contain the factor E/m, and this is the only one that survives for the case in which a decay product is massless. If a covariant decay amplitude requires the presence of a totally antisymmetric rank-4 tensor, one finds in addition that it contains an overall factor of the mass of the parent particle.

It should be noted that the approach of this paper is

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fundamentally different from that of the nonrelativistic Zemach formalism [7]. In the Zemach formalism, all the tensors are evaluated in their respective rest frames, and therefore the energy dependence of E/m never appears in the decay amplitudes. The purpose of this paper is to emphasize that the E/m factor is a necessary consequence of the relativistic wave functions of the decay products evaluated in the parent rest frame—and therefore cannot be neglected.

The ℓS-coupling amplitude has no meaning if one of the decay products is a photon, as in the J/ψ radiative decay. In this case one may start out with the simplest possible covariant amplitude incorporating the gauge invariance and then derive the energy dependence of the helicity-coupling amplitudes. This approach is to be preferred over the straightforward tensor approach, as the angular dependence is much more efficiently given in the helicity formalism. In an example in which a pseudovector (or more generally the states of unnatural spin-parity series) is produced in the radiative $J\psi$ decay, it is shown that there exist three covariant amplitudes which, while not independent, nevertheless contribute independently to the two helicity-coupling amplitudes in the problem.

The next two sections deal with the basic helicity and tensor formalisms and the representation of the helicity-coupling amplitudes in the tensor formalism. In addition, it is here that for the first time the concept is developed of the total intrinsic spin ($S = 0, 1, 2$) as a rank-2 tensor coupling to two spin-1 polarization four-vectors. The generalization is clear: if a decay involves a spin-2 and a spin-1 particle in the final state, the total intrinsic spin ($S = 1, 2, 3$) should be represented by a rank-3 tensor.

The remaining sections are devoted to a few basic but representative examples of two-body decay processes: $1 \to 1+0$, $1 \to 1+1$, $2 \to 1+0$, $1 \to 2+0$, $2 \to 2+0$, and $0 \to 2+0$. Most of the examples of the particle decays are taken from the recent compilation [8] by the Particle Data Group; however, other examples have been culled from the $\bar{p}p$ annihilations at rest. Clearly, these examples are meant for experimental physicists working on hadron spectroscopy. There are two nontrivial examples; the first concerns the amplitudes for $J/\psi \to f_1(1420)+\gamma$, which have been given in terms of three complex parameters even though there exist only two independent amplitudes, and the second deals with $\pi_2(1670) \to f_2(1270)+\pi$ in which the factor E/m appears as a polynomial in the expression for the helicity-coupling amplitudes.

The conclusions are given in Sec. X, and the Appendices deal with the problem of finding relationships among different covariant amplitudes which appear naturally in certain decay processes.

II. HELICITY-COUPLING AMPLITUDES

Consider a state with spin(parity)=$J(\eta_J)$ decaying into two states with $s(\eta_s)$ and $\sigma(\eta_\sigma)$. The decay amplitudes are given, in the rest frame of J, by

$$\begin{aligned}\mathcal{M}^J_{\lambda\nu}(\vartheta,\varphi;M) &\propto \langle\vartheta,\varphi,\lambda\nu|JM\lambda\nu\rangle\langle JM\lambda\nu|\ \mathcal{M}\ |JM\rangle\\ &\propto D^{J*}_{M\delta}(\varphi,\vartheta,0)F^J_{\lambda\nu}\end{aligned} \tag{1}$$

where $\mathcal{M}$ is the invariant operator for the decay, and λ and ν are the helicities of the two final-state particles s and σ with $\delta = \lambda - \nu$. The general decay amplitude is

$$\begin{aligned}\mathcal{M}^J(\vartheta,\varphi;M) &= \sum_{\lambda\nu}\mathcal{M}^J_{\lambda\nu}(\vartheta,\varphi;M)\\ &\propto \sum_{\lambda\nu} D^{J*}_{M\delta}(\varphi,\vartheta,0)F^J_{\lambda\nu}.\end{aligned} \tag{2}$$

The M is the z component of the spin J in a coordinate system fixed in the production system. The helicities λ and ν are rotational invariants by definition. For the decay amplitude $\mathcal{M}$ one has adopted a system of notation in which rotational invariants are given as indices while the noninvariants are listed as arguments. The direction of the break-up momentum of the decaying particle s is given by the angles ϑ and φ in the J rest frame. Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be the coordinate system fixed in the J rest frame. It is important to recognize, for applications to sequential decays, the exact nature of the body-fixed (helicity) coordinate system implied by the arguments of the D function given above. Let $\hat{\mathbf{x}}_h$, $\hat{\mathbf{y}}_h$, and $\hat{\mathbf{z}}_h$ be the helicity coordinate system fixed by the s decay. Then by definition $\hat{\mathbf{z}}_h$ describes the direction of the s in the J rest frame and the y axis is given by $\hat{\mathbf{y}}_h \propto \hat{\mathbf{z}} \times \hat{\mathbf{z}}_h$ and $\hat{\mathbf{x}}_h = \hat{\mathbf{y}}_h \times \hat{\mathbf{z}}_h$.

The helicity-coupling amplitude F^J given by

$$F^J_{\lambda\nu}\ \propto\ \langle JM\lambda\nu|\ \mathcal{M}\ |JM\rangle \tag{3}$$

is a rotational invariant. Parity conservation in the decay leads to the relationship

$$F^J_{\lambda\nu} = \eta_J\eta_s\eta_\sigma(-)^{J-s-\sigma}F^J_{-\lambda-\nu} \tag{4}$$

while, if the decay products s and σ are identical, the additional relationship

$$F^J_{\lambda\nu} = (-)^J F^J_{\nu\lambda} \tag{5}$$

holds for both integer and half-integer spins.

If the decay particles are both massive, the decay amplitude may be given in the ℓS-coupling scheme:

$$\begin{aligned}\mathcal{M}^J_{\ell S}(\vartheta,\varphi;m_1m_2M) &\propto \sum_{mm_s}\langle\vartheta,\varphi,m_1m_2|\ell mSm_s\rangle\langle\ell mSm_s|\ \mathcal{M}\ |JM\rangle\\ &\propto G^J_{\ell S}\ (sm_1\,\sigma m_2|Sm_s)\sum_m\ (\ell m\,Sm_s|JM)\ Y^\ell_m(\vartheta,\varphi)\end{aligned} \tag{6}$$

where $G^J_{\ell S}$ is the ℓS-coupling amplitude given by

$$\langle\ell mSm_s|\ \mathcal{M}\ |JM\rangle\ =\ (\ell m\,Sm_s|JM)\ G^J_{\ell S}. \tag{7}$$

It should be noted that m_1 and m_2 are measured in the canonical rest systems (reached by pure timelike Lorentz transformations from the J rest frame) of the two decay products. The $(\ell_1 m_1\, \ell_2 m_e|\ell_3 m_3)$ stands for the usual Clebsch-Gordan coefficients. The decay amplitude in the ℓS-coupling scheme contains two extra variables m_1 and m_2 that are not rotational invariants [compare with (1)]. The general decay amplitude is

$$\begin{aligned}\mathcal{M}^J(\vartheta,\varphi;M) &= \sum_{\ell S}\sum_{m_1 m_2} \mathcal{M}^J_{\ell S}(\vartheta,\varphi;m_1 m_2 M) \\ &\propto \sum_{\ell S}\sum_{m_1 m_2} G^J_{\ell S}\ (s m_1\, \sigma m_2|S m_s) \sum_m (\ell m\, S m_s|JM)\ Y^\ell_m(\vartheta,\varphi). \end{aligned} \tag{8}$$

It is common practice to give the amplitudes $G^J_{\ell S}$ a momentum dependence

$$G^J_{\ell S} \propto r^\ell \tag{9}$$

where $r = |\mathbf{r}|$ and $\mathbf{r}$ is the relative momentum between the two decay products in the J rest frame. This factor is often replaced with the Blatt-Weisskopf barrier factor [9] in partial-wave analyses. The barrier factors $B_\ell(p)$ are collected below for ease of reference:

$$\begin{aligned} B_0(p) &= 1, \\ B_1(p) &= \sqrt{\frac{2z}{z+1}}, \\ B_2(p) &= \sqrt{\frac{13z^2}{(z-3)^2+9z}}, \\ B_3(p) &= \sqrt{\frac{277z^3}{z(z-15)^2+9(2z-5)^2}}, \\ B_4(p) &= \sqrt{\frac{12746z^4}{(z^2-45z+105)^2+25z(2z-21)^2}}, \end{aligned} \tag{10}$$

where $z = (p/p_r)^2$ and p_r is a "scale" parameter in the problem which is presumably close to 0.1973 GeV/c, corresponding to the length of 1 fm. Note that one has adopted a normalization such that $B_\ell(p) = 1$ for $z = 1$.

It is instructive to write down (1) and (6) again for $\vartheta = \varphi = 0$

$$\begin{aligned} \mathcal{M}^J_{\lambda\nu}(0,0;\delta) &\propto F^J_{\lambda\nu}, \\ \mathcal{M}^J_{\ell S}(0,0;m_1 m_2 m_s) &\propto G^J_{\ell S}\ (s m_1\, \sigma m_2|S m_s) \\ &\quad \times (\ell 0\, S m_s|J m_s). \end{aligned} \tag{11}$$

By setting $m_1 = \lambda$, $m_2 = -\nu$ and $m_s = \delta$ in the latter expression above, one finds that the helicity-coupling amplitudes F^J are related to the ℓS-coupling amplitudes $G^J_{\ell S}$ via

$$F^J_{\lambda\nu} = \sum_{\ell S} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0\, S\delta|J\delta)(s\lambda\, \sigma\, {-\nu}|S\delta)\ G^J_{\ell S} \tag{12}$$

where the coupling amplitudes have been given the normalization

$$\sum_{\ell S} |G^J_{\ell S}|^2 = \sum_{\lambda\nu} |F^J_{\lambda\nu}|^2. \tag{13}$$

The formula (12) for the helicity-coupling amplitudes results from the usual scheme of coupling of the angular momenta but with the z axis chosen along the break-up momentum of the decay product s. Note that the orbital angular momentum ℓ has the zero z component in this case and the particle σ has the z component $-\nu$.

The decay amplitude (1) is simply given by the helicity-coupling amplitude itself if one sets $\vartheta = \varphi = 0$, as shown in (11). It is obvious now that the helicity-coupling amplitudes can be derived from the tensor formalism by restricting oneself to the four-vectors defined along the z axis. Let p, q, and k be the four-momenta for the states J, s, and σ with masses W, m, and μ

$$\begin{aligned} p^\alpha &= (p_0,\mathbf{p}), \quad p^2 = W^2, \quad q^\alpha = (q_0,\mathbf{q}), \\ q^2 &= m^2, \qquad k^\alpha = (k_0,\mathbf{k}), \quad k^2 = \mu^2, \end{aligned} \tag{14}$$

and let $r = |\mathbf{q}| - |\mathbf{k}|$ be the break-up four-momentum. Using the usual Lorentz metric $g_{\alpha\beta}$, one has

$$p_\alpha = g_{\alpha\beta}\, p^\beta = (p_0, -\mathbf{p}\ ) \tag{15}$$

and similarly for the other four-vectors. One uses, as usual, the notations p, q, k, and r to stand for *both* the four-momenta and the magnitudes of the three-momenta.

One may now write an explicitly covariant expression (Lorentz scalar) for the helicity-coupling amplitudes

$$\begin{aligned} F^J_{\lambda\nu} &= \sum_\alpha g_\alpha A_\alpha(\lambda\nu) \\ &\equiv \sum_\alpha g_\alpha A_\alpha\{p^n, r^\ell, \omega(\lambda), \varepsilon(-\nu), \phi^*(\delta)\} \end{aligned} \tag{16}$$

as a sum of functions A_α of five variables, provided all the four-momenta are defined along the z axis (i.e., no x and y components). As the momenta involved are all parallel with the z axis, this formula merely gives the momentum dependence of the helicity-coupling amplitudes but no angular dependence, as this is already contained in the D function in the expression (1). The variables α stand for the set $\{\ell, S\}$, and the constants g_α are the analogue of the $G^J_{\ell S}$ in (12). But the expression (16) is more general, since the expansion in α can be applied equally well even when ℓ is not well defined as in, for example, radiative J/ψ decays.

The covariant function A_α depends on p and r as well as the momentum-space wave functions (or tensors) $\phi^*(\delta)$, $\omega(\lambda)$ and $\varepsilon(-\nu)$ for the particles J, s, and σ, where δ, λ, and $-\nu$ are the z components of spin as defined before. Note that the complex conjugate of the J wave function appears in the above formula: it represents the initial state while those of s and σ correspond to the final states. As shown by examples in later sections, one may set $n=1$ or $n=0$ without loss of generality, depending on the intrinsic parities involved. In other words, the four-vector p is used in the covariant amplitudes at most once, if necessary, in order to satisfy the requirement of parity conservation. The covariant function A_α can depend on any multiples (up to ℓ) of r, reflecting orbital angular momenta allowed in the decay.

The expressions (9) and (12) show that the helicity-coupling amplitudes F^J depend only on r in the usual prescription. The formula (16) implies that the energy dependence on F^J is more complex, with the energies q_0 and k_0 appearing in the expression in addition to r, as the wave functions for s and σ are evaluated in the J rest frame, not in their respective rest frames. One may also note that, in addition, the masses W, m, and μ appear in the expression for F^J as a consequence of the covariant description of the decay amplitudes.

III. POLARIZATION FOUR-VECTORS

The polarization four-vector appropriate for the particle J is given first in the following; those for the decay products s and σ are of course similar.

The polarization four-vectors are in reality the spin-1 wave functions embedded in the momentum space. Thus, under a general rotation R, the four-vectors transform according to

$$\phi^\alpha(m) \to \sum_{m'} \phi^\alpha(m') D^1_{m'm}(R) \tag{17}$$

where m and m' are the z component of spin in an arbitrary coordinate system. In order for this to hold, the rest-frame wave functions should be given by

$$\begin{aligned} \boldsymbol{\phi}(\pm) &= \mp\tfrac{1}{\sqrt{2}}\ (1,\ \pm i,\ 0), \\ \boldsymbol{\phi}(0) &= \quad\ (0,\ \ 0,\ 1), \end{aligned} \tag{18}$$

and the time component is defined to be zero in the rest frame. These polarization four-vectors satisfy

$$\begin{aligned} p_\alpha \phi^\alpha(m) &= 0, \\ \phi^*_\alpha(m)\phi^\alpha(m') &= -\delta_{mm'}, \\ \sum_m \phi_\alpha(m)\phi^*_\beta(m) &= \tilde{g}_{\alpha\beta}, \end{aligned} \tag{19}$$

where

$$\tilde{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta}(W) = -g_{\alpha\beta} + \frac{p_\alpha p_\beta}{W^2}. \tag{20}$$

The last equation of (19) is the usual projection operator for spin-1 states.

The property (17) implies that the spin-2 wave functions should be written

$$\phi^{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1 1m_2|2m)\phi^\alpha(m_1)\phi^\beta(m_2) \tag{21}$$

which transform, under the rotation R:

$$\phi^{\alpha\beta}(m) \to \sum_{m'} \phi^{\alpha\beta}(m') D^2_{m'm}(R). \tag{22}$$

These spin-2 tensors satisfy

$$\begin{aligned} p^\alpha \phi_{\alpha\beta}(m) &= 0, \\ \phi_{\alpha\beta} &= \phi_{\beta\alpha}, \\ g^{\alpha\beta}\phi_{\alpha\beta} &= 0, \end{aligned} \tag{23}$$

which state that the spin-tensors are orthogonal to the momentum, symmetric, and traceless. These tensors satisfy, in addition,

$$\phi^*_{\alpha\beta}(m)\phi^{\alpha\beta}(m') = \delta_{mm'}, \tag{24}$$

$$\sum_m \phi_{\alpha\beta}(m)\phi^*_{\gamma\delta}(m) = \frac{1}{2}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} + \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}) - \frac{1}{3}\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta}. \tag{25}$$

One may note that the latter is the spin-2 projection operator.

In order to find connection of the tensor formalism with that of the ℓS-coupling scheme, one needs to develop the concept of total intrinsic spin S formed out of the s and σ polarization four-vectors and that of the pure orbital angular momentum ℓ built out of r. The key to a solution is contained in (6). One sees that this formula involves projection operators

$$\begin{aligned} P^{(S)} &= \sum_{m_s} |Sm_s\rangle\langle Sm_s|, \\ P^{(\ell)} &= \sum_m |\ell m\rangle\langle \ell m|. \end{aligned} \tag{26}$$

Consider now a wave function $\chi(m)$ which is to form the basis for constructing the ket state $|Sm_s\rangle$. One demands that this wave function have a zero time component in the J rest frame (very similar to ϕ) and the space components be given as in (18). The goal is to form rank-2 tensors, when coupled to ω and ε, project out pure spin S.

The desired rank-2 wave functions for spin 0, spin 1, and spin 2 are

$$\begin{aligned} \chi^{(0)}_{\alpha\beta}(0) &= \sum_{m_1 m_2} (1m_1 1m_2|00)\chi_\alpha(m_1)\chi_\beta(m_2), \\ \chi^{(1)}_{\alpha\beta}(m) &= \sum_{m_1 m_2} (1m_1 1m_2|1m)\chi_\alpha(m_1)\chi_\beta(m_2), \\ \chi^{(2)}_{\alpha\beta}(m) &= \sum_{m_1 m_2} (1m_1 1m_2|2m)\chi_\alpha(m_1)\chi_\beta(m_2), \end{aligned} \tag{27}$$

as analogues of the ket states $|Sm_s\rangle$. It is seen that spin-1 tensors are antisymmetric while the spin-2 tensors are symmetric and traceless. The projection operator [see (25) for spin 2] given by

$$P^{(S)}_{\alpha\beta\gamma\delta} = \sum_m \chi^{(S)}_{\alpha\beta}(m)\chi^{(S)*}_{\gamma\delta}(m) \tag{28}$$

leads to, after some algebra,

$$P^{(0)}_{\alpha\beta\gamma\delta} = \frac{1}{3}\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta},$$
$$P^{(1)}_{\alpha\beta\gamma\delta} = \frac{1}{2}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}), \quad (29)$$
$$P^{(2)}_{\alpha\beta\gamma\delta} = \frac{1}{2}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} + \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}) - \frac{1}{3}\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta}.$$

These result in the following pure-spin tensors:

$$\psi^{(0)}_{\alpha\beta} = (\tilde{\omega}\cdot\varepsilon)\tilde{g}_{\alpha\beta},$$
$$\psi^{(1)}_{\alpha\beta} = \tilde{\omega}_\alpha\tilde{\varepsilon}_\beta - \tilde{\varepsilon}_\alpha\tilde{\omega}_\beta, \quad (30)$$
$$\psi^{(2)}_{\alpha\beta} = \tilde{\omega}_\alpha\tilde{\varepsilon}_\beta + \tilde{\varepsilon}_\alpha\tilde{\omega}_\beta - \frac{2}{3}(\tilde{\omega}\cdot\varepsilon)\tilde{g}_{\alpha\beta},$$

where

$$(\tilde{\omega}\cdot\varepsilon) = (\tilde{\omega}\cdot\tilde{\varepsilon}) = (\omega\cdot\tilde{\varepsilon}), \quad (31)$$

$$\tilde{\omega}_\alpha = \tilde{g}_{\alpha\beta}\omega^\beta \quad (32)$$
$$= -\omega_\alpha + \frac{(p\cdot\omega)}{W^2}p_\alpha,$$

and similarly for $\tilde{\varepsilon}$ for any z components of spin for ω and ε. Note that there exist three independent space components in an antisymmetric tensor, while there are five independent space components in a symmetric and traceless tensor—reflecting spin states of spin-1 and spin-2 wave functions.

The analogue of the ket state $|\ell m\rangle$ may be constructed out of yet another polarization four-vector $\tau(m)$, defined to have zero time component in the J rest frame, since the orbital angular momentum is defined only in this frame. A "pure" S wave is characterized simply by the absence of $\mathbf{r}$ in the amplitude in the J rest frame. The spin-1 projection operator corresponding to $\tau(m)$ has already been given in the last equation of (19); this implies that the Lorentz metric, if required, should be replaced by the modified form of (20). The break-up four-momentum for $\ell = 1$ is then given by

$$\tilde{r}_\alpha = \tilde{g}_{\alpha\beta}r^\beta \quad (33)$$
$$= -r_\alpha + \left(\frac{m^2-\mu^2}{W^2}\right)p_\alpha$$

with *zero* time component in the J rest frame. Note that, in any frame, one has

$$(\tilde{r}\cdot\tilde{r}) = (r\cdot\tilde{r}) = (\mathbf{r}\cdot\mathbf{r}). \quad (34)$$

The spin-2 projection operator resulting from $\tau(m)$ has also been worked out—see (25). A pure D-wave orbital angular momentum is thus represented by a rank-2 tensor:

$$\tilde{t}^{(2)}_{\alpha\beta} = \tilde{r}_\alpha\tilde{r}_\beta - \frac{1}{3}(r\cdot\tilde{r})\tilde{g}_{\alpha\beta}. \quad (35)$$

Similar technique leads to a rank-3 tensor for a pure F-wave:

$$\tilde{t}^{(3)}_{\alpha\beta\gamma} = \tilde{r}_\alpha\tilde{r}_\beta\tilde{r}_\gamma - \frac{1}{5}(r\cdot\tilde{r})(\tilde{g}_{\alpha\beta}\tilde{r}_\gamma + \tilde{g}_{\beta\gamma}\tilde{r}_\alpha + \tilde{g}_{\gamma\alpha}\tilde{r}_\beta). \quad (36)$$

A rank-4 tensor for a pure G-wave is

$$\tilde{t}^{(4)}_{\alpha\beta\gamma\delta} = \tilde{r}_\alpha\tilde{r}_\beta\tilde{r}_\gamma\tilde{r}_\delta - \frac{1}{7}(r\cdot\tilde{r})(\tilde{g}_{\alpha\beta}\tilde{r}_\gamma\tilde{r}_\delta + \tilde{g}_{\beta\gamma}\tilde{r}_\alpha\tilde{r}_\delta + \tilde{g}_{\gamma\alpha}\tilde{r}_\beta\tilde{r}_\delta + \tilde{g}_{\alpha\delta}\tilde{r}_\beta\tilde{r}_\gamma + \tilde{g}_{\beta\delta}\tilde{r}_\gamma\tilde{r}_\alpha + \tilde{g}_{\gamma\delta}\tilde{r}_\alpha\tilde{r}_\beta)$$
$$+\frac{1}{35}(r\cdot\tilde{r})^2(\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta} + \tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\delta} + \tilde{g}_{\gamma\alpha}\tilde{g}_{\beta\delta}). \quad (37)$$

It is obvious that use of these tensors would lead to differences in the helicity-coupling amplitudes when compared to those resulting from the simplest covariant amplitudes. They are pointed out with a number of examples in the following sections.

Both the spin-1 and spin-2 wave functions are generic, in the sense that they describe both positive and negative intrinsic-parity states. One merely needs to know how these wave functions transform under parity operation. For the purpose, the following notations and conventions are useful: Under parity operation, the momenta transform according to

$$p^\alpha \to \bar{p}^{\ \alpha} = (p_0, -\mathbf{p}\ ), \qquad \bar{p}_\alpha = g_{\alpha\beta}\bar{p}^{\ \beta} = (p_0, \mathbf{p}\ ).$$

Let a, b, c, and d be any four-vector. Using the shorthand notations

$$(a\cdot b) = [a\cdot b] = g_{\alpha\beta}a^\alpha b^\beta,$$
$$(abcd) = [abcd] = \epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^{\ \delta} \quad (38)$$

one finds

$$(a\cdot b) = (\bar{a}\cdot\bar{b}\ ), \qquad (abcd) = -(\bar{a}\bar{b}\bar{c}\bar{d}\ ). \quad (39)$$

Define e via

$$e_\alpha = \epsilon_{\alpha\beta\gamma\delta}b^\beta c^\gamma d^{\ \delta}$$

which transforms under parity operation as a pseudovector, i.e.,

$$e^\alpha \to -\bar{e}^{\ \alpha} = (-e_0, \mathbf{e}\ ).$$

As the spin-1 wave functions should be pseudovectors, they transform under parity in exactly the same way as the four-vector e, i.e.,

$$\phi^\alpha(m) \to -\bar{\phi}^{\ \alpha}(m). \quad (40)$$

In another words, the time component changes sign, while the space components remain the same. Note the following transformation properties of the Lorentz scalars under parity operation:

$$(a \cdot \phi) \to -(\bar{a} \cdot \bar{\phi}) = -(a \cdot \phi),$$
$$(e \cdot \phi) \to +(\bar{e} \cdot \bar{\phi}) = +(e \cdot \phi)$$

where a and e are any vector and pseudovectors. From (40), one sees that the spin-2 wave functions transform, under parity, according to

$$\phi^{\alpha\beta} \to \bar{\phi}^{\,\alpha\beta}. \tag{41}$$

Suppose now that the second decay product σ is a photon, i.e., $\mu = 0$. One may take for the photon polarization vector $\varepsilon(\pm)$ the four-vectors (18) but without the zero z component of spin [i.e., $\varepsilon(0)$ not allowed], again setting the time component to zero (Coulomb gauge). Gauge invariance requires that any Lorentz scalar one writes down should vanish with the replacement $\varepsilon \to k$. The following two four-vectors, derived from the ε, satisfy this requirement:

$$\varepsilon^{(-)}_\alpha(\pm) = \epsilon_{\alpha\beta\gamma\delta} q^\beta k^\gamma \varepsilon^\delta(\pm), \tag{42}$$

$$\varepsilon^{(+)}_\alpha(\pm) = \varepsilon_\alpha(\pm) - \frac{[q \cdot \varepsilon(\pm)]}{(q \cdot k)} k_\alpha. \tag{43}$$

Here the superscripts $(-)$ and $(+)$ denote a vector and a pseudovector, respectively. They transform, under parity, according to

$$\varepsilon^{(-)}_\alpha(\pm) \to +\bar{\varepsilon}^{(-)}_\alpha(\pm), \tag{44}$$

$$\varepsilon^{(+)}_\alpha(\pm) \to -\bar{\varepsilon}^{(+)}_\alpha(\pm). \tag{45}$$

Let p, q, and k stand for the four-momenta for the particles J, s, and σ, and let ϕ, ω and ε stand for their spin-1 polarization four-vectors. In order to calculate the helicity-coupling amplitudes F^J in the prescription given in the expression (16), it is necessary to write down all the relevant momenta and the spin-1 polarization four-vectors along the z axis, in the rest frame of the decaying particle:

$$\begin{aligned} p^\alpha &= (\;W;\; 0,\; 0,\; 0\;), \\ q^\alpha &= (\;q_0;\; 0,\; 0,\; q\;), \\ k^\alpha &= (\;k_0;\; 0,\; 0,\; -q\;), \\ r^\alpha &= (\;q_0 - k_0;\; 0,\; 0,\; 2q\;), \end{aligned} \tag{46}$$

where $W = q_0 + k_0$, $q_0 = \sqrt{m^2 + q^2}$, $k_0 = \sqrt{\mu^2 + q^2}$, and $r = 2q$.

The relevant polarization four-vectors are given by

$$\begin{aligned} \phi^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} \;(\;0;\; 1,\; \pm i,\; 0\;), \\ \phi^\alpha(0) &= \;(\;0;\; 0,\; 0,\; 1\;), \\ \omega^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} \;(\;0;\; 1,\; \pm i,\; 0\;), \\ \omega^\alpha(0) &= \left(\;\frac{q}{m};\; 0,\; 0,\; \frac{q_0}{m}\;\right), \\ \varepsilon^\alpha(\pm) &= \pm\tfrac{1}{\sqrt{2}} \;(\;0;\; 1,\; \pm i,\; 0\;), \\ \varepsilon^\alpha(0) &= \left(\;-\frac{q}{\mu};\; 0,\; 0,\; \frac{k_0}{\mu}\;\right). \end{aligned} \tag{47}$$

Note that

$$[p \cdot \phi(\lambda)] = [q \cdot \omega(\lambda)] = [k \cdot \varepsilon(\lambda)] = 0 \tag{48}$$

for any λ.

In the following sections, a number of examples involving photons will be considered. Let σ be the photon. It is instructive to note here a general result concerning the Coulomb gauge adopted in this paper. Gauge invariance requires that one needs in the covariant amplitudes the photon polarization four-vectors given in (42) and (43). One may start out with the covariant amplitudes involving massive spin-1 states; it is seen that one must then perform the replacement [see (43) and (45)]

$$\varepsilon_\alpha(\pm) \to \varepsilon_\alpha(\pm) - \frac{[q \cdot \varepsilon(\pm)]}{(q \cdot k)} k_\alpha \tag{49}$$

in order to preserve parity conservation. In the particular case in which all the four-vectors are defined along the z axis, one finds that the correction term of (49) is zero since $[q \cdot \varepsilon(\pm)] = 0$. One may therefore conclude in this case that the helicity-coupling amplitudes with photons are identical to those with massive spin-1 particles in the final state. Note that these conclusions apply as well to the case in which the s is a photon:

$$\omega_\alpha(\pm) \to \omega_\alpha(\pm) - \frac{[k \cdot \omega(\pm)]}{(k \cdot q)} q_\alpha. \tag{50}$$

Since $[k \cdot \omega(\pm)] = 0$, the correction term vanishes. Gauge invariance is therefore automatic in this particular application; note that this conclusion holds also for the vector form of $\varepsilon(\pm)$ [see (42)] or $\omega(\pm)$.

IV. SPIN 1 → SPIN 1 + SPIN 0

One starts out with simple but practical examples. Depending on the intrinsic parities involved, the treatment can be divided into two categories.

A. $\omega(1600) \to \rho + \pi$

Let J, s, and σ stand for the $\omega(1600)$, the ρ, and the π. The net intrinsic parity is given by $\eta_J \eta_s \eta_\sigma = -1$. Because of parity conservation in the decay, the helicity-coupling amplitudes are as follows: $F^J_0 = 0$ and $F^J_+ = -F^J_-$. The general decay amplitude is given by

$$\mathcal{M}^J(\vartheta, \varphi, M) \propto F^J_+ \{D^{J*}_{M+}(\varphi, \vartheta, 0) - D^{J*}_{M-}(\varphi, \vartheta, 0)\}. \tag{51}$$

There is only one allowed orbital angular momentum, i.e., $\ell = 1$. The covariant amplitude takes on the form

$$A(\lambda) = [p\; \omega(\lambda)\; r\; \phi^*(\lambda)] \tag{52}$$

where the vectors are defined along the z axis. One finds that $A(0) = 0$ and $A(\pm) = \pm Wr$, so that

$$F^J_+ = -F^J_- = gWr \tag{53}$$

where $J = 1$ and g is an arbitrary complex constant. The presence of r reflects the P-wave decay; the factor W results from the covariant description of the decay amplitude. Note the treatment given here applies equally well to the decay $\omega \to \pi^0 + \gamma$.

B. $b_1(1235) \to \omega + \pi$

Let J, s, and σ stand for the $b_1(1235)$, the ω, and the π. The net intrinsic parity is given by $\eta_J \eta_s \eta_\sigma = +1$ and $F^J_\lambda = +F^J_{-\lambda}$. There are two allowed orbital angular momenta, i.e., $\ell = 0$ or $\ell = 2$. The helicity-coupling amplitudes have the following expansion:

$$\sqrt{2}F^J_+ = \sqrt{\frac{2}{3}}G^J_0 + \sqrt{\frac{1}{3}}G^J_2,$$
$$F^J_0 = \sqrt{\frac{1}{3}}G^J_0 - \sqrt{\frac{2}{3}}G^J_2, \tag{54}$$

where $J = 1$. Assume now that the $b_1(1235)$ decays predominantly $\omega\pi$. Then the $\omega\pi$ elastic scattering in the $J^P = 1^+$-wave is completely dominated by the $b_1(1235)$. It can be shown under this circumstances that the F^J_λ's are relatively real (see Sec. 5.2, Chung [6]), and hence the G^J_ℓ's are relatively real as well. According to the Particle Data Group [8], one has, experimentally,

$$\left|\frac{G^J_2}{G^J_0}\right| = 0.26 \pm 0.04. \tag{55}$$

The general decay amplitude is

$$\mathcal{M}^J(\vartheta,\varphi,M) \propto F^J_0 D^{J*}_{M0}(\varphi,\vartheta,0) + F^J_+\{D^{J*}_{M+}(\varphi,\vartheta,0) + D^{J*}_{M-}(\varphi,\vartheta,0)\}. \tag{56}$$

There are two covariant decay amplitudes corresponding to S and D waves in the problem,

$$A_0(\lambda) = [\omega(\lambda)\cdot\phi^*(\lambda)],$$
$$A_2(\lambda) = [r\cdot\omega(\lambda)][r\cdot\phi^*(\lambda)] - \frac{1}{3}r^2[\omega(\lambda)\cdot\phi^*(\lambda)]. \tag{57}$$

The helicity-coupling amplitudes are given by

$$F^J_\lambda = g_0 A_0(\lambda) + g_2 A_2(\lambda) \tag{58}$$

where g_0 and g_2 are arbitrary constants. Evaluating the covariant amplitudes, one obtains

$$F^J_+ = g_0 - \frac{1}{3}g_2 r^2,$$
$$F^J_0 = \left(\frac{q_0}{m}\right)\left(g_0 + \frac{2}{3}g_2 r^2\right), \tag{59}$$

where $J = 1$. Only in the limit $q_0/m \to 1$, the expressions of (59) reduce to those of (54) with the replacement

$$G^J_0 = \sqrt{3}\,g_0, \qquad G^J_2 = -\sqrt{\frac{2}{3}}\,g_2\,r^2. \tag{60}$$

V. SPIN 1 → SPIN 1 + SPIN 1

Four important and non-trivial examples are treated here; two of them are those of radiative J/ψ decays.

A. $J/\psi \to f_1(1420) + \omega$

Let J, s, and σ stand for the J/ψ, the $f_1(1420)$, and the ω. This decay, $1^- \to 1^+ + 1^-$, involves three helicity-coupling amplitudes F^J_{++}, F^J_{0+}, F^J_{+0}. There are two orbital angular momenta in this problem, i.e., $\ell = 0$ and $\ell = 2$. From (12), the F^J's are related to three $G^J_{\ell S}$'s through

$$\sqrt{2}F^{(1)}_{++} = \frac{1}{\sqrt{3}}G^{(1)}_{01} - \sqrt{\frac{2}{3}}\,G^{(1)}_{21},$$
$$\sqrt{2}F^{(1)}_{0+} = \frac{1}{\sqrt{3}}G^{(1)}_{01} + \frac{1}{\sqrt{6}}G^{(1)}_{21} - \frac{1}{\sqrt{2}}G^{(1)}_{22}, \tag{61}$$
$$\sqrt{2}F^{(1)}_{+0} = \frac{1}{\sqrt{3}}G^{(1)}_{01} + \frac{1}{\sqrt{6}}G^{(1)}_{21} + \frac{1}{\sqrt{2}}G^{(1)}_{22}.$$

Note that these satisfy (13).

Since $\eta_J \eta_s \eta_\sigma = +1$, the covariant amplitudes should not change sign under parity. It is best to write down the Lorentz scalars by observing the ℓS-coupling amplitudes. The $S = 1$ and $S = 2$ total intrinsic spins and an $\ell = 2$ orbital angular momentum may be reflected in the rank-2 tensors

$$\rho^{(\pm)}_{\alpha\beta} = \omega_\alpha\varepsilon_\beta \pm \varepsilon_\alpha\omega_\beta,$$
$$t_{\alpha\beta} = r_\alpha r_\beta. \tag{62}$$

The covariant amplitude corresponding to the $G^{(1)}_{01}$ with $\ell = 0$ is

$$A_1 = \frac{1}{2}(p\ \rho^{(-)}\ \phi^*) \tag{63}$$

where the parentheses indicate contraction of the four indices with the totally antisymmetric tensor, while that equivalent roughly to the $G^{(1)}_{21}$ may be written

$$A_2 = \frac{1}{2}(p\ \rho^{(-)}\ t\cdot\phi^*). \tag{64}$$

As always the dots signify contraction of neighboring indices. The Lorentz scalars corresponding, roughly, to the $G^{(1)}_{22}$ assume the form

$$A_3 = (p\ \rho^{(+)}\cdot t\ \phi^*). \tag{65}$$

Note that there exists another form for $G^{(1)}_{21}$:

$$A_4 = (p\ \rho^{(-)}\cdot t\ \phi^*). \tag{66}$$

Using the techniques developed in the previous section, one can show easily that these four amplitudes do not change sign under parity.

Not all four amplitudes above are independent; it is shown in Appendix A that one has

$$WA_4 = Wr^2A_1 + WA_2 + (q_0 - k_0)A_3 \tag{67}$$

evaluated in the J rest frame.

The amplitudes A_1, A_2, A_3, and A_4, specialized to the case in which all the vectors are defined along the z axis, are given by

$$\begin{aligned} A_1(\lambda\nu) &= [p\omega(\lambda)\varepsilon(-\nu)\phi^*(\delta)], \\ A_2(\lambda\nu) &= [pr\omega(\lambda)\varepsilon(-\nu)][r\cdot\phi^*(\delta)], \\ A_3(\lambda\nu) &= [pr\omega(\lambda)\phi^*(\delta)][r\cdot\varepsilon(-\nu)] + [pr\varepsilon(-\nu)\phi^*(\delta)][r\cdot\omega(\lambda)], \\ A_4(\lambda\nu) &= [pr\omega(\lambda)\phi^*(\delta)][r\cdot\varepsilon(-\nu)] - [pr\varepsilon(-\nu)\phi^*(\delta)][r\cdot\omega(\lambda)], \end{aligned} \tag{68}$$

where $\delta = \lambda - \nu$. Note that these A_i's conform to the prescription given for the Lorentz scalar in (16). Only one factor of p appears, while either none or two factors of r appear, in all the amplitudes.

Let g_1, g_2, g_3, and g_4 be any four arbitrary complex constants. Then, one may write

$$F^{(1)}_{\lambda\nu} = g_1 A_1(\lambda\nu) + g_2 A_2(\lambda\nu) + g_3 A_3(\lambda\nu) + g_4 A_4(\lambda\nu). \tag{69}$$

Eliminating A_4 through (67), one finds

$$F^{(1)}_{\lambda\nu} = (g_1 + g_4 r^2) A_1(\lambda\nu) + (g_2 + g_4) A_2(\lambda\nu) + \left[g_3 + g_4\left(\frac{q_0 - k_0}{W}\right)\right] A_3(\lambda\nu). \tag{70}$$

There are in principle three independent parameters in the problem corresponding to A_1, A_2, and A_3, but this formula shows that the parameters themselves must also depend on masses, energies and momenta. This "energy" dependence is brought out clearly only with the addition of an extra constant g_4. This need for an extra constant—it will be shown shortly—is eliminated when one takes amplitudes corresponding only to pure orbital angular momenta.

From (70) one finds, redefining sightly the g_i's,

$$\begin{aligned} F^{(1)}_{++} &= W(g_1 + g_2 r^2), \\ F^{(1)}_{0+} &= \left(\frac{W}{m}\right)[g_1 q_0 + (-g_3 + g_4) W r^2], \\ F^{(1)}_{+0} &= \left(\frac{W}{\mu}\right)[g_1 k_0 + (g_3 + g_4) W r^2]. \end{aligned} \tag{71}$$

This shows that the $F^{(1)}$'s depend on both the $\ell = 0$ and the $\ell = 2$ terms as in (61), but they depend also on the energies q_0 and k_0. These factors have so far been neglected in partial-wave analyses. Although the A_i's have been formulated to correspond roughly to the $G^J_{\ell S}$'s, their contribution to the $F^{(1)}$'s are different; the $G^{(1)}_{21}$ contributes to all three $F^{(1)}$'s, while the A_2 term appears only in $F^{(1)}_{++}$. Note that the constant g_4 allows for different amount of r^2 contribution to $F^{(1)}_{0+}$ and $F^{(1)}_{+0}$.

Suppose now that one wants to introduce pure S and D waves into the covariant amplitudes. This is achieved by replacing t in A_2, A_3, and A_4 by $\tilde{t}^{(2)}$ of (35). One may write

$$\begin{aligned} A_1 &= (p\omega\varepsilon\phi^*), \\ A_2 &= (pr\omega\varepsilon)(\tilde{r}\cdot\phi^*) - \frac{1}{3}(r\cdot\tilde{r})(p\omega\varepsilon\phi^*), \\ A_3 &= (pr\omega\phi^*)(\tilde{r}\cdot\varepsilon) + (pr\varepsilon\phi^*)(\tilde{r}\cdot\omega), \\ A_4 &= (pr\omega\phi^*)(\tilde{r}\cdot\varepsilon) - (pr\varepsilon\phi^*)(\tilde{r}\cdot\omega) + \frac{2}{3}(r\cdot\tilde{r})(p\omega\varepsilon\phi^*). \end{aligned} \tag{72}$$

It is shown in Appendix A that $A_2 = A_4$ in this case. One may therefore take A_1, A_2, and A_3 as independent amplitudes and obtain

$$\begin{aligned} F^{(1)}_{++} &= W\left(g_1 + \frac{2}{3} g_2 r^2\right), \\ F^{(1)}_{0+} &= W\left(\frac{q_0}{m}\right)\left(g_1 - \frac{1}{3} g_2 r^2 + g_3 r^2\right), \\ F^{(1)}_{+0} &= W\left(\frac{k_0}{\mu}\right)\left(g_1 - \frac{1}{3} g_2 r^2 - g_3 r^2\right). \end{aligned} \tag{73}$$

If the factors W, q_0/m and k_0/μ are taken out and if the substitutions

$$G^{(1)}_{01} = \sqrt{6} g_1, \quad G^{(1)}_{21} = -\frac{2}{\sqrt{3}} g_2 r^2, \quad G^{(1)}_{22} = -2 g_3 r^2 \tag{74}$$

are made, the $F^{(1)}$'s of (61) obtained in the ℓS-coupling scheme reduce to the $F^{(1)}$'s given above. An examination of (73) for the terms with g_2 and g_3 show that the covariant amplitudes A_2 and A_3 correspond to those with the total intrinsic spin $S = 1$ and $S = 2$, respectively. Note that A_1, A_2, and A_4 already contain the pure spin-1 tensor $\psi^{(1)}$ of (30) and the correction term of $\psi^{(2)}$ does not contribute to A_3. The reasons for the energy factors are clear: they come from the z component of the wave functions $\omega(0)$ and $\varepsilon(0)$, which have been evaluated in the J rest frame and not in their respective rest systems [see (47)].

B. $J/\psi \to f_1(1420) + \gamma$

Let J, s, and σ stand for the J/ψ, the $f_1(1420)$, and the γ. This decay involves two helicity-coupling amplitudes F^J_{++} and F^J_{0+}, as photons do not have zero helicities. The projection of the J/ψ spin along the break-up momentum is zero for F^J_{++}. This is to be contrasted to the case in which the J/ψ is produced in an e^+e^- collider; the spin projections in this case can only be either $+1$ or -1 along the direction of the e^+ or e^- beams [10].

The concept of orbital angular momentum does not exist when photons are involved in the final state, and the expansion of the $F^{(1)}$'s in terms of the $G^J_{\ell S}$'s cannot be carried out. Instead, as a starting point, one may impose gauge invariance on the A_i's defined in (63), (64), (65), and (66). They give, after the replacement $\varepsilon \to \varepsilon^{(+)}$ of (43),

$$\begin{aligned} A_1 &= (p\omega\varepsilon\phi^*) - \frac{(q\cdot\varepsilon)}{(q\cdot k)}(p\omega k\phi^*), \\ A_2 &= (pr\omega\varepsilon)(r\cdot\phi^*), \\ A_3 &= (pr\varepsilon\phi^*)(r\cdot\omega). \end{aligned} \tag{75}$$

Note that these amplitudes are parity conserving and that they all go to zero under $\varepsilon \to k$. (As pointed out earlier, the correction term of A_1 vanishes with the choice of the momenta along the z axis and Coulomb gauge.) Note also that both A_3 and A_4 of the previous section reduces to A_3 above. It is seen that the amplitudes A_2 and A_3 contain the $\varepsilon^{(-)}$ introduced in (42).

It is shown in Appendix B that there exists a relationship among the three A_i's. In the J rest frame, one has

$$Wr^2A_1 + WA_2 + 2q_0A_3 = 0. \tag{76}$$

It is instructive to write down explicitly the non-zero helicity components of A_i's with the vectors defined along the z axis. In this case the second term in A_1 disappears since $(q\cdot\varepsilon) = 0$, and one finds

$$\begin{aligned} A_1(++) &= iW, \\ A_2(++) &= -iWr^2, \\ A_1(0+) &= i\left(\frac{W}{m}\right)q_0, \\ A_3(0+) &= -i\left(\frac{W^2}{2m}\right)r^2. \end{aligned} \tag{77}$$

Note that these satisfy (76).

Using the same technique of the previous section, one may introduce three constants g_1, g_2 and g_3 and write the expression (69) again but eliminating A_2 via (76):

$$F^{(1)}_{\lambda\nu} = (g_1 - g_2r^2)A_1(\lambda\nu) + \left[g_3 - 2g_2\left(\frac{q_0}{W}\right)\right]A_3(\lambda\nu). \tag{78}$$

There are only two independent amplitudes, A_1 and A_3, in the problem, and one needs in general only two coefficients in the expression for $F^{(1)}$. But the above formula shows that they themselves are dependent on r^2 and q_0. One can insist that the g_i's be constants but then there are *three* constants in the problem. From (77), one finds

$$\begin{aligned} F^{(1)}_{++} &= W(g_1 + g_2r^2), \\ F^{(1)}_{0+} &= \left(\frac{W}{m}\right)(g_1q_0 + g_3Wr^2), \end{aligned} \tag{79}$$

with slight changes in g_1, g_2, and g_3. These equations show that both $F^{(1)}$'s have terms proportional to r^2 as in (71).

In a previous analysis of the J/ψ radiative decays [11], only the g_2 and g_3 terms have been retained, perhaps on the idea that the terms proportional to r^2 should be more important than those terms independent of r^2. To be more general, however, one should include all three terms and determine if they are required by the data. Note that one has adopted here a model in which each coefficient of the amplitudes A_1 and A_3 has been given an expansion in r^2 or q_0/W—resulting in three independent complex constants.

C. $\bar{p}p(^3S_1) \to K^*(892)\bar{K}^*(892)$

This decay process, $1^- \to 1^- + 1^-$, involves four helicity-coupling amplitudes and odd-ℓ orbital angular momenta ($\ell = 1$ or 3). They are given by, from (12),

$$\begin{aligned} \sqrt{2}F^{(1)}_{++} &= \sqrt{\frac{2}{3}}G^{(1)}_{10} - \sqrt{\frac{2}{15}}G^{(1)}_{12} + \frac{1}{\sqrt{5}}G^{(1)}_{32}, \\ \sqrt{2}F^{(1)}_{0+} &= \frac{1}{\sqrt{2}}G^{(1)}_{11} - \sqrt{\frac{3}{10}}G^{(1)}_{12} - \frac{1}{\sqrt{5}}G^{(1)}_{32}, \\ \sqrt{2}F^{(1)}_{+0} &= -\frac{1}{\sqrt{2}}G^{(1)}_{11} - \sqrt{\frac{3}{10}}G^{(1)}_{12} - \frac{1}{\sqrt{5}}G^{(1)}_{32}, \\ F^{(1)}_{00} &= -\frac{1}{\sqrt{3}}G^{(1)}_{10} - \frac{2}{\sqrt{15}}G^{(1)}_{12} + \sqrt{\frac{2}{5}}G^{(1)}_{32}. \end{aligned} \tag{80}$$

Again, it is seen that these equations satisfy (13).

Let J, s, and σ stand for $\bar{p}p$, K^*, and $\bar{K}^*$. As this process has $\eta_J\eta_s\eta_\sigma = -1$, the covariant amplitudes one writes down should change sign under parity. The covariant counterpart of the $G^{(1)}_{10}$ is

$$A_1 = (\omega\cdot\varepsilon)(r\cdot\phi^*). \tag{81}$$

Similarly, one may choose for $G^{(1)}_{11}$ and $G^{(1)}_{12}$

$$A_2 = (r\cdot\omega)(\varepsilon\cdot\phi^*) - (r\cdot\varepsilon)(\omega\cdot\phi^*) \tag{82}$$

and

$$A_3 = (r\cdot\omega)(\varepsilon\cdot\phi^*) + (r\cdot\varepsilon)(\omega\cdot\phi^*). \tag{83}$$

For $G^{(1)}_{32}$ one may take as the simplest possible choice the following:

$$A_4 = (r\cdot\omega)(r\cdot\varepsilon)(r\cdot\phi^*). \tag{84}$$

Note that all the A_i's have no dependence in p and the r factor appears either once or three times corresponding to $\ell = 1$ or $\ell = 3$.

Let g_1, g_2, g_3, and g_4 be any four arbitrary complex constants. Then, one may write as before

$$F^{(1)}_{\lambda\nu} = g_1A_1(\lambda\nu) + g_2A_2(\lambda\nu) + g_3A_3(\lambda\nu) + g_4A_4(\lambda\nu) \tag{85}$$

where $A_i(\lambda\nu)$'s are again defined with the vectors along the z axis. One obtains the following results, with a small modification of the g_i's,

$$\begin{aligned} F_{++}^{(1)} &= -4g_1 r, \\ F_{0+}^{(1)} &= \left(\frac{W}{m}\right)(\ g_2 + g_3)r, \\ F_{+0}^{(1)} &= \left(\frac{W}{\mu}\right)(-g_2 + g_3)r, \\ F_{00}^{(1)} &= \left(\frac{1}{m\mu}\right)[\ g_1(4q_0k_0 + r^2) - g_2W(q_0 - k_0) \\ &\qquad + g_3W^2 + g_4W^2r^2\ ]r. \end{aligned} \tag{86}$$

According to (80), each $F^{(1)}$ gets a contribution from the $\ell = 3$ amplitude $G_{32}^{(1)}$. In (86), however, only the $F_{00}^{(1)}$ amplitude has a term in r^3.

Covariant amplitudes corresponding to pure P and F waves and pure total intrinsic spins 0, 1, and 2 are

$$\begin{aligned} A_1 &= \ (r \cdot \psi^{(0)} \cdot \phi^*), \\ A_2 &= \ (r \cdot \psi^{(1)} \cdot \phi^*), \\ A_3 &= \ (r \cdot \psi^{(2)} \cdot \phi^*), \\ A_4 &= \frac{1}{2}(\psi^{(2)} : \tilde{t}^{(3)} \cdot \phi^*), \end{aligned} \tag{87}$$

where the colon signifies contraction of two neighboring indices. The A_i's may be expressed as

$$\begin{aligned} A_1 &= (\tilde{\omega} \cdot \varepsilon)(\tilde{r} \cdot \phi^*), \\ A_2 &= (\tilde{r} \cdot \omega)(\tilde{\varepsilon} \cdot \phi^*) - (\tilde{r} \cdot \varepsilon)(\tilde{\omega} \cdot \phi^*), \\ A_3 &= (\tilde{r} \cdot \omega)(\tilde{\varepsilon} \cdot \phi^*) + (\tilde{r} \cdot \varepsilon)(\tilde{\omega} \cdot \phi^*) - \frac{2}{3}(\tilde{\omega} \cdot \varepsilon)(\tilde{r} \cdot \phi^*), \\ A_4 &= (\tilde{r} \cdot \omega)(\tilde{r} \cdot \varepsilon)(\tilde{r} \cdot \phi^*) \\ &\quad - \frac{1}{5}(\tilde{r} \cdot r)[(\tilde{\omega} \cdot \varepsilon)(\tilde{r} \cdot \phi^*) + (\tilde{r} \cdot \varepsilon)(\tilde{\omega} \cdot \phi^*) \\ &\qquad + (\tilde{r} \cdot \omega)(\tilde{\varepsilon} \cdot \phi^*)]. \end{aligned} \tag{88}$$

In the J rest frame all the scalars in these amplitudes are reduced to those involving three-momenta only. Note, in particular, that one has

$$A_2 = (\boldsymbol{\omega} \times \boldsymbol{\varepsilon}) \cdot (\mathbf{r} \times \boldsymbol{\phi}^*) \tag{89}$$

in the J rest frame—a familiar result. However, the polarization four-vectors for s and σ are not evaluated in their respective rest frames. The helicity-coupling amplitudes are

$$\begin{aligned} F_{++}^{(1)} &= \left(-g_1 + \frac{2}{3}g_3 + \frac{1}{5}g_4r^2\right) r, \\ F_{0+}^{(1)} &= \left(\frac{q_0}{m}\right)\left(g_2 + g_3 - \frac{1}{5}g_4r^2\right) r, \\ F_{+0}^{(1)} &= \left(\frac{k_0}{\mu}\right)\left(-g_2 + g_3 - \frac{1}{5}g_4r^2\right) r, \\ F_{00}^{(1)} &= \left(\frac{q_0k_0}{m\mu}\right)\left(g_1 + \frac{4}{3}g_3 + \frac{2}{5}g_4r^2\right) r. \end{aligned} \tag{90}$$

Aside from the energy dependent factors, these helicity-coupling amplitudes are seen to be identical to (80) with the following substitutions:

$$\begin{aligned} G_{10}^{(1)} &= -\sqrt{3}g_1r, \quad G_{11}^{(1)} = 2g_2r, \\ G_{12}^{(1)} &= -2\sqrt{\frac{5}{3}}g_3r, \quad G_{32}^{(1)} = \sqrt{\frac{2}{5}}g_4r^3. \end{aligned} \tag{91}$$

D. $J/\psi \to \hat{\rho}(1405) + \gamma$

This decay mode, so far unobserved, may be considered a special case of the preceding example, in the sense that the second decay product is massless, i.e., $\mu = 0$. Gauge invariance is imposed by substituting ε with $\varepsilon^{(+)}$ in (81), (82), (83), and (84). Note that this substitution preserves parity. The results are

$$\begin{aligned} A_1 &= (\omega \cdot \varepsilon)(r \cdot \phi^*) - \frac{(q \cdot \varepsilon)}{(q \cdot k)}(\omega \cdot k)(r \cdot \phi^*), \\ A_2 &= (r \cdot \omega)(\varepsilon \cdot \phi^*) - \frac{(q \cdot \varepsilon)}{(q \cdot k)}(r \cdot \omega)(k \cdot \phi^*). \end{aligned} \tag{92}$$

Note that A_4 is zero and A_3 reduces to A_2.

Once again one considers the special case in which the vectors are along the z axis. In this case, as before, one finds that the factor $(q \cdot \varepsilon) = 0$, so that the second terms in both A_1 and A_2 drop out. In terms of two arbitrary complex constants g_1 and g_2, one obtains

$$\begin{aligned} F_{++}^{(1)} &= \ g_1 r, \\ F_{0+}^{(1)} &= \left(\frac{W}{m}\right) g_2 r. \end{aligned} \tag{93}$$

This is very similar to the first two equations of (86).

VI. SPIN 2 → SPIN 1 + SPIN 0

One may use, for this problem, the same notations and symbols introduced in Sec. IV. Thus, the J, s, and σ stand for the spin-2, spin-1, and spin-0 particles, respectively. As before, the treatment of this decay can be grouped into two separate categories depending on the intrinsic parities involved.

A. Net intrinsic parity $\eta_J\eta_s\eta_\sigma = +1$

An example of this case is provided by the decay $a_2(1320) \to \rho + \pi$. There exists only one helicity amplitude $F_{+}^{(2)}$ corresponding the D wave allowed in the decay. Let $t = rr$ be the rank-2 tensor for the D wave.

Then, the covariant amplitude may be written, for the vectors defined along the z axis,

$$\begin{aligned} A(m) &= [p\ \omega(m)\ t \cdot \phi^*(m)] \\ &= \sum_{m_1 m_2} (1m_1 1m_2|2m)[r \cdot \phi^*(m_1)] \\ &\qquad \times [p\ \omega(m)\ r\ \phi^*(m_2)]. \end{aligned} \tag{94}$$

The only nonzero component of A is

$$A(+) = \frac{1}{\sqrt{2}}[r \cdot \phi^*(0)][p\ \omega(+)\ r\ \phi^*(+)] \tag{95}$$

to be evaluated in the J rest frame. From (46) and (47) one obtains for the helicity-coupling amplitude,

$$F_+^{(2)} = gWr^2 \tag{96}$$

where g is an arbitrary complex constant. The form of the amplitude (94) shows that introduction of a pure D wave does not alter the result (96).

Suppose now that the particle s is a photon. Indeed, such a decay has been observed in $a_2(1320) \to \pi + \gamma$. Gauge invariance requires that one must replace ω as given in (50), but the amplitude (95) does not change. The helicity-coupling amplitude (96) is thus appropriate for photons as well.

B. Net intrinsic parity $\eta_J \eta_s \eta_\sigma = -1$

The decay $\pi_2(1670) \to \rho + \pi$ is an example. There are two orbital angular momenta $\ell = 1$ and $\ell = 3$. The helicity-coupling amplitudes are

$$\sqrt{2}F_+^{(2)} = \sqrt{\frac{3}{5}}G_1^{(2)} + \sqrt{\frac{2}{5}}G_3^{(2)}, \tag{97}$$
$$F_0^{(2)} = \sqrt{\frac{2}{5}}G_1^{(2)} - \sqrt{\frac{3}{5}}G_3^{(2)}.$$

The covariant amplitudes corresponding to G_1 and G_3 are

$$A_1 = (\omega \cdot \phi^* \cdot r), \tag{98}$$
$$A_2 = (r \cdot \omega)(r \cdot \phi^* \cdot r).$$

Again, taking the vectors along the z axis, one gets

$$A_1(m) = \sum_{m_1 m_2} (1m_1 1m_2|2m)[\omega(m) \cdot \phi^*(m_1)][r \cdot \phi^*(m_2)],$$
$$A_2(m) = [r \cdot \omega(m)] \sum_{m_1 m_2} (1m_1 1m_2|2m)[r \cdot \phi^*(m_1)] \times [r \cdot \phi^*(m_2)]. \tag{99}$$

The nonzero components are, from (46) and (47),

$$A_1(+) = \frac{1}{\sqrt{2}}[\omega(+) \cdot \phi^*(+)][r \cdot \phi^*(0)],$$
$$A_1(0) = \sqrt{\frac{2}{3}}[\omega(0) \cdot \phi^*(0)][r \cdot \phi^*(0)], \tag{100}$$
$$A_2(0) = \sqrt{\frac{2}{3}}[r \cdot \omega(0)][r \cdot \phi^*(0)]^2.$$

Introducing two complex constants g_1 and g_2, one may set

$$F_m^{(2)} = g_1 A_1(m) + g_2 A_2(m) \tag{101}$$

which gives

$$F_+^{(2)} = \frac{1}{\sqrt{2}} g_1 r, \tag{102}$$
$$F_0^{(2)} = \sqrt{\frac{2}{3}}\left(\frac{1}{m}\right)(g_1 q_0 + g_2 W r^2) r.$$

Consider now the case in which the state s is a photon. There exists only one helicity-coupling amplitude $F_+^{(2)}$. One finds that carrying out the replacement (50) does not change the amplitude, so that the form of the helicity-coupling amplitude remains the same.

One may insist that only the "pure" orbital angular momenta be used in the covariant amplitudes. In this case, one needs to replace r by $\tilde{r}$ and one obtains

$$A_1 = (\omega \cdot \phi^* \cdot \tilde{r}), \tag{103}$$
$$A_2 = (\omega \cdot \tilde{t}^{(3)} : \phi^*).$$

In the J rest frame, they assume the form

$$A_1(m) = \sum_{m_1 m_2} (1m_1 1m_2|2m)[\boldsymbol{\omega}(m) \cdot \boldsymbol{\phi}^*(m_1)][\mathbf{r} \cdot \boldsymbol{\phi}^*(m_2)], \tag{104}$$
$$A_2(m) = \sum_{m_1 m_2} (1m_1 1m_2|2m) \left\{[\mathbf{r} \cdot \boldsymbol{\omega}(m)][\mathbf{r} \cdot \boldsymbol{\phi}^*(m_1)][\mathbf{r} \cdot \boldsymbol{\phi}^*(m_2)] - \frac{2}{5} r^2 [\boldsymbol{\omega}(m) \cdot \boldsymbol{\phi}^*(m_1)][\mathbf{r} \cdot \boldsymbol{\phi}^*(m_2)]\right\}.$$

From this one obtains

$$F_+^{(2)} = \frac{1}{\sqrt{2}}\left(g_1 - \frac{2}{5} g_2 r^2\right) r, \tag{105}$$
$$F_0^{(2)} = \sqrt{\frac{2}{3}}\left(\frac{q_0}{m}\right)\left(g_1 + \frac{3}{5} g_2 r^2\right) r.$$

Note that, if the q_0/m factor is neglected, one can make the equations above identical to those in (97) with the following substitutions:

$$G_1^{(2)} = \sqrt{\frac{5}{3}} g_1 r, \quad G_3^{(2)} = -\sqrt{\frac{2}{5}} g_2 r^3. \tag{106}$$

The energy factor which breaks this symmetry is a necessary consequence of the fact that the wave function ω is being evaluated in the J rest system.

VII. SPIN 1 → SPIN 2 + SPIN 0

As before, one may use the notations of Sec. IV: J, s, and σ stand for spin 1, spin 2, and spin 0, respectively.

A. Net intrinsic parity $\eta_J\eta_s\eta_\sigma = +1$

An example of this type of decay would be $\bar{p}p(^3S_1) \to a_2(1320) + \pi$. There is just one helicity-coupling amplitude $F_+^{(1)}$ corresponding to $\ell = 2$. The covariant amplitude is

$$A(m) = \sum_{m_1m_2} (1m_11m_2|2m)[r\cdot\omega(m_1)][pr\omega(m_2)\phi^*(m)] \tag{107}$$

and

$$A(+) = \frac{1}{\sqrt{2}}[r\cdot\omega(0)][pr\omega(+)\phi^*(+)]. \tag{108}$$

The helicity-coupling amplitude is then

$$F_+^{(1)} = g\left(\frac{W^2}{m}\right)r^2 \tag{109}$$

for an arbitrary complex constant g. If a pure D wave is introduced into the amplitude, one gets

$$F_+^{(1)} = gW\left(\frac{q_0}{m}\right)r^2. \tag{110}$$

B. Net intrinsic parity $\eta_J\eta_s\eta_\sigma = -1$

This case is exemplified by the process $\bar{p}p(^3P_1) \to f_2(1270) + \pi$. There are two helicity-coupling amplitudes corresponding to two orbital angular momenta $\ell = 1$ and $\ell = 3$:

$$\sqrt{2}F_+^{(1)} = -\sqrt{\frac{3}{5}}G_1^{(1)} - \sqrt{\frac{2}{5}}G_3^{(1)},$$
$$F_0^{(1)} = -\sqrt{\frac{2}{5}}G_1^{(1)} + \sqrt{\frac{3}{5}}G_3^{(1)}. \tag{111}$$

The covariant amplitudes are

$$A_1(m) = \sum_{m_1m_2}(1m_11m_2|2m)[r\cdot\omega(m_1)][\omega(m_2)\cdot\phi^*(m)],$$
$$A_2(m) = \sum_{m_1m_2}(1m_11m_2|2m)[r\cdot\omega(m_1)][r\cdot\omega(m_2)]\times[r\cdot\phi^*(m)], \tag{112}$$

which lead to

$$A_1(+) = \frac{1}{\sqrt{2}}[r\cdot\omega(0)][\omega(+)\cdot\phi^*(+)],$$
$$A_1(0) = \sqrt{\frac{2}{3}}[r\cdot\omega(0)][\omega(0)\cdot\phi^*(0)], \tag{113}$$
$$A_2(0) = \sqrt{\frac{2}{3}}[r\cdot\omega(0)]^2[r\cdot\phi^*(0)].$$

From these one finds

$$F_+^{(1)} = \frac{1}{\sqrt{2}}g_1\left(\frac{W}{m}\right)r,$$
$$F_0^{(1)} = \sqrt{\frac{2}{3}}\left(\frac{W}{m^2}\right)(g_1q_0 + g_2Wr^2)r, \tag{114}$$

for arbitrary complex constants g_1 and g_2.

If pure P and F waves are used in the amplitudes, one obtains

$$A_1(m) = \sum_{m_1m_2}(1m_11m_2|2m)[\mathbf{r}\cdot\boldsymbol{\omega}(m_1)][\boldsymbol{\omega}(m_2)\cdot\boldsymbol{\phi}^*(m)], \tag{115}$$
$$A_2(m) = \sum_{m_1m_2}(1m_11m_2|2m)\left\{[\mathbf{r}\cdot\boldsymbol{\omega}(m_1)][\mathbf{r}\cdot\boldsymbol{\omega}(m_2)][\mathbf{r}\cdot\boldsymbol{\phi}^*(m)] - \frac{2}{5}r^2[\mathbf{r}\cdot\boldsymbol{\omega}(m_1)][\boldsymbol{\omega}(m_2)\cdot\boldsymbol{\phi}^*(m)]\right\},$$

evaluated in the J rest frame. The helicity-coupling amplitudes are

$$F_+^{(1)} = \frac{1}{\sqrt{2}}\left(\frac{q_0}{m}\right)\left(g_1 - \frac{2}{5}g_2r^2\right)r,$$
$$F_0^{(1)} = \sqrt{\frac{2}{3}}\left(\frac{q_0}{m}\right)^2\left(g_1 + \frac{3}{5}g_2r^2\right)r. \tag{116}$$

Note that, as a result of the spin-2 character of the s, one finds multiple factors of q_0/m in $F^{(1)}$. Aside from these factors, the substitution

$$G_1^{(1)} = -\sqrt{\frac{5}{3}}g_1r, \quad G_3^{(1)} = \sqrt{\frac{2}{5}}g_2r^3 \tag{117}$$

into (111) leads to the results given above.

VIII. SPIN 2 → SPIN 2 + SPIN 0

There are in total five orbital angular momenta, from $\ell = 0$ to $\ell = 4$. In this section, only pure orbital angular momenta are treated for brevity.

The problem again divides into two distinct cases.

A. Net intrinsic parity $\eta_J\eta_s\eta_\sigma = -1$

An example of this case is provided by the decay $\bar{p}p(^3P_2) \to f_2(1270) + \pi$. There are two helicity-coupling amplitudes $F_2^{(2)}$ and $F_1^{(2)}$ corresponding to $\ell = 1$ and $\ell = 3$:

$$\sqrt{2}F_2^{(2)} = -\frac{2}{\sqrt{5}}G_1^{(2)} - \frac{1}{\sqrt{5}}G_3^{(2)}, \tag{118}$$
$$\sqrt{2}F_1^{(2)} = -\frac{1}{\sqrt{5}}G_1^{(2)} + \frac{2}{\sqrt{5}}G_3^{(2)}.$$

The $G^{(2)}$'s are the ℓS-coupling amplitudes.

The covariant amplitudes corresponding to pure orbital angular momenta are

$$A_1 = (pr\ \omega \cdot \phi^*), \tag{119}$$
$$A_2 = (pr\ \tilde{r} \cdot \omega\ r \cdot \phi^*) - \frac{1}{5}(r \cdot \tilde{r})(pr\ \omega \cdot \phi^*),$$

with the spin-2 tensors given by

$$\omega_{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1\ 1m_2|2m)\omega_\alpha(m_1)\omega_\beta(m_2), \tag{120}$$
$$\phi_{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1\ 1m_2|2m)\phi_\alpha(m_1)\phi_\beta(m_2).$$

Using the same techniques as before, one finds

$$F_2^{(2)} = W\left(g_1 - \frac{1}{5}g_2 r^2\right) r, \tag{121}$$
$$F_1^{(2)} = W\left(\frac{q_0}{m}\right)\left(\frac{1}{2}g_1 + \frac{2}{5}g_2 r^2\right) r,$$

for two arbitrary complex constants g_1 and g_2. Again, neglecting the mass and energy factors, the substitutions

$$G_1^{(2)} = -\sqrt{\frac{5}{2}}g_1 r, \quad G_3^{(2)} = \sqrt{\frac{2}{5}}g_2 r^3 \tag{122}$$

into (118) lead to the results given above.

B. Net intrinsic parity $\eta_J\eta_s\eta_\sigma = +1$

An example for this case is $\pi_2(1670) \to f_2(1270) + \pi$. There are three helicity-coupling amplitudes corresponding to $\ell = 0$, $\ell = 2$, and $\ell = 4$, which are given by

$$\sqrt{2}F_2^{(2)} = \sqrt{\frac{2}{5}}G_0^{(2)} + \frac{2}{\sqrt{7}}G_2^{(2)} + \frac{1}{\sqrt{35}}G_4^{(2)},$$
$$\sqrt{2}F_1^{(2)} = \sqrt{\frac{2}{5}}G_0^{(2)} - \frac{1}{\sqrt{7}}G_2^{(2)} - \frac{4}{\sqrt{35}}G_4^{(2)}, \tag{123}$$
$$F_0^{(2)} = \frac{1}{\sqrt{5}}G_0^{(2)} - \sqrt{\frac{2}{7}}G_2^{(2)} + 3\sqrt{\frac{2}{35}}G_4^{(2)},$$

in terms of the ℓS-coupling amplitudes.

The covariant amplitudes corresponding to pure orbital angular momenta are, with the spin-2 tensors of (120),

$$A_1 = (\omega\ :\phi^*),$$
$$A_2 = (\tilde{r} \cdot \omega \cdot \phi^* \cdot r) - \frac{1}{3}(r \cdot \tilde{r})(\omega\ :\phi^*), \tag{124}$$
$$A_3 = (\tilde{r} \cdot \omega \cdot \tilde{r})(r \cdot \phi^* \cdot r) - \frac{4}{7}(r \cdot \tilde{r})(\tilde{r} \cdot \omega \cdot \phi^* \cdot r) + \frac{2}{35}(r \cdot \tilde{r})^2(\omega\ :\phi^*).$$

These lead to the following forms for the helicity-coupling amplitudes:

$$F_2^{(2)} = g_1 - \frac{1}{3}g_2 r^2 + \frac{2}{35}g_3 r^4,$$
$$F_1^{(2)} = \left(\frac{q_0}{m}\right)\left(g_1 + \frac{1}{6}g_2 r^2 - \frac{8}{35}g_3 r^4\right), \tag{125}$$
$$F_0^{(2)} = \frac{2}{3}\left(\frac{q_0}{m}\right)^2\left(g_1 + \frac{2}{3}g_2 r^2 + \frac{17}{35}g_3 r^4\right) + \frac{1}{3}\left(g_1 - \frac{1}{3}g_2 r^2 + \frac{2}{35}g_3 r^4\right),$$

where g_1, g_2, and g_3 are three arbitrary complex constants.

One encounters here for the first time a helicity-coupling amplitude which is not a monomial in q_0/m but a polynomial. If and only if one sets $q_0/m = 1$, these formula are seen to be identical to (123) after the substitutions

$$G_0^{(2)} = \sqrt{5}g_1, \quad G_2^{(2)} = -\frac{1}{3}\sqrt{\frac{7}{2}}g_2 r^2, \tag{126}$$
$$G_4^{(2)} = 2\sqrt{\frac{2}{35}}g_3 r^4.$$

One may now generalize these results. For the case in which one of the decay product is a spin-0 particle, the helicity-coupling amplitude F_λ^J is in general a polynomial of order $s - |\lambda|$ in q_0/m, where s is the spin of a decay product and λ is its helicity.

IX. SPIN 0 → SPIN 2 + SPIN 0

The decay $\eta_c(2980) \to a_2(1230) + \pi$ provides an example for this process. There is one helicity-coupling amplitude for $\ell = 2$ and $\eta_J\eta_s\eta_\sigma = +1$ always. The covariant amplitude is

$$A = (\tilde{r} \cdot \omega \cdot \tilde{r}) \tag{127}$$

for a pure D wave. Note that the correction term for a pure D-wave orbital angular momentum does not contribute, as the spin-2 tensor ω is itself traceless. The helicity-coupling amplitude is

$$F_0^{(0)} = \left(\frac{q_0}{m}\right)^2 g r^2 \tag{128}$$

where g is a complex constant.

X. CONCLUSIONS

This paper is not concerned with theory and is thus devoid of dynamics. It is concerned solely with the covariance requirement as applied to the decay processes of hadrons—more specifically, one wishes to write down the decay amplitudes with a minimal set of mass, energy and momentum dependence satisfying the requirement of special relativity and rotational invariance.

It is shown that the most economical way to incorporate the covariance requirement is to separate out the angular dependence from the covariant tensor formalism and deal only with the four-vectors defined along the z axis. It has been found in the process that the helicity-coupling amplitudes contain not only the usual angular-momentum barrier factors but also the energy factors from the daughter particles, as well as the mass of the parent particles themselves in some cases. Consider a hadron of spin J decaying into a daughter particle with spin s and its helicity λ and a spin-0 particle. The helicity-coupling amplitude F^J_λ is in general a polynomial of order $s - |\lambda|$ in q_0/m, where q_0 and m are the energy and the mass of the daughter particle. The following two examples illustrate further this energy dependence.

Consider as a first example the decay $b_1(1235) \to \omega + \pi$, treated in Sec. IV B. At the b_1 mass, one has $W = 1.235$ GeV and $q_0/m \simeq 1.10$, not too far from 1. Therefore, the effect of neglecting the factor has at most 20% effect on the branching ratio of $|F^{(1)}_0|^2$ over $|F^{(1)}_+|^2$. The factor q_0/m is in addition a function of W over the b_1 Breit-Wigner shape, ranging from 1.08 to 1.12 as the mass W goes from 1.185 to 1.285 GeV. Consider next an example given in Sec. VI B, i.e., the decay $\pi_2(1670) \to \rho + \pi$, with its branching ratio quoted as 31%. The factor q_0/m which appears in the expression for $F^{(2)}_0$ [see (105)] is 1.31, far different from 1; therefore, it can have a major impact on the branching ratio for the helicity amplitudes. As the width for the $\pi_2(1670)$ is 0.250 GeV, the factor q_0/m also varies significantly, ranging from 1.21 to 1.42 as the π_2 mass W goes from 1.47 to 1.87 GeV.

Most, if not all, partial-wave analysis programs did not incorporate this energy dependence and therefore violates the covariance requirement. It is seen, however, that the corrections must be small for a hadronic state not far from the threshold of a given decay channel. The main purpose of this paper is to show that the correction factors can be significant as, for example, in the decay $\pi_2(1670) \to \rho + \pi$.

In addition, the techniques have been worked out in this paper as to how one may apply the present formalism to the hadronic decays involving photons in the final state. It is shown that highly non-trivial complications arise if one wishes to keep careful track of the mass and energy dependence in the decay amplitudes. The example given in Sec. V B illustrate this point; although the decay $J/\psi \to f_1(1420) + \gamma$ involves two helicity-coupling amplitudes, they must be given in terms three complex constants. This fact has never been appreciated so far in the analyses involving J/ψ radiative decays.

Finally, it may be worth emphasizing that for the first time a technique has been developed by which the concept of total intrinsic spin can be incorporated into the covariant tensor formalism—see the projection operators for S in Sec. III. It is shown that these projection operators have rank $2(s_1 + s_2)$, where s_1 and s_2 are the spins out of which the total intrinsic spin S is to be formed. This technique provides a crucial link between the helicity-coupling and ℓS-coupling amplitudes within the covariant tensor formalism.

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APPENDIX A: AMPLITUDES FOR $J/\psi \to f_1(1420) + \omega$

It is the purpose of this Appendix to derive a relationship among A_1, A_2, A_3, and A_4 of Sec. V A:

$$\begin{aligned} A_1 &= (p\omega\varepsilon\phi^*), \\ A_2 &= (pr\omega\varepsilon)(r\cdot\phi^*), \\ A_3 &= (pr\omega\phi^*)(r\cdot\varepsilon) + (pr\varepsilon\phi^*)(r\cdot\omega), \\ A_4 &= (pr\omega\phi^*)(r\cdot\varepsilon) - (pr\varepsilon\phi^*)(r\cdot\omega). \end{aligned} \tag{A1}$$

As these are Lorentz invariants, it is sufficient to show a relationship in the J rest frame. In this frame, the three-vectors $\mathbf{k}$ and $\mathbf{q} = -\mathbf{k}$ are allowed to have an arbitrary direction, and the z components of spin are left to go over all the allowed values.

Using the relations $W = q_0 + k_0$, $\mathbf{r} = -2\mathbf{k}$ and

$$(p\cdot\phi) = (q\cdot\omega) = (k\cdot\varepsilon) = 0,$$

the scalar products are, in the J rest frame,

$$\begin{aligned} (r\cdot\varepsilon) &= \left(\frac{W}{k_0}\right)(\mathbf{k}\cdot\boldsymbol{\varepsilon}), \\ (r\cdot\omega) &= \left(\frac{W}{q_0}\right)(\mathbf{k}\cdot\boldsymbol{\omega}), \\ (r\cdot\phi^*) &= 2(\mathbf{k}\cdot\boldsymbol{\phi}^*). \end{aligned} \tag{A2}$$

It is convenient to introduce a shorthand notation

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}). \tag{A3}$$

The following combination of A_1 and A_2 can be expressed in different ways:

$$\begin{aligned} 4k^2A_1 + A_2 &= 4W[k^2(\boldsymbol{\omega}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*) - (\mathbf{k}\cdot\boldsymbol{\phi}^*)(\mathbf{k}\ \boldsymbol{\omega}\ \boldsymbol{\varepsilon})] \\ &= 4W[(\boldsymbol{\omega}\times\boldsymbol{\varepsilon})\mathbf{k}(\boldsymbol{\phi}^*\times\mathbf{k})] \\ &= 4W[(\mathbf{k}\cdot\boldsymbol{\omega})(\mathbf{k}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*) \\ &\quad -(\mathbf{k}\cdot\boldsymbol{\varepsilon})(\mathbf{k}\ \boldsymbol{\omega}\ \boldsymbol{\phi}^*)], \end{aligned} \tag{A4}$$

while A_3 and A_4 can be written

$$\begin{aligned} -k_0(\ A_3 + A_4) &= 4W^2(\mathbf{k}\cdot\boldsymbol{\varepsilon})(\mathbf{k}\ \boldsymbol{\omega}\ \boldsymbol{\phi}^*), \\ q_0(-A_3 + A_4) &= 4W^2(\mathbf{k}\cdot\boldsymbol{\omega})(\mathbf{k}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*). \end{aligned} \tag{A5}$$

From these one finds

$$WA_4 = 4Wk^2A_1 + WA_2 + (q_0 - k_0)A_3. \tag{A6}$$

Multiplying both sides of this equation by W, one may express it in an explicitly covariant form:

$$\begin{aligned} W^2A_4 = [W^2 - (m-\mu)^2][W^2 - (m+\mu)^2]A_1 \\ +W^2A_2 + (m^2-\mu^2)A_3. \end{aligned} \tag{A7}$$

One now turns to the case in which pure D waves are used in the covariant amplitudes. They are, in the J rest frame,

$$\begin{aligned} A_1 &= W(\boldsymbol{\omega}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*), \\ A_2 &= W\left[(\mathbf{r}\cdot\boldsymbol{\phi}^*)(\mathbf{r}\ \boldsymbol{\omega}\ \boldsymbol{\varepsilon}) - \frac{1}{3}r^2(\boldsymbol{\omega}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*)\right], \\ A_3 &= W[(\mathbf{r}\cdot\boldsymbol{\varepsilon})(\mathbf{r}\ \boldsymbol{\omega}\ \boldsymbol{\phi}^*) + (\mathbf{r}\cdot\boldsymbol{\omega})(\mathbf{r}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*)], \\ A_4 &= W\left[(\mathbf{r}\cdot\boldsymbol{\varepsilon})(\mathbf{r}\ \boldsymbol{\omega}\ \boldsymbol{\phi}^*) - (\mathbf{r}\cdot\boldsymbol{\omega})(\mathbf{r}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*) + \frac{2}{3}r^2(\boldsymbol{\omega}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*)\right]. \end{aligned} \tag{A8}$$

Using the same technique of (A4), it can be shown that $A_2 = A_4$.

APPENDIX B: AMPLITUDES FOR $J/\psi \to f_1(1420) + \gamma$

It is the purpose of this Appendix to derive a relationship among A_1, A_2, and A_3 of Sec. V B. Introducing A_0, one may form

$$\begin{aligned} A_0 &= (q\cdot k)A_1 \\ &= (q\cdot k)(p\omega\varepsilon\phi^*) - (q\cdot\varepsilon)(p\omega k\phi^*), \\ A_2 &= (pr\omega\varepsilon)(r\cdot\phi^*), \\ A_3 &= (pr\varepsilon\phi^*)(r\cdot\omega). \end{aligned} \tag{B1}$$

Using the relations $k_0 = k$, $W = q_0 + k$, and

$$(p\cdot\phi) = (q\cdot\omega) = (k\cdot\varepsilon) = 0$$

the scalar products are, in the J rest frame,

$$\begin{aligned} (q\cdot k) &= Wk, \\ (q\cdot\varepsilon) &= \left(\frac{W}{k}\right)(\mathbf{k}\cdot\boldsymbol{\varepsilon}), \\ (r\cdot\omega) &= \left(\frac{W}{q_0}\right)(\mathbf{k}\cdot\boldsymbol{\omega}), \\ (r\cdot\phi^*) &= 2(\mathbf{k}\cdot\boldsymbol{\phi}^*). \end{aligned} \tag{B2}$$

The amplitudes A_i may now be written

$$\begin{aligned} A_0 &= \left(\frac{W^2}{k}\right)[k^2(\boldsymbol{\omega}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*) - (\mathbf{k}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\omega}\ \mathbf{k}\ \boldsymbol{\phi}^*)] \\ &= \left(\frac{W^2}{k}\right)[(\boldsymbol{\varepsilon}\times\mathbf{k})\ \mathbf{k}\ (\boldsymbol{\omega}\times\boldsymbol{\phi}^*)] \\ &= \left(\frac{W^2}{k}\right)[(\mathbf{k}\cdot\boldsymbol{\phi}^*)(\mathbf{k}\ \boldsymbol{\omega}\ \boldsymbol{\varepsilon}) + (\mathbf{k}\cdot\boldsymbol{\omega})(\mathbf{k}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*)], \\ A_2 &= -4W(\mathbf{k}\cdot\boldsymbol{\phi}^*)(\mathbf{k}\ \boldsymbol{\omega}\ \boldsymbol{\varepsilon}), \\ A_3 &= -2\left(\frac{W^2}{q_0}\right)(\mathbf{k}\cdot\boldsymbol{\omega})(\mathbf{k}\ \boldsymbol{\varepsilon}\ \boldsymbol{\phi}^*), \end{aligned} \tag{B3}$$

which lead to

$$4kA_0 + WA_2 + 2q_0A_3 = 0 \tag{B4}$$

and the equation given in (76). Noting the relationship $W^2 + m^2 = 2Wq_0$, one may write down an explicitly covariant expression

$$(W^2 - m^2)^2A_1 + W^2A_2 + (W^2 + m^2)A_3 = 0 \tag{B5}$$

valid in any frame.

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