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Perturbation spectra from intermediate inflation

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We investigate models of 'intermediate' inflation, where the scale factor $a(t)$ grows as $a(t) = \exp(At^f)$, $0 < f < 1$, A constant. These solutions arise as exact analytic solutions for a given class of potentials for the inflaton ϕ. For a simpler class of potentials falling off as a power of ϕ they arise as slow-roll solutions, and in particular they include, for $f = \frac{2}{3}$, the class of potentials which give the Harrison-Zel'dovich spectrum. The perturbation spectral index n can be greater than unity on astrophysical scales. It is also possible to generate substantial gravitational waves while keeping the scalar spectrum close to scale invariance; this latter possibility performs well when confronted with most observational data.

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I. INTRODUCTION

Power-law and exponential inflationary universes are well studied. Exact solutions exist in both cases and they are created by exponential and constant scalar field potentials respectively [1, 2]. Exact solutions can also be found for "intermediate" inflationary universes in which the scale factor expands as [3]

$$a(t) = \exp\left(At^f\right), \quad 0 < f < 1, \;\; A > 0 \quad \text{constants.} \tag{1}$$

These models possess an array of interesting properties, particularly with regard to the perturbation spectra they generate, and we shall here present some of these properties.

The $k = 0$ Friedmann universe containing a scalar field ϕ with potential $V(\phi)$ obeys the equations ($8\pi G = c = \hbar = 1$)

$$3H^2 = \dot{\phi}^2/2 + V(\phi), \tag{2}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V' \tag{3}$$

where $H = \dot{a}/a$ is the Hubble parameter, and throughout overdots indicate derivatives with respect to time and primes are derivatives with respect to ϕ.

An exact solution of Eqs. (2) and (3) of the form of Eq. (1) exists [3] with

$$V(\phi) = \frac{8A^2}{(\beta+4)^2}\left[\frac{\phi}{(2A\beta)^{1/2}}\right]^{-\beta}\left[6 - \frac{\beta^2}{\phi^2}\right] \tag{4}$$

where $\beta = 4(f^{-1} - 1)$ and

$$\phi = \left(2A\beta t^f\right)^{1/2}. \tag{5}$$

For later use, we note that this allows one to write

$$H(\phi) = Af(2\beta A)^{\beta/4}\phi^{-\beta/2}. \tag{6}$$

The potential which gives rise to this solution is shown in Fig. 1. It is negative for $0 < \phi^2 < \beta^2/6$, increases up to a maximum at $\phi^2 = \beta(\beta + 2)/6$ and then falls asymptotically to zero as $1/\phi^\beta$ as $\phi \to \infty$. The solution exists anywhere on this potential for $\phi > 0$, by choice of the appropriate initial velocity for ϕ. In particular, to the left of the maximum of the potential the field must be given a rapid velocity in order to cross the maximum and reach the far side of the potential.

Note that although the solution is valid anywhere on the potential, it is not always inflating. In fact, the condition for inflation ($\ddot{a} > 0$) is only satisfied when $\phi^2 > \beta^2/2$, which guarantees that we must be in the region of the potential where it is positive. For $\beta > 1$ (i.e., $f < 4/5$), inflation can only occur beyond the maximum of the potential.[1]

This form for $a(t)$ also arises when one solves the equations of motion in the slow-roll approximation (see below) with a simple power-law potential

$$V(\phi) = \frac{48A^2}{(\beta+4)^2}(2A\beta)^{\beta/2}\,\phi^{-\beta}. \tag{7}$$

Such a potential bears qualitative similarity to the exponential potentials of power-law inflation. Here we note that the solutions for $\phi(t)$ and $H(\phi)$ obtained for Eq. (7) in the slow-roll approximation are identical to those obtained in the exact solution, Eqs. (5) and (6), and we shall exploit this later.

In some ways the slow-roll solution is more interesting than the exact solution. In particular it arises from a much simpler form of the potential, requiring only its asymptotic properties. It also possesses one rather cu-

[1] This raises an issue about the use of exact solutions — one must be wary to use them only when they serve as attractors for the system. In this sense, the inflation that one can gain for $4/5 < f < 1$ to the left of the maximum must be regarded as only a curiosity; the initial condition which serves to fire the scalar field up the potential and over the maximum is certainly not typical and the generic behavior would be to roll down the left hand side of the potential. Contrarily, the inflation as the potential rolls down to the right of the maximum is generic.

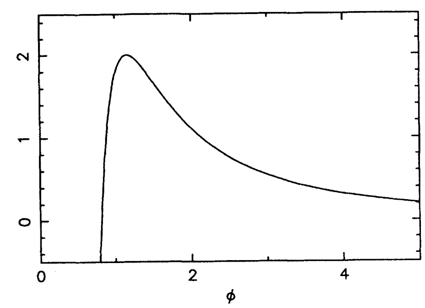

FIG. 1. The potential which gives exact intermediate inflation, for sample parameter choices $\beta = 2$ ($f = 2/3$) and $A = 1$. Generically, the maximum is located at $\phi^2 = \beta(\beta+2)/6$.

rious property, which is that no inflation occurs in the earliest stages of the scalar rolling down the potential. While $\phi^2 < \beta^2/2$, the potential is too steep. Only when the field reaches the asymptotic region of the potential can inflation begin. If one assumes, following the usual 'chaotic inflation' philosophy, that the initial scalar energy is at the Planck boundary, then inflation will always begin when the field reaches this value.

For intermediate inflation the slow-roll conditions become increasingly well satisfied with time and so, like power-law inflation, there is no natural end to inflation within the model. As with power-law inflation one expects this state of affairs to be remedied by modifications to the potential which create a minimum at a finite scalar field value. Intermediate inflation would then arise only in a region of the potential. Another possibility would be that, akin to Jordan-Brans-Dicke extended inflation [4] for power-law inflation, intermediate inflation could arise in the conformal frame of an extended inflation model and inflation could end via bubble nucleation. Examples of this in scalar-tensor gravity theories have been given by Barrow and Maeda [5] and Barrow [6].

Before progressing, we formalize what we mean by the slow-roll approximation. Noting that $2\dot{H} = -\dot{\phi}^2$, and assuming that during inflation $\dot{\phi}$ never passes through zero so that we may divide by it, substitution yields more useful forms of Eqs. (2) and (3). These are the Hamilton-Jacobi equations [7]

$$(H')^2 - \frac{3}{2}H^2 = -\frac{1}{2}V(\phi), \tag{8}$$

$$\dot{\phi} = -2H'. \tag{9}$$

In this formalism, it is possible to treat $H(\phi)$ as the fundamental quantity to be chosen, rather than the more usual $V(\phi)$ [8].

This formalism also allows a simple expression of the slow-roll conditions to be made, one which is more fundamental than the version commonly seen involving V'/V and V''/V. Define slow-roll parameters ϵ and η by[2]

$$\epsilon \equiv 3\,\frac{\dot{\phi}^2/2}{V + \dot{\phi}^2/2} = 2\left(\frac{H'}{H}\right)^2, \tag{10}$$

$$\eta \equiv -3\,\frac{\ddot{\phi}}{3H\dot{\phi}} = 2\frac{H''}{H}. \tag{11}$$

The slow-roll approximation is the assumption that both ϵ and $|\eta|$ are small, and they correspond to the ability to neglect the first term in Eq. (8) and in its ϕ derivative respectively. With these definitions, the condition for inflation to occur, $\ddot{a} > 0$, is *precisely* equivalent to $\epsilon < 1$.

II. PERTURBATION SPECTRA FROM INTERMEDIATE INFLATION

It has long been recognized that inflation typically gives rise to a spectrum of density perturbations close to the scale-invariant Harrison-Zel'dovich form [11]. Recent improved observations [12] require that deviations from this form be taken very seriously. Further, the possibility that large-angle microwave background anisotropies may have contributions not only from density perturbations, but also from gravitational wave modes [13] which are excited during inflation, must be taken into consideration [10, 14].

We shall simply quote standard results. The spectra of scalar and transverse-traceless tensor perturbations are given [15, 10] by the expressions

$$P_{\mathcal{R}}^{1/2}(k) = \left.\left(\frac{H^2}{2\pi|\dot{\phi}|}\right)\right|_{aH=k} = \left.\left(\frac{H^2}{4\pi|H'|}\right)\right|_{aH=k}, \tag{12}$$

$$P_g^{1/2}(k) = \left.\left(\frac{H}{2\pi}\right)\right|_{aH=k}, \tag{13}$$

where $\mathcal{R}$ is the perturbation in the spatial curvature. The expressions on the right are to be evaluated when the comoving scale k leaves the horizon during inflation. These results hold to first order in the slow-roll approximation, and we shall assume them throughout.[3] Useful quantities are the spectral indices, which are scale dependent in general. They can be calculated from the above to first order in the slow-roll parameters, as

$$n \equiv 1 + \frac{d\ln P_{\mathcal{R}}}{d\ln k} = 1 - 4\epsilon_* + 2\eta_*, \tag{14}$$

$$n_g \equiv \frac{d\ln P_g}{d\ln k} = -2\epsilon_*, \tag{15}$$

where n and n_g are the scalar and gravitational wave spectral indices respectively, and the star indicates that the slow-roll parameters should be evaluated when the appropriate scale passes outside the horizon during inflation (that is, at the value of the scalar field when the scale k leaves the horizon during inflation). The flat Harrison-Zel'dovich spectrum corresponds to $n = 1$.

The most important quantity concerning the gravi-

[2]These definitions, as employed in [9], differ from those used in [10].

[3]Recently, the next order corrections in slow roll to these expressions have been calculated [16], but are rather cumbersome and will not be needed here.

tational wave modes is the extent to which they influence large-angle microwave anisotropies [on small angular scales ($< 2°$), the gravitational waves would have been within the horizon at the time of last scattering and their redshifting would have reduced their significance]. Thus, we decompose the temperature fluctuation field into spherical harmonics:

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{l,m} a_{lm} Y^l_m(\theta,\phi). \tag{16}$$

In a given inflationary model, it is simple to calculate the contribution to the variances of the a_{lm}, which are m independent due to rotational invariance and can be denoted Σ_l^2. The general expressions are given as integrals over the power spectra, but in the slow-roll approximation, the relative contribution R_l of scalar and tensor modes is well approximated [10] by the simple expression

$$R_l \equiv \frac{\Sigma_l^2(\text{tensor})}{\Sigma_l^2(\text{scalar})} = 12.4\epsilon_l \tag{17}$$

where ϵ_l indicates that the slow-roll parameter is to be evaluated at the scale corresponding to the lth multipole, $k = lH_0/2$.

Details of all the above expressions can be found in [10]. If the slow-roll conditions are well satisfied, then the spectrum must be close to flat and the contribution from gravitational waves to the Cosmic Background Explorer (COBE) signal must be small.

The benefit of the $H(\phi)$ behavior being the same for both the exact solution and slow-roll solution is immediately apparent — they give rise to the same density perturbation spectra. Indeed, one need not ask what the origin of the particular form of $H(\phi)$ was. One can compute the slow-roll parameters during inflation; these are given by

$$\epsilon = \frac{\beta^2}{2\phi^2}, \tag{18}$$

$$\eta = \beta(1+\beta/2)\frac{1}{\phi^2}. \tag{19}$$

They therefore possess a mild scale dependence, whereas in power-law inflation from an exponential potential they would be constant in time.

The amplitude of the scalar spectrum will depend on the amplitude of the potential, and is to be fixed by observations such as COBE. More interesting is the scale dependence of the scalar spectrum. This is given from Eq. (14) by

$$n = 1 - \frac{\beta(\beta-2)}{\phi^2}. \tag{20}$$

Recall that $\beta = 2$ corresponds to $f = 2/3$.

This spectrum offers properties which are unusual in inflation, where the typical behavior (exhibited by polynomial chaotic inflation, power-law inflation, natural inflation, extended inflation, etc.) is for a spectrum with $n < 1$, so that a COBE normalized spectrum has reduced small-scale power compared to a similarly normalized scale-invariant spectrum. However, provided $0 < \beta < 2$ ($1 > f > 2/3$), the spectrum Eq. (20) offers a value of n greater than 1, albeit with a scale dependence we address below.

As one expects, the limit of exponential expansion ($f \to 1$) gives rise to a flat Harrison-Zel'dovich spectrum, though were one to compute the amplitude one would find it diverging in this limit. Much more interesting is the case $f = 2/3$. This too offers a flat spectrum, but now with finite (in fact freely selectable) amplitude. The potential therefore corresponds to that giving (in the slow-roll approximation) an exactly flat spectrum. Although this potential has arisen before on general grounds [17, 9], we are not aware of it having been identified as that giving intermediate inflation before.

The gravitational spectral index is only of modest interest; more significant is the relative contribution R_l of tensors and scalars to the COBE signal. This is just

$$R_l \simeq 12.4\epsilon_l = 6.2\beta^2/\phi^2 \tag{21}$$

where ϕ is evaluated at the time the scale corresponding to the lth multipole leaves the horizon. In general, the relative amplitude is related to the scalar spectral index by

$$n = 1 - \frac{\beta-2}{6.2\beta}R_l\,, \tag{22}$$

where n is given on the scale corresponding to the lth multipole.[4]

For typical choices of β, if we are on the part of the potential where the spectrum is close to flat, there are few gravitational waves, in keeping with the slow-roll conditions. However, for the exact Harrison-Zel'dovich case $\beta = 2$, the relative contribution depends simply on the value of ϕ when the relevant scales leave the horizon. In terms of the slow-roll parameters, one is arranging $\eta_* = 2\epsilon_*$, without requiring that either separately be small. As ϕ is only constrained by the inflation condition $\phi^2 > \beta^2/2$, the gravitational waves can in fact be the dominant contributor to COBE. This is also possible for other choices of β close to 2.

We get departures from "almost" scale invariance with negligible gravitational waves only for small values of ϕ. Thus, for cosmological interest, one needs to be on this part of the potential as large-scale structure scales leave the horizon during inflation. With standard reheating, this will correspond to around 60 e-foldings from the end of inflation. The number of e-foldings of intermediate inflation between scalar field values ϕ_1 and ϕ_2 is given by

$$N(\phi_1,\phi_2) = -\frac{1}{2}\int_{\phi_1}^{\phi_2}\frac{H}{H'}d\phi = \frac{1}{2\beta}\left(\phi_2^2-\phi_1^2\right). \tag{23}$$

Without knowing how inflation ends, one cannot draw any further conclusions from this, because the number of

[4]Compare with the power-law inflation result $n = 1 - R_l/6.2$. In that case n and R_l are both scale independent.

TABLE I. Examples of inflationary models giving rise to different types of prediction for the perturbation spectra.

	Small gravitational-wave contribution to COBE	Large gravitational-wave contribution to COBE
Nearly flat spectra	Polynomial chaotic inflation [18]	Intermediate inflation ($f \simeq 2/3$) [3]
Tilted spectra	Natural inflation [19]	Power-law inflation (small power) [2]

e-foldings to complete inflation is unknown.

Suppose that we assume inflation begins at the smallest ϕ value which permits it, so $\phi_1^2 = \beta^2/2$. As discussed after Eq. (7), this seems very reasonable in the slow-roll case. The spectral index and gravitational wave contributions can now be expressed in terms of the number of e-foldings N_b which have passed since the *beginning* of inflation:

$$n = 1 - \frac{2\beta - 4}{4N_b + \beta}, \tag{24}$$

$$R_l = \frac{12.4\beta}{4N_b + \beta}. \tag{25}$$

One expects that the total amount of inflation must exceed the 60 e-foldings mentioned above by a factor of at least a few[5] (one often sees 70 as the minimum number required to solve the flatness problem). This sets upper limits on the $|n-1|$ and R_l as a function of β. For example, for the Harrison-Zel'dovich spectrum case $\beta = 2$ this would imply that the gravitational wave contribution to COBE is subdominant ($R_l < 0.6$) though certainly not insignificant. Figure 2 illustrates the possibilities.

This also allows us to address the scale dependence of the spectral index — would one be able to see deviations from power-law behavior across observable scales? Potentially observable scales, those in the linear regime at the present, stretch from about $3000h^{-1}$ Mpc down to $8h^{-1}$ Mpc, which corresponds to about 6 e-foldings of scale. Putting this into the expressions above, we see that the typical scale dependence of n is rather limited, even in cases where the deviation from $n = 1$ is dramatic (notably large β).

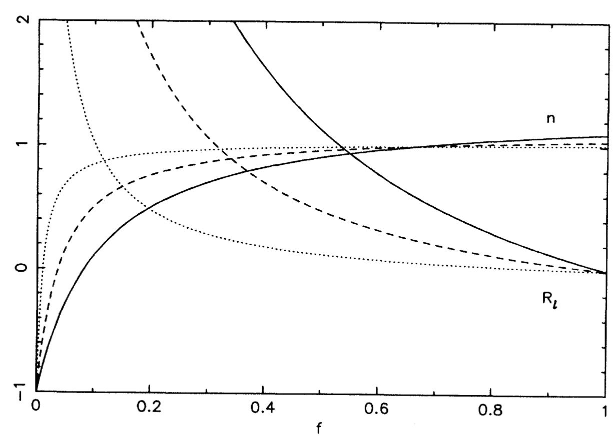

FIG. 2. The tilt n and gravitational wave contribution R_l are indicated as functions of f, with the number of e-foldings N_b since the start of inflation chosen as 10 (solid), 25 (dashed), and 100 (dotted). As N_b is increased, the predictions progressively approach $n = 1$ and $R_l = 0$. Note in particular the exact Harrison-Zel'dovich $f = 2/3$ case, which always gives $n = 1$ but can have gravitational wave contributions ranging from substantial to negligible depending upon N_b. The $f \to 1$ limit should be treated with care, as the relation between ϕ and N_b is singular in that limit as $H' \to 0$ in Eq. (23).

III. DISCUSSION

We have investigated models of intermediate inflation. Intermediate inflation arises as the slow-roll solution to potentials which fall off asymptotically as a power law in ϕ, and can be modeled by an exact cosmological solution. This model bears many qualitative similarities to power-law inflation: like power-law inflation, there is no natural end to inflation and a mechanism must be introduced in order to bring inflation to an end. Also, as with power-law inflation, intermediate inflation offers the possibility of density perturbation and gravitational wave spectra which differ significantly from the usual inflationary prediction of a nearly flat spectrum with negligible gravitational waves.

In particular, two interesting types of behavior can be obtained which do not arise in traditional inflationary models: (1) It is possible for the spectral index to exceed unity over large scale structure scales; (2) intermediate inflation contains the class of models which generate the exact Harrison-Zel'dovich spectrum (in the slow-roll approximation). Within this class, there exist models which produce substantial gravitational waves despite the flatness of the spectrum.

This latter case is of particular interest, because it completes a square of possible inflationary predictions for the tilt of the density spectrum and the influence of gravitational waves. This is illustrated in Table I. This model would perform well on confrontation with large-scale structure data in a cold dark matter model, as the gravitational wave contribution could explain the unexpectedly large amplitude of the COBE result, especially should the true result prove to lie towards the lower end of the COBE range. The most troublesome data would be the clustering data such as the APM survey [20]; one would have to resort to astrophysical effects (e.g., cooperative galaxy formation [21]) in order to explain this.

However, we must remark that the models which give these deviations from the usual predictions are rather special, in that we are relying on the minimum amount of inflation occurring so that large-scale structure scales cross the horizon at a time when the slow-roll conditions were still not well obeyed. This corresponds to a nontrivial constraint on when modifications to the po-

[5]Of course the total amount of inflation could be enormous, in which case we are guaranteed the slow-roll limit.

tential entered to end inflation. However, this is still quite a strong result because it implies that obtaining a nearly scale-invariant spectrum with significant gravitational waves, while possible in principle, is very difficult in practice because the scale invariance in the density perturbation spectrum implies that the relative contribution from gravitational waves falls as inflation proceeds.

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