

Quantum theory of black holes

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A solvable two-dimensional conformally invariant midisuperspace model for black holes is obtained by imposing spherical symmetry in four-dimensional conformally invariant Einstein gravity. The Wheeler-DeWitt equation for the theory is solved exactly to obtain the unique quantum wave functional for an isolated black hole with a fixed mass. By suitably relaxing the boundary conditions, a nonperturbative ansatz is obtained for the wave functional of a black hole interacting with its surroundings.

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One of the most important developments in field theory in the last two decades was the discovery of the quantum-mechanical instability of black holes due to Hawking radiation [1]. This discovery provided a tantalizing, and still poorly understood, link between two previously distinct branches of physics, namely, gravitation theory and thermodynamics [2]. In addition, questions surrounding the end point of black hole radiation have touched on the foundations of our understanding of both quantum mechanics and thermodynamics [3].

Most calculations of black hole radiation involve matter fields quantized on a classical curved background. Recently the back reaction of the quantum matter fields on the gravitational field has been studied semiclassically in a class of two-dimensional (2D) models which exhibit many features in common with 4D gravity [4]. Another interesting method [5] for studying black hole thermodynamics uses the Euclidean action for black holes to approximate functional integral expressions for the relevant thermodynamic partition functions. Unfortunately, although many interesting results have been obtained, neither approach has yet provided a resolution to the question of the end point of gravitational collapse, which appears to lie outside the realm of the validity of the semiclassical approximation.

The purpose of this Rapid Communication is to present a completely different and inherently nonperturbative approach to these issues. In particular, we study a two-dimensional conformally invariant midisuperspace model for black holes in which the gravitational field can be quantized exactly. The analysis of the exact quantum theory for such a model can in principle provide information about the validity of the semiclassical approximation, the significance of back-reaction effects, and ultimately the nature of the end point of decay by Hawking radiation. The model is obtained by imposing spherical symmetry in conformally invariant four-dimensional Einstein gravity [6]. It is important to stress that the 4D theory is classically equivalent to Einstein gravity so that the model in principle makes direct contact with physical, four-dimensional black holes. A semiclassical analysis has shown that the "matter fields" in the theory give rise to Hawking radiation with the usual tempera-

ture [6]. A related model has also been analyzed [7,8] using the methods of Ref. [4]. In what follows, the theory will be quantized exactly using techniques first applied by Henneaux [9] to Jackiw-Teitelboim 2D gravity [10].

We start from the classical action for a scalar field conformally coupled to gravity in four dimensions:

$$I^{(4)}[\phi, \hat{g}_{ab}] = \kappa \int d^4\hat{x} \sqrt{-\hat{g}} (\phi^2 \hat{R} + 6\hat{g}^{ab} \hat{\nabla}_a \phi \hat{\nabla}_b \phi), \quad (1)$$

where $\{a, b = 0, 1, 2, 3\}$ and $\kappa = 1/16\pi G$. Without loss of generality, we will normalize the vacuum expectation value of the scalar field to unity. Equation (1) is invariant under the conformal transformations: $\hat{g}_{ab} \rightarrow e^{2\sigma} \hat{g}_{ab}$ and $\phi \rightarrow e^{-\sigma} \phi$. The theory is equivalent to Einstein gravity classically, and (perturbatively) at the quantum level as well [11].

A midisuperspace model for black holes is obtained by imposing exact spherical symmetry with a four-metric:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + \lambda^2(x) d\Omega^2, \quad (2)$$

where the fields, including the matter field ϕ , are now functions only of $x^\mu = \{r, t\}$ and $d\Omega^2$ is the standard line element on the two-sphere with volume 4π . The reduced action

$$I^{(2)} = 24\pi\kappa \int d^2x \sqrt{-g} \left[\frac{1}{3} \tau R(g) + g^{\mu\nu} \nabla_\mu \tau \nabla_\nu \psi + \frac{1}{3} \frac{e^{3\psi}}{\sqrt{2\tau}} \right] \quad (3)$$

describes a two-dimensional, conformally invariant field theory with two "matter" fields $\tau := \frac{1}{2}\lambda^2\phi^2$ and $e^{3\psi} := \lambda\phi^3$ [12]. In terms of this parametrization, conformal transformations take $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$ and $\psi \rightarrow \psi - \frac{2}{3}\sigma$, while τ is invariant. The field equations obtained by varying Eq. (3) are equivalent to those obtained by imposing spherical symmetry on the equations obtained from the four-dimensional action (1) [13]. Consequently, Birkhoff's theorem in four dimensions guarantees that the two-dimensional theory is classically solvable. Up to diffeomorphisms and conformal transformations, there exists only a one-parameter family of solutions:

$$\begin{aligned} \tau &= \frac{1}{2}r^2, \\ e^{3\psi} &= r, \\ ds^2 &= -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2. \end{aligned} \tag{4}$$

These solutions describe black holes of mass m , with four-dimensional dilaton $\phi^2=1$.

The theory based on the action (3) was first used in [6] to derive an expression for Hawking radiation in a semi-classical approximation by computing the trace anomaly of the ‘‘matter’’ fields τ and ψ . Here we present the results of the exact quantization of the full theory, using the methods of Henneaux [9]. Details will be given elsewhere [15].

As with all diffeomorphism invariant theories [14], the Hamiltonian is, up to a surface term, a linear combination of constraints:

$$H = \int dr \left\{ \frac{1}{2G} \sigma \mathcal{G} + M \mathcal{F} + \Lambda \Pi_\beta \right\} + H_{\text{ADM}}, \tag{5}$$

where the Lagrange multipliers σ and M are related to the lapse and shift functions, and the constraints

$$\mathcal{F} = \alpha' \Pi_\alpha + \tau' \Pi_\tau - 2\Pi'_\alpha \approx 0, \tag{6}$$

$$\mathcal{G} = 2\tau'' - \alpha'\tau' - (2G)^2 \Pi_\alpha \Pi_\tau - \frac{e^\alpha}{\sqrt{2\tau}} \approx 0 \tag{7}$$

generate spatial diffeomorphisms, and time translations, respectively. The Arnowitt-Deser-Misner (ADM) energy [14] is

$$H_{\text{ADM}} = \frac{1}{2G} \int dr (\sigma\tau' - 2\sigma\alpha'\tau)'. \tag{8}$$

In the above, the primes denote differentiation with respect to r , and we have defined the fields $\alpha := 2\rho + 3\psi$ and $\beta := 2\rho - 3\psi$ where $e^{2\rho} = g_{11}$ represents the conformal mode of the two-metric in our parametrization. The field α is conformally invariant while the ‘‘pure (conformal) gauge’’ component β has disappeared from the Hamiltonian, as required. Π_α , Π_τ , and Π_β are momenta canonically conjugate to α , τ , and β , respectively. In this parametrization the generator of conformal transformations is simply Π_β and the conformal mode can be trivially eliminated without affecting the subsequent discussion.

It can easily be verified that for the solutions given in Eq. (4), $H_{\text{ADM}} = m/G$ [16]. Moreover, the generator of time translations given above is well defined for all configurations which approach (4) asymptotically.

As in Ref. [9] we avoid potential factor ordering problems associated with quadratic momentum constraints by first solving the constraints classically. The result is

$$\Pi_\alpha = \frac{1}{2G} Q(\alpha, \tau), \tag{9}$$

$$\Pi_\tau = \frac{1}{2G} \frac{2\tau'' - \alpha'\tau' - e^\alpha/\sqrt{2\tau}}{Q(\alpha, \tau)}, \tag{10}$$

where

$$Q := [(\tau')^2 + (C - \sqrt{2\tau})e^\alpha]^{1/2}. \tag{11}$$

The parameter C is a constant of integration that deter-

mines the allowed classical, static solutions: they are of the form Eq. (4) with $m = C/2$.

This completes the discussion of the essential classical features of the model. The quantum theory in the functional Schrödinger representation will now be constructed using the so-called ‘‘Dirac approach,’’ in which one first quantizes the theory in the unreduced configuration space, and then imposes the constraints as operator constraints on physical states. The states in the unreduced theory are arbitrary functionals $\psi[\alpha, \tau]$ of the fields $\alpha(r)$ and $\tau(r)$. Conjugate momenta are defined as functional derivatives:

$$\begin{aligned} \hat{\Pi}_\alpha &:= -i\hbar \frac{\delta}{\delta\alpha(r)}, \\ \hat{\Pi}_\tau &:= -i\hbar \frac{\delta}{\delta\tau(r)}. \end{aligned} \tag{12}$$

These operators are formally self-adjoint with respect to the inner product [17]:

$$\langle \psi | \psi \rangle := \int \prod_r [d\alpha(r)][d\tau(r)] \psi^*[\alpha, \tau] \psi[\alpha, \tau]. \tag{13}$$

In the Dirac approach physical states are functionals $\psi[\alpha, \tau]$ that obey the operator constraints, which now take the form

$$\begin{aligned} -i\hbar \frac{\delta}{\delta\alpha(r)} \psi[\alpha, \tau] &= \frac{1}{2G} Q(\alpha, \tau) \psi[\alpha, \tau], \\ -i\hbar \frac{\delta}{\delta\tau(r)} \psi[\alpha, \tau] &= \frac{1}{2G} \frac{2\tau'' - \alpha'\tau' - e^\alpha/\sqrt{2\tau}}{Q(\alpha, \tau)} \psi[\alpha, \tau]. \end{aligned} \tag{14}$$

These equations can be functionally integrated (see Ref. [15]) to yield the unique (up to total divergences) physical state in the theory:

$$\Psi[\alpha, \tau] = \exp \left[\frac{i}{m_{\text{Pl}}^2} \int dr \left\{ Q + \frac{\tau'}{2} \ln \left[\frac{\tau' - Q}{\tau' + Q} \right] \right\} \right], \tag{15}$$

where $m_{\text{Pl}} = \sqrt{\hbar G}$ is the Planck length.

This solution, which is one of the main results of this paper, has several interesting properties: It is invariant under spatial diffeomorphisms, and (trivially) under conformal transformations. In addition, $\psi=1$ for the classical solution in Eq. (4). Moreover, classically forbidden field configurations which have imaginary momenta [$Q^2 < 0$: cf. Eq. (10)] yield wave functions whose amplitudes are exponentially damped. Finally, we note that if the fields $\alpha(r)$ and $\tau(r)$ obey suitable boundary conditions as $r \rightarrow \infty$, namely $\tau(r) \rightarrow \frac{1}{2}r^2[1 + O(1/r^2)]$ and $\alpha(r) \rightarrow r + 2m + O(1/r)$, then the state Eq. (15) is an eigenstate of the ADM Hamiltonian with eigenvalue $m/G = C/2G$. These boundary conditions effectively restrict consideration to fields configurations with classical ADM energy equal to m/G .

As usual in quantum gravity, physical interpretation of the wave function Eq. (15) requires considerable care. For one thing, since the inner product on the Hilbert space is given by a functional integral it is not obvious that the state is normalizable (even after all field configurations related by spatial diffeomorphisms are factored out). However, since it is the only state in the phys-

ical Hilbert space, we will assume that its associated probability amplitude does in principle contain information about relative probabilities of quantum mechanically allowed field configurations.

Another important point is the fact that the quantum theory as constructed contains no physical degrees of freedom, and hence no unconstrained observables. The solution as given above therefore cannot directly yield information about Hawking radiation, or gravitational collapse. In order to gain insight into these questions, it is necessary to know how matter can ultimately be incorporated into the model. In the following we will present an ansatz that attempts to mimic the effect of interactions with its surroundings by putting the black hole in a box of radius $R \gg m$, and relaxing the boundary conditions on the fields to include configurations that have ADM energy $M \neq m$. The wave functional Eq. (15) is still a solution of the quantum constraints Eq. (14). For simplicity we neglect local fluctuations and restrict consideration to fields of the form

$$\begin{aligned} \tau &= \frac{1}{2}r^2, \\ e^\alpha &= \frac{r^2}{r-2M}, \end{aligned} \tag{16}$$

with support only in the region $r > r_0$, where $r_0 \ll 2m$. These fields correspond to black holes of mass M , expressed in Schwarzschild coordinates. For fixed $C = 2m$, they do not correspond to classical solutions unless $M = m$. We will now evaluate the wave functional Eq. (15) for these configurations. The quantum state is therefore a function of the single variable M . As one might anticipate, by relaxing the boundary conditions so as to allow exchange of energy with an external source, we have abandoned the self-adjointness of the Hamiltonian: its action takes states out of the physical Hilbert space. Nonetheless we will see that the resulting wave function has some interesting properties.

The physically relevant information in the wave function [18] is contained in the (unnormalized) probability amplitude $P[M] := \exp\{- (2/\hbar)\text{Im}S[M]\}$. For $M > m$,

$$P[M] = \exp\left\{-\frac{3}{4m_{\text{Pl}}^2}(2M-2m)^{3/2}\sqrt{R} + O(m/R)\right\}, \tag{17}$$

and, for $0 \leq M < m$,

$$\begin{aligned} P[M] = \exp\left\{-4\frac{m^2}{m_{\text{Pl}}}\left[\frac{\pi}{2} - \arctan\left(\frac{M}{m-M}\right)\right]^{1/2} \right. \\ \left. - \frac{M^2}{m^2}\left(\frac{m-M}{M}\right)^{1/2} \right. \\ \left. + \frac{1}{2}\frac{M(2m-2M)}{m^2}\left(\frac{m-M}{M}\right)^{1/2}\right\}. \end{aligned} \tag{18}$$

Figure 1 contains graphs of the probability amplitude $P[M]$ for different values of the classical mass m . The simplifications made above yield a probability amplitude with remarkable properties.

(i) The amplitude is finite and well behaved in the limit that $r_0 \rightarrow 0$. The classical curvature singularity has disappeared from the quantum amplitude.

(ii) The amplitude is continuous and smooth (with zero slope) at $m = M$, as long as $R \neq \infty$.

(iii) The amplitude is peaked at the classical mass, and the width decreases rapidly with increasing $m > m_{\text{Pl}}$.

(iv) The relative probability of configurations with mass $M > m$ (compared to $M = m$) is exponentially suppressed, with exponent proportional to the spatial volume.

(v) The relative probability of configurations with mass $M = m$ and $M = 0$ is

$$\frac{P[m]}{P[0]} = \exp\left\{2\pi\frac{m^2}{m_{\text{Pl}}}\right\}. \tag{19}$$

It is interesting to note that if one interprets this relative probability thermodynamically in terms of the number of (equally probable) microstates with mass m (assum-

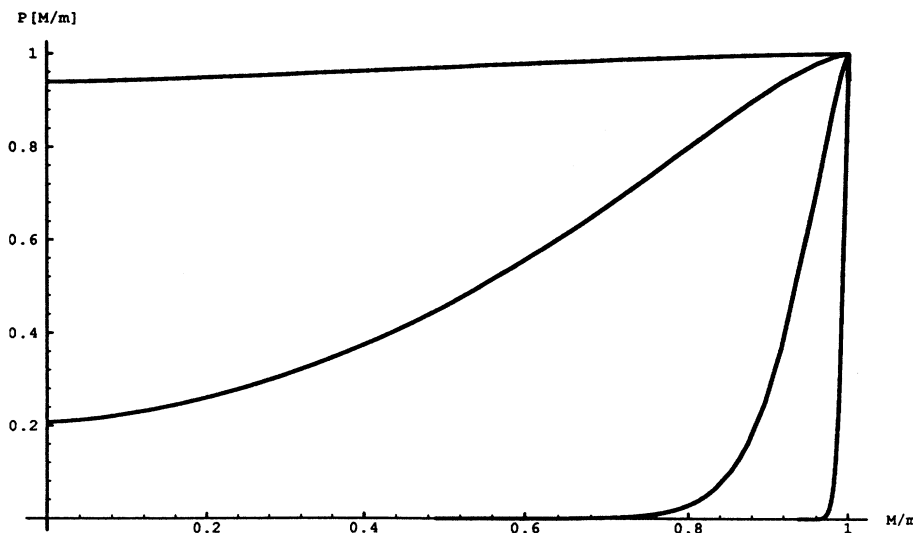


FIG. 1. Plot of the (unnormalized) probability amplitude $P[M/m]$ for black holes of mass $m/m_{\text{Pl}} = \{0.1, 0.5, 2, 10\}$, showing the sharp decrease in width as m increases.

ing a unique zero mass state), then one gets an expression for entropy for a black hole of mass m : $S = k \ln(P[m]/P[0]) = 2\pi m^2/m_{\text{pl}}^2$. This is a factor of 2 smaller than the standard value for the entropy of a black hole.

Although Eqs. (17) and (18) and their interpretation are highly speculative, the results outlined above seem to suggest that 2D models may provide a nonperturbative basis for the study of black hole radiation. In order to answer questions concerning the end point of gravitational collapse, it is of course necessary to understand how to incorporate matter self-consistently into the model. It is also important to establish whether other 2D models

(such as the one used in Ref. [4]) can be quantized using these techniques. Finally, one would like to have a better understanding of the derived probability amplitude and its physical interpretation in the context of quantum gravity. These questions are currently under investigation.

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- [17] An interesting feature of the present model is the fact [15] that the choice of functional measure in Eq. (13) has no effect on the final probability amplitude described by physical quantum states as long as the momentum operators are formally self-adjoint with respect to that measure.
- [18] It is shown in [15] that the measure induced from Eq. (13) on the space of M is trivial, up to formally infinite contributions that can be regulated to zero.