PHYSICAL REVIEW D

PARTICLES, FIELDS, GRAVITATION, AND COSMOLOGY

THIRD SERIES, VOLUME 47, NUMBER 2 15 JANUARY 1993

RAPID COMMUNICATIONS

Rapid Communications are intended for important new results which deserve accelerated publication, and are therefore given priority in editorial processing and production. A Rapid Communication in **Physical Review D** *should be no longer than five printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but because of the accelerated schedule, publication is generally not delayed for receipt of corrections unless requested by the author.*

Sachs-Wolfe effect in a local and gauge-invariant formalism

J. C. R. Magueijo

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

(Received 22 July 1992)

A formula relating the cosmic microwave background temperature anisotropies and the cosmological perturbations responsible for them is derived in a local and gauge-invariant formalism.

PACS number(s): 98.80.Hw

I. INTRODUCTION

With the recent observation of anisotropies in the cosmic microwave background radiation (CMBR) [1,2] it has become of paramount importance to work out which CMBR distortion patterns are predicted by each theory of structure formation. Finding what is what in the CMBR fingerprints will ultimately decide which structure formation theories are viable and which are not.

Cosmological perturbation theory is the underlying language in which these theories are expressed and important technical and conceptual problems arise in setting up frameworks conveniently adapted to each theory of structure formation. For instance, theories based on quantum-produced fluctuations in inflationary scenarios normally make use of the Bardeen gauge-invariant formalism ([3], also [4–7]), in which the technical aspects of the theory take their simplest form and the gauge problem is overcome. Unfortunately, Bardeen's formalism is nonlocal (since it makes use of the geometrical splitting) and therefore its variables are misleading whenever the perturbations are well localized. Generally, this happens in theories with non-Gaussian statistics, the most blatant case being theories based on topological defects. A framework which is both local and gauge invariant then becomes desirable, and it has indeed been set up [9–11, 8]. The field equations are more complicated in it, but the work in [8], bridging the two formalisms, enables the calculations to be done in Bardeen's theory, the more physical local and gauge-invariant variables eventually being derived from them.

It is naturally of extreme importance to find formulas linking the CMBR anisotropies to the cosmological perturbations that generated them, in each of these formalisms. Such formulas were initially derived in a gauge-dependent formalism [12], and more recently have been derived for the Bardeen theory, in phase space for all types of perturbations [13], and in configuration space for scalar perturbations [14]. In this paper, we briefly review and complement this earlier work, and work out an analogous formula for the local and gauge-invariant formalism developed in [8]. In Sec. II the gauge invariance of several measures of the anisotropy is discussed and the general setting of the problem presented. In Sec. III a formula for the anisotropy using the gauge-dependent variables is given. We spell out some of the steps in the derivation in [12], and pay particular attention to the problem of the perturbed photon spectrum. In Sec. IV an original derivation for the gauge-invariant formula for the anisotropy is given, the results of [14] being extended to cover the case of all types of perturbations described in configuration space. Finally in Sec. V the results of Sec. IV are recombined so as to produce a formula for the anisotropy in terms of the local and gauge-invariant variables defined in [8], and we briefly discuss the practical and conceptual superiority of the formula obtained.

In order to keep review material to a minimum we refer the reader to [8] for a full explanation of the notation used in Secs. III, IV, and V. Throughout this paper units are used in which $G=c=\hbar=1$.

II. THE GAUGE INVARIANCE OF THE CMBR TEMPERATURE ANISOTROPIES

Finding a gauge-invariant formula for the CMBR temperature fluctuation $\delta T/T$ involves addressing the issue of the gauge-invariance of $\delta T/T$ itself. This has been the source of considerable confusion in the literature. Both [12] and [14] failed to recognize that even though $\delta T/T(\mathbf{x},\tau,\theta,\phi)$ (the CMBR temperature measured by an observer at point $\mathbf{x}$ and time τ in the direction defined by the polar angles θ and ϕ) is observable, it is a gauge-dependent quantity. In fact, $\delta T/T(\mathbf{x},\tau,\theta,\phi)$ has a time-dependent background contribution and, consequently, under a gauge transformation of the form

$$x^{\mu}\rightarrow x^{\mu}+\xi^{\mu} \tag{1}$$

it transforms as

$$\Delta_g\frac{\delta T}{T}=\frac{1}{T}\mathcal{L}_{\xi}T=\frac{\dot{T}}{T}\xi^0=-\frac{\dot{a}}{a}\xi^0\neq 0\ . \tag{2}$$

This example makes the point that, contrary to the statements made in [14], the measurability of a quantity does not imply its gauge invariance. In fact, one normally associates a measurable quantity with a field of observers whose space-time indexing can change under gauge transformations. It follows that $\delta T/T$ can never be written solely in terms of gauge-invariant perturbation variables and hence it is not a good quantity to look at in a gauge-invariant formalism. One should rather look at quantities which are both observable and gauge invariant such as the temperature anisotropy

$$\Delta\frac{\delta T}{T}(\tilde{\theta},\tilde{\phi},\theta,\phi)=\frac{\delta T}{T}(\tilde{\theta},\tilde{\phi})-\frac{\delta T}{T}(\theta,\phi) \tag{3}$$

or any of its differential versions, such as

$$^{(2)}\Delta\frac{\delta T}{T}=\left[\partial_\theta^2+\frac{1}{\sin^2\theta}\partial_\phi^2\right]\frac{\delta T}{T} \tag{4}$$

or

$$\aleph=\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi\partial\theta}\right]\frac{\delta T}{T}\ . \tag{5}$$

In this paper we will seek formulas with the structure

$$\mathcal{M}\left[\frac{\delta T}{T}\right]=A|_i+B|_f+\int_i^f d\lambda C\ , \tag{6}$$

where $\mathcal{M}$ is some function of $\delta T/T$ and A, B, and C are cosmological perturbation variables defined at the last scattering surface ($|_i$), at the observation event ($|_f$), and on the null geodesic connecting these two events (λ being an affine parameter on it), respectively. By making $\mathcal{M}$, say $\Delta\delta T/T(\tilde{\theta},\tilde{\phi},\theta,\phi)$, one obtains a quantity which is invariant under gauge transformations applied on any equal-time hypersurface (and not only on the observation event hypersurface), and therefore A, B, and C can be made gauge invariant. One will also see that by choosing $\mathcal{M}$ to be $\aleph$, one can make A, B, and C functions of the local and gauge invariant variables introduced in [9–11] and [8].

III. A FORMULA FOR THE SACHS-WOLFE EFFECT IN TERMS OF GAUGE-DEPENDENT VARIABLES

We first derive an expression for $\delta T/T$ in terms of gauge-dependent quantities. In principle what one should do is to integrate a Liouville equation in curved space-time and obtain the perturbed spectrum of photon energies observed in a specific direction. However, photons are essentially collisionless and non-self-gravitating after last scattering, and so the whole physical content of the Liouville equation is in the geodesic equation for a background metric fixed by the matter. Therefore we will simply integrate the geodesic equation for a generic photon energy.

The photon energy distribution at last scattering is assumed to be thermal, or more precisely, a Bose-Einstein distribution with $\mu=0$. The zeroth-order cosmological expansion redshifts the temperature like $T=T_0/a$, leaving the spectrum undistorted, and with $\mu=0$. Now suppose that the energy shift caused by the cosmological perturbations acting on photons traveling from the last scattering surface to the observation event takes the form

$$\delta E(E)=c+dE$$

where c and d are small numbers of the same order of magnitude. Then, to first order, the thermal spectrum will be preserved, with the new parameters

$$\tilde{T}=T(1+d)\ , \tag{7}$$

$$\tilde{\mu}=\mu(1+d)+c\ . \tag{8}$$

In this section we will see that the linearized cosmological perturbations always induce δE with $c=0$. Therefore, they do not bring about any chemical potential perturbation, and the temperature fluctuation has the form

$$\frac{\delta T}{T}=\frac{\delta E}{E}\ , \tag{9}$$

where $\delta E/E$ can be evaluated for photons with any zeroth-order energy.

So let us consider a universe filled with matter with four-velocity

$$u^M=[(1-A)\partial_\tau+v_L^{Mi}\partial_i]/a \tag{10}$$

and a radiation component which becomes collisionless after a specific time (the "last scattering time," denoted by $|_i$). We will follow individual photons generically characterized by the energy-momentum four-vector

$$k^{\mu}=\frac{\nu}{a^2}(n^{\mu}+\delta n^{\mu}) \tag{11}$$

with $n^{\mu}=(1,\mathbf{n})$ and $|\mathbf{n}|^2=1$. Parametrizing the perturbed metric as

$$\begin{aligned}ds^2&=a^2(\tau)(\eta_{\mu\nu}+h_{\mu\nu})dx^{\mu}dx^{\nu}\\&=a^2(\tau)\{-(1+2A)d\tau^2+2B_i^L dx^i d\tau\\&\qquad+[(1+2H_L)\gamma_{ij}+2H_{Tij}^L]dx^i dx^j\}\ ,\end{aligned} \tag{12}$$

we find that the energy measured by an observer (attached to the matter) is

$$E=-k\cdot u^M|_f=\frac{\nu}{a}(1+A+\delta n^0-n^k v_k^{ML}-n^k B_k^L)|_f \ . \tag{13}$$

Now write

$$\delta n^0|_f=\delta n^0|_i+\int_i^f d\lambda \frac{d\delta n^0}{d\lambda} \tag{14}$$

with $d\lambda=n^\mu dx_\mu$, and notice that if

$$k^\mu=\frac{dx^\mu}{d\lambda},\quad \frac{Dk^\mu}{d\lambda}=0,\quad k^2=0 \ , \tag{15}$$

then $\hat{D}\hat{k}^\mu/d\hat{\lambda}=0$, with $\hat{k}^\mu=dx^\mu/d\hat{\lambda}=\Omega^{-2}k^\mu$ and $\hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$. Applying this theorem with $\Omega^2=a^{-2}$ and performing an affine transformation setting the zeroth-order energy of the photon, we can derive the result

$$\delta n^0|_i^f=\int_i^f d\lambda\frac{d\delta n^0}{d\lambda}=n^\mu h_{\mu 0}|_i^f-\frac{1}{2}\int_i^f d\lambda n^\mu n^\nu \dot{h}_{\mu\nu} \ . \tag{16}$$

Next note that the photons are thermalized in the free falling rest frame of the matter at last scattering, where the observed photon energy is given by $E=-k\cdot u^M|_i$. An intrinsic photon energy density fluctuation (taking equal values in the cosmological and matter frames) may exist, and according to the Stephan-Boltzmann equation it is related to a temperature fluctuation as $\delta T/T=\frac{1}{4}\delta^\gamma$. Thus each photon suffers an energy shift such that $\delta E/E=\delta T/T$, and so

$$\frac{1}{4}\delta^\gamma|_i=(A+\delta n^0-n^k v_k^{LM}-n^k B_k^L)|_i \ . \tag{17}$$

Combining (13), (16), and (17) we finally conclude that

$$\frac{\delta E}{E}=\frac{\delta T}{T}=\left[A+\frac{1}{4}\delta^\gamma\right]|_i-n^k v_k^{LM}|_i^f -\frac{1}{2}\int_i^f d\lambda n^\mu n^\nu \dot{h}_{\mu\nu} \ , \tag{18}$$

$$\delta\mu=0 \ . \tag{19}$$

In closing, note that the absence of a chemical potential perturbation is a generic nonlinear feature of the Sachs-Wolfe effect [the line integral in (18)]. It relates to the invariance of the geodesic equation under affine transformations, which, in turn, is related to the equivalence principle as applied for photons. It is conceivable that the experimental bounds on $\delta\mu$ may be used as a test for the equivalence principle, if one can assume that no chemical potential perturbation exists on the last scattering surface. We will elaborate on this comment elsewhere [15].

IV. A FORMULA FOR THE SACHS-WOLFE EFFECT IN TERMS OF BARDEEN VARIABLES

Equation (18) can be used to produce a formula for $\Delta\delta T/T(\tilde{\theta},\tilde{\phi},\theta,\phi)$ in terms of gauge-dependent variables. This can be done by adding to it

$$0=-\int_i^f d\lambda\frac{d\alpha}{d\lambda}+\alpha|_i^f \tag{20}$$

with

$$\alpha=\dot{\sigma}-n^k(D_k\dot{H}^T+2\dot{H}_k^T) \tag{21}$$

and subtracting the result for two arbitrary directions of observation (θ,ϕ) and $(\tilde{\theta},\tilde{\phi})$ [14,13]. The most linear way of identifying the gauge-invariant expressions so obtained consists in using the longitudinal-vector gauge ($B=H_T=H_T^i=0$) for which $\alpha=0$ and $\Psi=A$, $\Phi=H_L$, $D_s^\gamma=\delta^\gamma$, $V^M=v^M$, $V_i^M=v_i^M$, and $\Psi_i=B_i$, and in which (18) becomes

$$\Delta\frac{\delta T}{T}(\tilde{\theta},\tilde{\phi},\theta,\phi)=\left[\left[\frac{1}{4}\delta^\gamma+A\right]|_i-n^k(v_k^M-D_k v^M)|_i^f -\int_i^f d\lambda(-\dot{A}+\dot{H}_L+n^k\dot{B}_k +n^k n^l\dot{H}_{kl}^T)\right]|_{\theta,\phi}^{\tilde{\theta},\tilde{\phi}} \ . \tag{22}$$

This leads straightforwardly to

$$\Delta\frac{\delta T}{T}(\tilde{\theta},\tilde{\phi},\theta,\phi)=\left[\left[\frac{1}{4}D_s^\gamma+\Psi\right]|_i-n^k(V_k^M-D_k V^M)|_i^f -\int_i^f d\lambda(-\dot{\Psi}+\dot{\Phi}+n^k\dot{\Psi}_k+n^k n^l\dot{H}_{kl}^T)\right]|_{\theta,\phi}^{\tilde{\theta},\tilde{\phi}} \tag{23}$$

which includes the scalar, vector, and tensor contributions to the anisotropies.

V. THE SACHS-WOLFE EFFECT IN A LOCAL AND GAUGE-INVARIANT FORMALISM

We will finally write a formula of the form (6), where A, B, and C are functions of local and gauge-invariant variables. As announced before, it is convenient to take the measure $\mathcal{M}$ to be the variable $\aleph$. This variable captures all the Tesseral harmonics ($l\neq|m|, m\neq 0$) of $\delta T/T$ and is therefore suitable for most practical purposes (note that the dipole, for instance, is left out, but in most discussions one subtracts it off anyway). Now notice that when the angular derivatives $\partial^2/\partial\hat{\phi}\,\partial\theta=(1/\sin\theta)\partial^2/\partial\phi\,\partial\theta$ act on a formula of the form of (6) they are converted into spatial derivatives transverse to n:

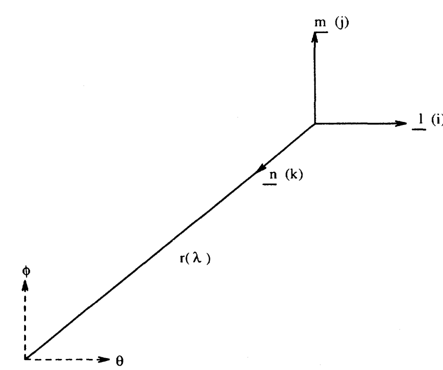

FIG. 1. Conventions used for the angles and tetrad indices.

$$\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi\,\partial\theta}\mathcal{M}\left[\frac{\delta T}{T}\right]=r^2\Delta_{ij}A|_i+r^2\Delta_{ij}B|_f+\int_i^f d\lambda r^2(\lambda)\Delta_{ij}C\ , \tag{24}$$

where we have used the conventions indicated in Fig. 1 for the indexing of the orthonormal frame induced at a point at coordinate distance $r(\lambda)$ by the angular system (θ,ϕ) at the observation point $\mathcal{O}$. Note also that

$$\partial_\theta \mathbf{n}=-\mathbf{l},\quad \partial_\theta \mathbf{l}=\mathbf{n},\quad \partial_\theta \mathbf{m}=0\ ,$$
$$\partial_{\hat\phi}\mathbf{n}=-\mathbf{m},\quad \partial_{\hat\phi}\mathbf{m}=\mathbf{n},\quad \partial_{\hat\phi}\mathbf{l}=0\ ,$$

and that therefore $\partial^2_{\hat\phi\theta}\mathbf{n}=0$. Then, applying

$$1/\sin\theta\frac{\partial^2}{\partial\phi\,\partial\theta}$$

to $\delta T/T$ as written in (23) and adding to it the expression

$$0=r^2\alpha_{ij}|_i^f-\int_i^f d\lambda\left[r^2\frac{d\alpha_{ij}}{d\lambda}-2r\alpha_{ij}\right]\ , \tag{25}$$

where we have used $dr^2/d\lambda=2rn^r=-2r$ and where we choose

$$\alpha_{ij}=-D_{(i}\dot\Psi_{j)}+\ddot H_{ij}^T-2D_{(i}(n^k\dot H^T_{j)k})+n^kD_k\dot H^T_{ij}\ , \tag{26}$$

one obtains, after a rather laborious calculation,

$$\aleph=\left[r^2\left[\frac{1}{4}\mathcal{D}^{s\gamma}_{ij}+\Psi_{ij}+2D_{(i}(n^kV^M_{j)k})-n^kD_kV^M_{ij}\right]\right]\Big|_i-2\int_i^f d\lambda r^2(n^k\varepsilon_{(i|k}{}^m\dot H_{|j)m}-\dot E_{ij})\ . \tag{27}$$

Let us justify the choice of α_{ij} employed in the derivation of (27). The reason why nonlocality sneaks into formula (23) lies on the integration by parts (20) required for making A, B, and C in (18) gauge invariant. Since α is not local the resulting gauge-invariant variables are necessarily nonlocal. Now, since

$$\Delta_{ij}\alpha+\alpha_{ij}=\dot K_{ij}-2D_{(i}(n^k\dot H^{TL}_{j)k})+n^kD_k\dot H^{TL}_{ij} \tag{28}$$

is local, one knows that gauge-invariance is achieved without breaking locality in the calculation of $\aleph$.

Formula (27) casts the generation of CMBR anisotropies by the cosmological perturbations in its clearest form: only local and gauge invariant quantities are involved. We find three types of contributions to $\aleph$ at the last scattering surface: $\mathcal{D}^{s\gamma}_{ij}$ represents the intrinsic photon energy density fluctuation, Ψ_{ij} represents a gravitational redshift effect, and the remaining terms are a Doppler shift effect from possible peculiar velocities of the matter at last scattering. Whenever the peculiar gravitational field is time dependent one has also to consider the line integral containing the electric and magnetic parts of the Weyl tensor (E_{ij} and H_{ij}, respectively): this is the so-called Sachs-Wolfe effect. In order to isolate this effect one can set $\mathcal{D}^{s\gamma}_{ij}=\Psi_{ij}=V^M_{ij}=0$ at last scattering, a statement which now respects both gauge-invariance and locality.

In conclusion, we stress that in addition to its practical importance, formula (27) also makes an important point of principle, as it casts the Sachs-Wolfe effect in a formalism which circumvents the gauge problem without giving away the essentially local nature of its underlying physics. Indeed locality and causality are somewhat hidden in the formulas derived in [13] and [14]. Consider, for instance, a cosmological perturbation confined inside a compact domain Ω (which may be the causal future of the birth of a seed, as is the case of compensated topological defects studied in [16] and [17]). In general, Bardeen's variables do not become trivial outside Ω and consequently the various geometrical contributions to $\delta T/T$ for a photon which has always been outside Ω may be nonvanishing. Obviously these geometrical contributions add up to zero, a somewhat contorted path to a physically sensible result which follows immediately from a formula for $\delta T/T$ in terms of local and gauge-invariant variables.

ACKNOWLEDGMENTS

I wish to thank A.-C. Davis and J. M. Stewart for discussion in connection with the subject of this paper. This work was financially supported by a grant from the Commission of the European Communities (Science Programme).

[1] G. Smoot *et al.*, Astrophys. J. **396**, L1 (1992).
[2] E. L. Wright *et al.*, "Interpretation of the CMB anisotropy detected by the COBE DMR," COBE Report, 1992 (unpublished).
[3] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[4] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[5] R. Durrer and N. Straumann, Helv. Phys. Acta **61**, 1027 (1988).
[6] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[7] J. M. Stewart, Class. Quantum Grav. **7**, 1169 (1990).
[8] J. C. R. Magueijo, Phys. Rev. D **46**, 3360 (1992).
[9] G. F. R. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989).
[10] G. F. R. Ellis, J. Hwang, and M. Bruni, Phys. Rev. D **40**, 1819 (1989).
[11] G. F. R. Ellis, J. Hwang, and M. Bruni, Phys. Rev. D **42**, 1035 (1990).
[12] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).
[13] M. Panek, Phys. Rev. D **34**, 416 (1986).
[14] R. Durrer, Phys. Rev. D **42**, 2533 (1990).
[15] J. C. R. Magueijo, "Testing the weak equivalence principle with the cosmic microwave background spectrum," Report No. DAMTP/92-52, 1992 (unpublished).
[16] J. C. R. Magueijo, Phys. Rev. D **46**, 1368 (1992).
[17] S. Veeraraghavan and A. Stebbins, Astrophys. J. Lett. (to be published).