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#### Is quantum spacetime foam unstable?

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A very simple wormhole geometry is considered as a model of a mode of topological fluctuation in Planck-scale spacetime foam. The quantum dynamics of the hole reduces to quantum mechanics of one variable, the throat radius, and admits a WKB analysis. The hole is quantum-mechanically unstable: It has no bound states. Wormhole wave functions must eventually leak to large radii. This suggests that stability considerations along these lines may place strong constraints on the nature and even the existence of spacetime foam.

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#### I. INTRODUCTION

Some 35 years ago Wheeler [1] made a remarkable suggestion, based on dimensional arguments: On Planck-length scales spacetime fluctuates quantum mechanically so randomly and violently that it develops all kinds of microscopic topological structures, such as “wormholes,” although on larger scales it appears smooth and simply connected. It is distressing that after so many years our knowledge of quantum gravity is still far from being able to confirm or disprove the existence of this “spacetime foam.”

Obstacles to analyzing this conjecture are apparent. It is well known [2] that a Lorentzian manifold must become singular or degenerate at points of topological change, or admit closed timelike paths. Formulation of field theory on such a manifold is plainly problematic. Such difficulties might be avoided by treating Euclidean manifolds [3]; unlike the Euclideanization of ordinary field theory, equivalent to the Lorentzian formulation in the sense of contour integration, Euclidean quantum gravity is physically different from Lorentzian. Indeed, maybe spacetime is intrinsically Euclidean on the Planck scale, characterized by Euclidean quantum foam, and  $(3+1)$ -dimensional Lorentzian spacetime only emerges

after a transition to a classical regime. However, Euclidean quantum gravity has fundamental difficulties of its own [4], most notably failure of the Euclidean action to be positive definite, the problem of interpretation, and recovery of Lorentzian spacetime.

Both Euclidean and Lorentzian versions of quantum foam have received much recent attention. The focus on the Euclidean version is on its possible role in determining fundamental constants [5], while for the Lorentzian version it has been suggested that a microscopic wormhole might be extracted from the foam to produce a traversable macroscopic wormhole or a time machine [6]. But though both sets of ideas are ingenious and have far-reaching consequences for other areas of physics, neither sheds much light on the actual existence or structure of the foam itself.

Here we present a simple analysis probing the stability of spacetime foam, to examine the constraints placed on its existence and structure by the apparent absence of topological structure to spacetime on macroscopic scales. We picture Lorentzian spacetime filled with many different sorts of microscopic wormholes, fluctuating into existence, living for microscopic time periods, and pinching off. At moments of birth and pinch-off of holes Euclideanization may or may not be needed; we do not treat

the actual points of topological change here.

Some of these structures are easily modeled classically: Wormholes can be constructed by excising a “world tube” from some  $(3+1)$ -dimensional spacetime and joining this to another such spacetime, with a corresponding excision [7]. These are extreme versions of situations in which the curvature in a wormhole throat is greater than that of the surrounding spacetime; here the curvature at the join or throat is a  $\delta$ -function distribution. This approximation is very useful for simplifying the dynamics. The stress-energy at the throat is determined by the Einstein field equations in the form of junction conditions [8]. This necessarily violates the weak energy condition, with negative energy density (in some reference frames) somewhere in the throat. This is not itself a fatal flaw of the wormholes [6,9]—while it might help account for the absence of macroscopic holes, it does not rule out the possibility of quantum, Planck-scale ones, nor does it guarantee that microscopic holes would not grow in size.

To make the analysis tractable we treat the simplest such wormhole: that obtained by excising a spherical region, with time-varying radius, from two Minkowski spaces and joining them [7]. The quantum-gravitational dynamics of the model reduces to quantum mechanics of a single degree of freedom, the throat radius, yielding a “minisuperspace model” for spacetime foam. The quantum wormhole is described by a wave function depending on that radius and time, as defined in the external flat spaces. If the wave function is localized about some Planck-scale radius at some initial time, what will be its subsequent evolution?

There is no standard approach to the quantization of a system like this. Evolution in a time coordinate defined by the Minkowski spaces external to the wormhole is at issue. Hence the familiar Dirac quantization procedure, giving rise to a time-independent wave function solving the Wheeler-DeWitt equation [10], is not suitable. Instead we impose the Hamiltonian constraint classically, using it to reduce the phase space of the system. We construct an action for the dynamics in the reduced phase space, and quantize the system by using this action in a Feynman path integral. The resulting propagator for wormhole wave functions is evaluated in a WKB approximation.

The results indicate that these wormholes are quantum-mechanically unstable: Though the classical evolution of the throat radius may be bounded, the quantum propagator admits no decomposition into contributions from any spectrum of bound and continuum states. Rather its behavior is akin to that of a “leaking” system, such as a particle confined by finite walls. This implies that wormhole wave functions must eventually “leak” to arbitrarily large throat-radius values. (Such quantum instability of a classically stable object is familiar, as in the particle case. So too in a gravitational context: Classically stable black holes are subject to Hawking evaporation.) The wormholes thus suggest a possible unstable mode of spacetime foam, microscopic topological structure growing eventually to macroscopic size. Numerical calculations of wave-function evolution show that the time scale of this instability might be very long, in terms of the

Planck scales appropriate to the model—though perhaps not on scales of observational significance.

This simple analysis thus points up a line of inquiry potentially of great significance. If more detailed, comprehensive analyses substantiate the existence of an unstable mode, then that together with the observed absence of macroscopic wormholes might indicate that spacetime does not possess microscopic topological structure, of Lorentzian signature, after all. Lorentzian spacetime foam could be inconsistent with known gravitational and quantum theory and observation.

## II. WORMHOLE QUANTUM MECHANICS

A classical, spherically symmetric “Minkowski wormhole” [7] is constructed by excising a sphere of radius  $r = R(t)$ , with  $R$  some function of a Minkowski time coordinate  $t$ , from two copies of Minkowski spacetime, identifying the two boundary surfaces  $r = R(t)$ , and incorporating an appropriate surface-layer stress-energy on the boundary to satisfy the Einstein field equations. Off the boundary both exterior spacetime regions are flat and empty, so the field equations are satisfied trivially. On the boundary, now the throat of the wormhole, the Einstein equations are equivalent to the junction conditions [8]

$$S_j^i = \frac{1}{8\pi} [K_j^i - \delta_j^i K_m^m], \quad (1)$$

where  $S_j^i$  is the surface stress-energy tensor and the right-hand side is the discontinuity in the extrinsic curvature  $K_j^i$ , minus its trace  $K_m^m$ , across the boundary. (Units with  $G = 1$ , as well as  $\hbar = c = 1$ , are used throughout.) For this wormhole geometry the junction conditions take the form

$$S_{\tau\tau} = -\frac{1}{2\pi R} \frac{1}{(1-\dot{R}^2)^{1/2}} \quad (2a)$$

and

$$S_{\theta\theta} = \frac{1}{4\pi} \left[ \frac{R}{(1-\dot{R}^2)^{1/2}} + \frac{R^2 \ddot{R}}{(1-\dot{R}^2)^{3/2}} \right], \quad (2b)$$

where overdots denote derivatives with respect to Minkowski-coordinate time  $t$  (in a frame in which the boundary sphere expands or contracts but does not translate), and the boundary coordinates  $\tau$  and  $\theta$  are proper time, related to coordinate time via  $d\tau = (1-\dot{R}^2)^{1/2} dt$ , and polar angle, respectively. These give the classical equation of motion for the wormhole, once an equation of state relating the surface density  $\sigma = S_{\tau\tau}$  and pressure  $p = S_{\theta\theta}/R^2$  of the matter on the throat is specified.

The equation of state could be chosen to make the equation of motion simple. For example, the choice  $p = -\sigma/2$  would imply  $\dot{R} = 0$ . The quantization of the system thus described is trivial: The wave function evolves as that of a free particle. A wave function initially concentrated about some  $R$  value will disperse to infinity. However, we do not expect such a wormhole, which evolves classically with its throat radius either

fixed, or expanding or collapsing linearly, to correspond to those fluctuating into existence in spacetime foam.

Instead we choose an equation of state such that the equation of motion describes expansion from zero radius to some maximum value and recollapse. The classical behavior of the model thus accords with that, e.g., of a Schwarzschild wormhole, and that expected of a foamlke fluctuation. Specifically, we use

$$p = -\sigma/4, \quad (3)$$

which yields

$$2R\ddot{R} - \dot{R}^2 + 1 = 0. \quad (4)$$

The solutions of this equation are parabolic trajectories:

$$R_{cl}(t) = \frac{1}{\alpha} \left[ 1 - \frac{\alpha^2(t-t_0)^2}{4} \right], \quad (5)$$

where  $\alpha$  and  $t_0$  are constants.

The quantum dynamics of the wormhole can be described via a Feynman path integral. An action corresponding to Eq. (4), obtained from the integral of the scalar curvature of the wormhole geometry, is

$$S = \int \left[ R\dot{R} \ln \left| \frac{1+\dot{R}}{1-\dot{R}} \right| - 2R \right] dt. \quad (6)$$

Reduced to the single dynamical variable  $R$ , the system resembles a point particle in one dimension, with a complicated “kinetic term” in the action. (In this respect it is similar to a relativistic free particle [11].) The wormhole is described by a wave function  $\psi(R, t)$ , the evolution of which may be given as

$$\psi(R, t) = \int G[R, t; R_0, 0] \psi_0(R_0) dR_0. \quad (7)$$

The propagator is given by

$$G[R, t; R_0, 0] = \int_C e^{iS[R(t)]} \mathcal{D}[R(t)], \quad (8)$$

with  $C$  denoting the class of paths included in the path integral. All paths moving forward in  $t$ , with  $R(t) \geq 0$ , are included. The latter restriction can be implemented as for a point particle confined to a half space, i.e., as if there were an infinite potential wall at  $R=0$ . This implies the boundary condition  $\psi(0, t)=0$ . By imposing this condition we exclude consideration of topology-changing processes, wormhole creation or disappearance, at  $R=0$ . But this will not alter the implications of the wormhole dynamics for the stability of spacetime foam, as indicated below.

The propagator (8), with action (6), can be evaluated approximately. In the WKB limit the path integral is dominated by the contributions of classical paths and small fluctuations about those paths; it takes the form [12]

$$G[R, t; R_0, 0] \sim \sum_{\text{classical paths}} \left[ \frac{i}{2\pi} \frac{\partial^2 S[R_{cl}]}{\partial R \partial R_0} \right]^{1/2} e^{iS[R_{cl}]}. \quad (9)$$

The classical paths in the sum include the trajectory of form (5) between the initial and final values, owing

to the restriction  $R \geq 0$ , paths between those values which are piecewise of form (5) but which “bounce” one or more times at  $R=0$ , the bounce times determined by the requirement that these paths too be extrema of  $S$ . That condition takes the form of a cubic equation for the bounce time of a single-bounce trajectory, yielding one or three such paths, and a quartic equation for the bounce times of multiple-bounce trajectories, yielding four or two paths with a given number of bounces up to a maximum number. Hence the WKB approximation for  $G$  can be written

$$G^{(WKB)} = \sum_{n=0}^{n_{\max}(R_0, R, t)} \sum_k G_n^{(k)}, \quad (10)$$

where  $n$  is the number of bounces,  $k$  labels the  $n$ -bounce paths, and  $G_n^{(k)}$  is the corresponding contribution. Each of these is of the form on the right-hand side of Eq. (9); the prefactors and classical actions are complicated functions of  $R_0, R, t, n$ , and the bounce times, but they can be obtained explicitly in closed form [13]. The relative phases of the contributions are determined by the boundary condition  $\psi(0, t)=0$ . The propagator is nonvanishing outside the light cone and, hence, acausal, because spacelike as well as timelike paths are included in the path integral. This is in accord with, e.g., the suggestion of Hartle [14] that acausal histories should be included in path integrals for quantum gravity. It also accords with the case of the relativistic point particle, for which spacelike paths must be included in the path integral to obtain agreement with the results of canonical quantization [11].

The result reveals the quantum instability of the wormhole. The “ground-state energy” of the hole should follow from the Feynman-Kac [15] formula

$$E_0 = - \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \ln G[R, -i\tau; R, 0]. \quad (11)$$

But the result we obtain for the propagator (1) indicates that the  $\tau \rightarrow \infty$  behavior of  $G$  is

$$G \sim - \left[ \frac{1}{\pi i} \right]^{1/2} \sum_{n=1}^{\lfloor \tau/4R \rfloor} \frac{e^{i\tau^2/(4n)}}{n^{1/2}}. \quad (12)$$

Hence the right-hand side of Eq. (11) does not approach a definite limit: Its real part tends to zero while its imaginary part oscillates. This implies that the wormhole has no spectrum of bound states. Such behavior is reminiscent of systems, e.g., with “inverted” potentials diverging to negative infinity, or of metastable systems such as a particle confined by finite walls. The former case corresponds to rapid growth of a wormhole to large size; the latter to eventual “leaking” to large size, though the wormhole might remain near its initial size for a long time. It is the latter behavior which appears to characterize our model. The evolution via the propagator (10) of a wormhole wave function is illustrated in Fig. 1. In this example the initial wave function  $\psi_0(R_0)$  is simply a real Gaussian centered at  $R_0=10$ , with standard deviation  $1/\sqrt{2}$ , all quantities in Planck units. The wave packet collapses to  $R \approx 0$  and rebounds to  $R=10$  to begin again, following a bouncing classical trajectory piecewise

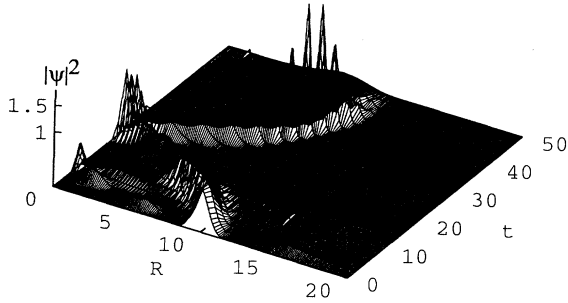


FIG. 1. Evolution of a wormhole wave function  $\psi(R, t)$ , as effected by the propagator  $G^{(\text{WKB})}$ ; the squared magnitude of  $\psi$  is shown. The initial wave function used is  $\psi(R, 0) = (2/\pi)^{1/4} \exp[-(R-10)^2]$ . All quantities are in Planck units.

of form (5). Its behavior over many oscillations is indicated by the asymptotic behavior of the propagator [13]: In the limit  $t \gg R, R_0$ , the largest contributions to  $G^{(\text{WKB})}$  in Eq. (10) are certain of the  $n = n_{\text{max}}$  terms, which give rise to caustics at intervals corresponding to classical bouncing. The singularities in the propagator at these points are integrable; they yield a peak in the wave function which follows a classical trajectory, but with amplitude decreasing as  $t^{-1/2}$ . The other terms in Eq. (10) give a combined contribution to the wave function, at radii near that of the initial peak, which appears to fluctuate, without dying away, for at least some hundreds of thousands of classical bounce times. The wormhole behaves not unlike an  $\alpha$  particle, which may oscillate millions of times within a nucleus before escaping to infinity.

### III. CONCLUSIONS

Spherically symmetric Minkowski wormholes [7] provide a very simple model of a mode of topological fluctuation in Lorentzian spacetime foam, and suggest a mode unstable against growth to macroscopic size. The quantum-gravitational dynamics of these wormholes is reduced to quantum mechanics of one variable, the throat radius, by describing the matter at the wormhole throat with a suitable equation of state and imposing the Hamiltonian constraint classically to reduce the phase space of the system. A corresponding reduced action is used in a Feynman path integral to obtain the propagator for wormhole wave functions; this is evaluated in the WKB approximation. The result shows that although the classical evolution of a wormhole may be bounded, i.e., stable, the hole nonetheless has no stable bound quantum states, and will eventually grow to large size by quantum “diffusion.”

Many systems exhibit similar behavior. For a particle with the familiar quadratic kinetic term in the action, the form of the potential determines whether such diffusion or spreading occurs: A potential well with walls falling off at large distances will allow a classically bound particle to leak out via quantum tunneling, while one which increases monotonically with distance will not. For these wormholes, with more complicated action (6), so simple an analysis is not possible. The more detailed examination of the wormhole propagator described here is needed

to show the instability.

The existence of an unstable mode of fluctuation, such as suggested by this minisuperspace analysis, would have profound implications. Since a macroscopic structure of wormholes is not observed, i.e., spacetime appears smooth and topologically trivial on all observable scales, it could indicate that spacetime does not possess (Lorentzian) foamlike structure on Planck scales. Whatever features might characterize the quantum behavior of spacetime, topological structures such as wormholes unstable against growth could not appear.

The stability of spacetime foam, then, needs more comprehensive study, to go beyond the limitations of our present calculations. The most fundamental of these is our restriction of the gravitational degrees of freedom to those of the spherically symmetric Minkowski wormhole, i.e., the use of a minisuperspace model for topological structure. In fact our model is even more restricted than the usual minisuperspace models [3], since the matter in the hole is treated not as a dynamical field but via an equation of state. Moreover, we use the particular equation of state (3) to simplify the calculations; other choices give rise to somewhat different dynamics.<sup>1</sup> Also we analyze the model via quantization in the reduced phase space. In the absence of a general framework for quantum-gravity calculations, this method seems best suited to the problem. It does differ markedly, though, from the Wheeler-DeWitt approach [3,10]. Here we use the particular reduced action (6); other forms corresponding to the classical equation (4) are possible, leading to different descriptions of the wormhole’s quantum behavior [16]. Our calculations are carried out in the WKB approximation. This is certainly expected to be valid in the late-time limits in which the instability is manifest. And WKB calculations of quantum instabilities in classically stable systems, tunneling processes, for example, are well known. But with no exact solution for comparison it is difficult to confirm the accuracy of the approximation. Finally, we implement the restriction that throat radii are non-negative as for a particle in a half space, with the boundary condition  $\psi(0, t) = 0$ . Other implementations might be used, the most general condition being only that  $\psi$  entail no current in the  $-R$  direction at  $R = 0$ .

This last assumption eliminates from consideration topology-changing processes, e.g., wormhole creation or disconnection, at  $R = 0$ . Including these would add an entirely new dimension to the problem, entailing a second-quantized (actually third-quantized, outside the constraints of a minisuperspace model) treatment of wormholes. But the possible instability of spacetime

<sup>1</sup>For example, if the familiar “dust” equation of state,  $p = 0$ , is used, the classical dynamics of the wormhole is only slightly different: The equation of motion is  $R\ddot{R} - \dot{R}^2 + 1 = 0$  instead of Eq. (4), and the classical trajectories are sine functions instead of the parabolas (5). The quantum dynamics is amenable to a different treatment than that described here [16].

foam, then an equilibrium population of wormholes continually fluctuating in and out of existence, should remain, even as the continual formation and dissolution of  $\alpha$  particles in a heavy nucleus does not alter its instability against eventual decay.

Even with all our assumptions the calculations are dauntingly difficult [13]. But it is to be hoped that further work along these lines will provide valuable insight into the quantum dynamics of spacetime.

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