RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 47, NUMBER 4 15 FEBRUARY 1993

L matrix for the massive Thirring model

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(Received 13 April 1992)

As new results for the massive Thirring model, the L matrix and the algebraic relations for its action-angle variables are given. It is shown most directly that this model which describes self-interacting relativistic fermions in one-dimensional space is a quantum integrable system.

PACS number(s): 11.10.Lm, 03.70.+k

I. INTRODUCTION

In this note we present the L matrix for the massive Thirring model [1] defined by the Hamiltonian

$$H=\int_0^M dx\{\phi^\dagger(-i\sigma_3)\partial_x\phi+m\phi^\dagger\sigma_1\phi -2\sin(2c)\phi_1^\dagger\phi_2^\dagger\phi_2\phi_1\}\ , \tag{1}$$

where σ_i are the Pauli matrices. The fermion fields $\phi(x)=(\phi_1(x),\phi_2(x))$ satisfy the usual equal time anticommutation relation $\{\phi_\mu(x),\phi_\nu^\dagger(y)\}=\delta_{\mu\nu}\delta(x-y)$; the indices refer to right- or left-moving particles. The constant $c\le\pi/2$ gives the strength of interaction between the fermions. As a solvable and simple model the massive Thirring model has attracted much interest (see Refs. [2–11]).

The L matrix corresponding originally to the Lax pair for nonlinear partial differential equations is an operator that determines a "canonical" transformation of the field variables $(\phi^\dagger,\phi)$ to variables of action-angle type, which are the matrix elements of the so-called monodromy matrix. The integrability can be shown by constructing an R matrix, which gives information about the algebraic relations between these action-angle variables. Hence the L matrix and the R matrix are crucial for the study of an integrable system.

The Thirring model is equivalent to the quantum sine-Gordon model [2]. For this bosonic model the L matrix and R matrix have been given by Sklyanin, Takhtadzhyan, and Faddeev [4]. We think it is an interesting problem to construct these matrices also for its fermionic counterpart especially since supermatrices are the tools which are not so common with integrable systems.

Here we will study the L and R matrices for the Thirring model. The method we used originally is to find the L matrix by taking the continuum limit from the R matrix of an inhomogeneous six-vertex model. It is known that the six-vertex model can be used to construct the Bethe ansatz solution of the massive Thirring model [8] and that the Hamiltonian should be connected to a six-vertex model with staggered weights [10]. Actually, we used this method before to derive the L matrices for the sine-Gordon model [13] and a bosonic system with Hamiltonian similar to Eq. (1) [12]. Since in the present case the derivation is lengthy and similar to the one in [12], we use here a different approach. Given the result, we have to prove that we have found the L matrix of the Thirring model. So we have to find the R matrix starting from the L matrix in order to show the integrability. Since the L matrix determines the monodromy matrix, we have to check whether the latter really contains the Hamiltonian given above as one of the simple conserved entities.

II. L MATRIX

The action-angle variables or, more specifically, the monodromy matrix $T(u)=T(x=M|u)$, where M is the length of the system, are defined by the differential equation

$$\frac{dT(x|u)}{dx}=:L(x|u)T(x|u): \tag{2}$$

and the boundary condition $T(x=0|u)=I$, the identity matrix. The colons mean normal ordering of the Fermi operators and u is the spectral parameter.

We found, for the L matrix after taking the continuum limit and introducing fermion fields,

$$L(x|u)=i\tfrac{1}{2}m\sinh u\,\tau_3+\Sigma(x)+S(x|u)\ . \tag{3}$$

Here we distinguish the Pauli matrices in classical space τ_i and those in quantum space σ_i used before. For Σ and S we have

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$$\Sigma(x)=\begin{bmatrix}-\phi^\dagger(x)(1-e^{ic\sigma_3})\phi(x) & 0\\ 0 & -\phi^\dagger(x)(1+e^{-ic\sigma_3})\phi(x)\end{bmatrix}, \tag{4}$$

$$S(x|u)=i\sqrt{m}\,\mathrm{sinc}\begin{bmatrix}0 & e^{-u/2}\phi_1(x)-e^{u/2}\phi_2(x)\\ -e^{-u/2}\phi_1^\dagger(x)+e^{u/2}\phi_2^\dagger(x) & 0\end{bmatrix}. \tag{5}$$

We have to choose the L matrix as a supermatrix with its row or column parities $p(1)=0$ and $p(2)=1$; consequently, $T(x|u)$ as a solution of Eq. (2) is also a supermatrix.

The integrability of the Thirring model can be shown by finding the Yang-Baxter equation for the monodromy matrix $T(u)$:

$$R(u-v)T(u)\overset{\otimes}{s}T(v)=T(v)\overset{\otimes}{s}T(u)R(u-v), \tag{6}$$

where the tensor product indicated by the symbol $\overset{\otimes}{s}$ is of the superform $(A\overset{\otimes}{s}B)_{i,j}^{k,l}=(-1)^{p(j)[p(k)+p(i)]}A_{i,k}B_{j,l}$. The matrix R has the same form as for the six-vertex model; note, however, that the spectral parameter u is imaginary in the usual six-vertex model

$$R(u)=\begin{bmatrix}\sinh(\frac{u}{2}+ic) & 0 & 0 & 0\\ 0 & \sinh ic & \sinh\frac{u}{2} & 0\\ 0 & \sinh\frac{u}{2} & \sinh ic & 0\\ 0 & 0 & 0 & \sinh(\frac{u}{2}+ic)\end{bmatrix}. \tag{7}$$

In the following let us prove the Yang-Baxter equation (6). We can rewrite the differential equation (2) as an integral equation for $0\le x\le M$:

$$T(x|u)=\exp\left[i\frac{m}{2}x\sinh u\,\tau_3\right]+\int_0^x dz\exp\left[i\frac{m}{2}(x-z)\sinh u\,\tau_3\right]\times:[\Sigma(z)+S(z|u)]T(z|u):, \tag{8}$$

from which we can get more easily two auxiliary equations

$$\phi_\nu(x)T(x|u)=\tau_3T(x|u)\tau_3\phi_\nu(x)+\tfrac{1}{2}:E_\nu(x|u)T(x|u):, \tag{9}$$

$$T(x|u)\phi_\nu^\dagger(x)=\phi_\nu^\dagger(x)\tau_3T(x|u)\tau_3+\tfrac{1}{2}:F_\nu(x|u)\tau_3T(x|u)\tau_3:,$$

with

$$\begin{aligned}E_\nu(x|u)&=-\phi_\nu(x)(1-\tau_3e^{ic\tau_3(-1)^{\nu+1}})+i\sqrt{m}\,\mathrm{sinc}\,(-1)^\nu e^{(-1)^\nu u/2}\tau^-,\\ F_\nu(x|u)&=-\phi_\nu^\dagger(x)(1-\tau_3e^{ic\tau_3(-1)^{\nu+1}})-i\sqrt{m}\,\mathrm{sinc}\,(-1)^\nu e^{(-1)^\nu u/2}\tau^+.\end{aligned} \tag{10}$$

Defining the tensor product in the Yang-Baxter equation (6) as

$$K(x|u,v)=T(x|u)\overset{\otimes}{s}T(x|v), \tag{11}$$

one can get using the last Eqs. (9) and (10) a differential equation for this tensor product,

$$\partial_xK(x|u,v)=:L(x|u,v)K(x|u,v): \tag{12}$$

with a L matrix depending now on two spectral parameters u and v:

$$L(x|u,v)=L(x|u)\overset{\otimes}{s}1+1\overset{\otimes}{s}L(x|v)+\sum_\nu F_\nu(x|u)\overset{\otimes}{s}E_\nu(x|v). \tag{13}$$

One can show only by an explicit calculation that

$$R(u-v)L(x|u,v)=L(x|v,u)R(u-v). \tag{14}$$

The equation means that R matrix can exchange u and v in $L(x|u,v)$ in a manner the Yang-Baxter relation postulates. Equations (11)–(13) above defining the tensor product $K(x|u,v)$ from $L(x|u,v)$ are of course also valid if the spectral parameter u and v are exchanged. This implies that an equation such as Eq. (14) must hold also for K:

$$R(u-v)K(x|u,v)=K(x|v,u)R(u-v), \tag{15}$$

which is the Yang-Baxter relation (6) taking $x=M$.

Hence the transfer matrix $t(u)=T(u)_{11}-T(u)_{22}$ which is the supertrace of the $T(u)$ must commute for different spectral parameters, i.e.,

$$[t(u),t(v)]=0. \tag{16}$$

So the definition of the L matrix (3) generates a quantum integrable system. The main problem is now to show that this quantum system is the massive Thirring model.

III. HAMILTONIAN AND MOMENTUM

Here we will find the Hamiltonian of the Thirring model (1) from the transfer matrix $t(u)$. This shows directly that the L matrix (3) gives the Thirring model. Using an integral form of Eq. (2) similar to (8), however, and expanding T with respect to S given by Eq. (5), one has

$$\tilde{T}(x|u)=:Q:+\int_0^x dz\,e^{-imz\tau_3\sinh u}:Q(z)S(z|u)\tilde{T}(z|u):, \tag{17}$$

where

$$\tilde{T}(x|u)=\exp\left[-i\frac{m}{2}x\tau_3\sinh u\right]T(x|u)\ ,$$

$$Q(z)=\exp\left[\int_z^x dy\ \Sigma(y)\right]\ ,$$

and

$$Q=Q(z=0)\ .$$

In order that such an expansion makes sense one must add an imaginary part to the spectral parameter, for example, $u\rightarrow u\pm i\pi/2$ to have simple expressions. Thus from

$$T(x|u\pm i\pi/2)=\exp\left[i\frac{m}{2}x\tau_3\sinh(u\pm i\pi/2)\right]:Q:+\cdots$$

$$=\exp\left[\mp\frac{m}{2}x\tau_3\cosh u\right]:Q:+\cdots \tag{18}$$

we can see that the transfer matrix $t(u\pm i\pi/2)$ decreases or increases rapidly for $u\rightarrow\pm\infty$. After iterating Eq. (17) we see that the expansions

$$\tilde{T}(x|u-i\pi/2)_{11}=:Q_{11}:+m\ \text{sinc}\int_0^x dz_1\int_0^{z_1}dz_2 e^{-m(z_1-z_2)\cosh u}:\Phi_-(z_1)\Phi_-^\dagger(z_2)Q_{11}:+\cdots \tag{19a}$$

and

$$\tilde{T}(x|u+i\pi/2)_{22}=:Q_{22}:+m\ \text{sinc}\int_0^x dz_1\int_0^{z_1}dz_2 e^{-m(z_1-z_2)\cosh u}:\Phi_+^\dagger(z_1)\Phi_+(z_2)Q_{22}:+\cdots \tag{19b}$$

have nontrivial limits for $u\rightarrow\pm\infty$, where

$$\Phi_\pm(z)=Q(z)_{11}[e^{-u/2\mp i\pi/4}\phi_1(z)-e^{u/2\pm i\pi/4}\phi_2(z)]Q^{-1}(z)_{22}$$

and

$$\Phi_\pm^\dagger(z)=Q(z)_{22}[e^{-u/2\mp i\pi/4}\phi_1^\dagger(z)-e^{u/2\pm i\pi/4}\phi_2^\dagger(z)]Q^{-1}(z)_{11}\ .$$

By choosing the sign of the imaginary part of u we have picked out the exponentially growing contributions. Neglecting the decreasing part related to $\exp(-e^{\pm u})$ for $u\rightarrow\pm\infty$, the transfer matrix multiplied by

$$\exp\left[\pm i\frac{m}{2}M\sinh\left[u\pm i\frac{\pi}{2}\right]\right]$$

is simply $\tilde{T}(u-i\pi/2)_{11}$ or $\tilde{T}(u+i\pi/2)_{22}$ in this limiting case. We combine the contributions defining the generators

$$G_\pm(u)=\lim_{u\rightarrow\pm\infty}\left\{e^{\pm ic}\ln\left[\exp\left[-i\frac{m}{2}M\sinh(u-i\pi/2)\right]t(u-i\pi/2)\right]-e^{\mp ic}\ln\left[\exp\left[i\frac{m}{2}M\sinh(u+i\pi/2)\right]t(u+i\pi/2)\right]\right\}\ . \tag{20}$$

The generators $G_\pm$ can be calculated from Eqs. (19) using partial integrations repeatedly. In this way one obtains a series in $e^{\mp u}$ for $u\rightarrow\pm\infty$:

$$G_\pm(u)=\sum_{s\geq 0}C_{\pm s}e^{\mp su}\ . \tag{21}$$

The calculation is tedious but straightforward. Every term of the expansions $\tilde{T}_{11}$ and $\tilde{T}_{22}$ has a contribution to $G_\pm$, even to the first order $C_{\pm 1}$. Fortunately, all coefficiencies of the factor $e^{\pm u}$ can be summed up and give us the wanted results

$$C_{+1}=\frac{8\ \text{sinc}}{m}\int_0^M dz\left[i\phi_2^\dagger\partial_z\phi_2+\frac{m}{2}(\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1)-\sin(2c)\phi_1^\dagger\phi_2^\dagger\phi_2\phi_1\right] \tag{22}$$

and

$$C_{-1}=\frac{8\ \text{sinc}}{m}\int_0^M dz\left[-i\phi_1^\dagger\partial_z\phi_1+\frac{m}{2}(\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1)-\sin(2c)\phi_1^\dagger\phi_2^\dagger\phi_2\phi_1\right]\ . \tag{23}$$

It can be seen that the generators G_- and G_+ give the conserved quantities for right- and left-moving particles respectively. The Thirring system includes both contributions of right and left particles. The sum of the two first order coefficients is just the Hamiltonian of the Thirring model (1) whereas the difference is the momentum

$$P=-i\int_0^M dz\ \phi^\dagger(z)\partial_z\phi(z)\ . \tag{24}$$

The other coefficients $C_{\pm s}$ ($s=0,2,3,\ldots$), if they are not zero, should give other conserved quantities of the Thirring model. These are not easy to calculate and cannot be studied here.

IV. CONCLUSION

In this note we have described the L and R matrices for the Thirring model which were not given before for a fermionic relativistic theory. For quantum inverse scattering transformations the L and R matrices are important operators, for example, for studying the algebraic Bethe ansatz and the inverse problem of the Thirring model in the sense of the works [4] and [14]. Also it is interesting to note that the Yang-Baxter equation (6) for the monodromy matrix has a superstructure, i.e., the diagonal elements of the monodromy matrix are of bosonic type, whereas the off-diagonal ones are of fermionic type. Hence Eq. (6) gives us also a graded and deformed algebra or a graded quantum algebra (see Refs. [15,16]).

ACKNOWLEDGMENTS

One of the authors (Y.K.Z.) wishes to thank W. Nahm, E. Olmedilla, V. Rittenberg, and Al. B. Zamolodchikov for discussion. He is also grateful to the Alexander von Humboldt Foundation for financial support.

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