$NN \to \Delta N$ transition amplitude analysis in optimal formalism

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A method is developed for determining the $NN \to \Delta N$ transition amplitudes. The quadratic relations existing between all the spin observables of the transition are presented. The study is performed in the optimal formalism, and applied to the cases of transversity and helicity frames. For each case, a methodology is given for selecting a set of 31 observables, which determines the 16 magnitudes of the amplitudes and 15 independent relative phases between them.

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I. INTRODUCTION

In view of the role of the isobaric resonance in intermediate energy physics [1–4], a more precise knowledge of the $NN \to \Delta N$ transition is of importance. Experimentally, information can be extracted from the $NN \to NN\pi$ reactions [5]. A set of spin observables of the Δ production on a wide energy range has been provided by the Argonne data [6]. In recent years, the spin structure of this transition has been investigated [2, 7, 8] in order to interpret the data already obtained and to prepare the framework of experimental and theoretical programs in view of a better knowledge of the reaction. More particularly, the complete determination of the transition amplitudes is an aim to be kept in mind.

A systematic formalism for the spin observables has been developed in Refs. [2, 7, 8], according to the polarization states of the four particles involved in the reaction. It uses a spin-space decomposition of the transition matrix analogous to the Wolfenstein representation in NN elastic scattering. The 16 complex spin amplitudes $f_i(\theta_\Delta)$ and $g_i(\theta_\Delta)$ are somewhat similar to the spin-nonflip and spin-flip amplitudes of pion-nucleon scattering. This decomposition is convenient for studying nuclear reactions in intermediate-energy physics. Use is made of this spin-space decomposition for tackling problems such as nucleon-nucleus scattering [4] and nuclear Δ production [3], by eikonal models. At a different level, still along this line, the iterated pion-exchange model of Kloet and Silbar [9] has been used to generate theoretical $f_i(\theta_\Delta)$ and $g_i(\theta_\Delta)$ spin amplitudes. They have been tested [10] against Argonne experimental data.

Any formalism describing the transition matrix and observables presents observables in terms of bilinear combinations of amplitudes ("bicoms"), on the one hand, and yields linear and nonlinear relationships among observables, on the other hand. In general, however, the matrix connecting observables and bicoms is far from diagonal, and thus a given observable depends on many bicoms and vice versa.

In the absence of constraints, 32 amplitudes are needed to describe the studied reaction. Parity conservation reduces the number of independent complex amplitudes to 16 and consequently we are facing $16^2 = 256$ linearly independent bicoms, which is also the number of observables. Therefore, these observables are not independent but are related to each other in a nonlinear way. Taking into account quadratic relations between linearly independent observables yields a number of 31 independent observables only. The amplitude analysis requires the determination of the 16 magnitudes and 15 independent relative phases. Whereas a single set of observables determines the magnitudes, the 15 independent relative phases can be obtained by many different sets of 15 observables. The selection of such sets is closely linked with the analysis of the quadratic relations.

In a previous paper [8], the optimal formalism of Goldstein, Moravcsik, and Arash [11–14], has been applied to the study of the $NN \to \Delta N$ transition. It optimally diagonalizes the matrix connecting observables and bicoms and consequently is well adapted to the phenomenological determination of amplitudes.

In optimal formalism, as far as "primary observables" are concerned, in which the spin projection of each particle state is specified, polarization structure analysis yields bicom-observable relations in a particularly simple form. Yet, it is much simpler to perform experiments in which some particles are unpolarized, which correspond to averaged spin states and leads to a redefinition of the observables in terms of "secondary observables." Unfortunately, for these "secondary observables," the complexity of the relations with bicoms increases. Among all the possibilities for quantization directions, helicity and transversity frames play an important role and are particularly studied in the framework of optimal formalism.

The purpose of the present paper is to investigate the quadratic relations existing between all the spin observables of the $NN \to \Delta N$ transition, and to propose an efficient method for an amplitude analysis. Our study is performed in the optimal formalism, and applied to the case of transversity and helicity frames. Taking account of parity invariance reduces the number of independent amplitudes in a quite different way for transversity and helicity frames. For each case, a methodology is developed for selecting a set of 31 observables which determines the 16 magnitudes of the amplitudes and 15 independent relative phases between them, the overall phase being irrelevant, in the present case.

The paper is organized as follows. Section II is devoted to a brief summary of the optimal formalism for the $NN \to \Delta N$ transition and to the nonlinear relationships between observables. Sections III and IV present a method for determining the magnitudes and the independent relative phases of the transition amplitudes, in the transversity and helicity frames, respectively.

II. QUADRATIC RELATIONS IN OPTIMAL FORMALISM

Optimal formalism, developed by Goldstein, Moravcsik and Arash [11–14], is applied to the transition $N_1N_2 \to \Delta_1 N_2'$. Here, a brief recall of notations and basic formulas is given. More details can be found in [8]. The transition matrix is written

$$M = \sum_{\lambda l \Lambda L} D(\lambda, l; \Lambda, L)\delta^{\lambda l} \otimes \delta^{\Lambda L}, \tag{2.1}$$

where the $D(\lambda, l; \Lambda, L)$'s are the amplitudes and the $\delta^{\lambda l}$ and $\delta^{\Lambda L}$ the spin-momentum tensors referring to particles Δ_1 and N_1, N_2', and N_2, respectively. The indices λ, l, Λ, and L are the magnetic projections along the $\hat{\mathbf{z}}$ quantization axis of each particle, Δ_1, N_1, N_2', and N_2, respectively. The reaction is completely described by a set of 32 amplitudes $D(\lambda, l; \Lambda, L)$, each of which being a function of energy and scattering angle.

The spin observables are defined by

$$\mathcal{L}(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = \mathrm{Tr}(Q^{\xi\omega H_q}\rho^{\Xi\Omega H_Q} M \rho^{uvH_p}\rho^{UVH_P} M^\dagger), \tag{2.2}$$

where u and v characterize the spin-space matrix of particle N_1, U, and V that of N_2, ξ, and ω being for Δ_1, Ξ, and Ω for N_2'. Each of the two indices, for a given particle of spin s, takes $(2s+1)$ values from 1 to $(2s+1)$, which are related to magnetic projections along the quantization axis. The H's can be either "real" (R) or "imaginary" (I), for off-diagonal elements of the density matrix. For diagonal elements, H is only "real" and the label (R) may be omitted for sake of simplicity. The indices p, P, q, Q are equal to $+1$ or -1, H_1 standing for R and H_{-1} for I. The ρ and Q operators, describing initial polarizations and measured final polarizations, denote all the spin-space operators required to generate spin observables of the reaction. As usual, putting Eq. (2.1) into Eq. (2.2), it is easy to see that spin observables are given by bilinear combinations of amplitudes, called "bicoms."

In the optimal formalism, the choice of observables and amplitudes provides observable-bicom relations as simple as possible. For this, the δ's in Eq. (2.1) are chosen to have only one nonzero element and the ρ and Q operators to be "minimally Hermitian," so that the corresponding matrices have minimal number of nonzero elements compatible with the Hermiticity requirement. All spin-space matrices corresponding to the $N_1N_2 \to \Delta_1 N_2'$ transition are given in the Appendix.

Then, the relationship between spin observables of Eq. (2.2) and bicoms may be written

$$\begin{aligned}\mathcal{L}(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = 2pPSH_{pPqQ}[\, & D(\{\xi\},\{u\};\{\Xi\},\{U\})D^*(\{\omega\},\{v\};\{\Omega\},\{V\})\\ & +pD(\{\xi\},\{v\};\{\Xi\},\{U\})D^*(\{\omega\},\{u\};\{\Omega\},\{V\})\\ & +PD(\{\xi\},\{u\};\{\Xi\},\{V\})D^*(\{\omega\},\{v\};\{\Omega\},\{U\})\\ & +pPD(\{\xi\},\{v\};\{\Xi\},\{V\})D^*(\{\omega\},\{u\};\{\Omega\},\{U\})\\ & +QD(\{\xi\},\{u\};\{\Omega\},\{U\})D^*(\{\omega\},\{v\};\{\Xi\},\{V\})\\ & +pQD(\{\xi\},\{v\};\{\Omega\},\{U\})D^*(\{\omega\},\{u\};\{\Xi\},\{V\})\\ & +PQD(\{\xi\},\{u\};\{\Omega\},\{V\})D^*(\{\omega\},\{v\};\{\Xi\},\{U\})\\ & +pPQD(\{\xi\},\{v\};\{\Omega\},\{V\})D^*(\{\omega\},\{u\};\{\Xi\},\{U\})],\end{aligned} \tag{2.3}$$

where $S=+1$ for $\Sigma I = 0,3,4$ and $S=-1$ for $\Sigma I = 1,2$, ΣI being number of I indices among H_p, H_P, H_q, H_Q or also number of -1 among p, q, P, Q. The symbol H_{pPqQ} is equal to "real" or "imaginary," if the product $pPqQ$ is $+1$ or -1, respectively. The symbol $\{u\} = \frac{1}{2}, -\frac{1}{2}$ for $u = 1, 2$, respectively, and similarly for $\{v\}$, $\{U\}$, $\{V\}$, $\{\Xi\}$, and $\{\Omega\}$. The symbol $\{\xi\} = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ for $\xi = 1, 2, 3, 4$, respectively, and similarly for $\{\omega\}$.

For spin-$\frac{1}{2}$ particle, the uvH_p argument (and also, UVH_P and $\Xi\Omega H_Q$ arguments) may be written as two diagonal arguments 11 and 22 and two off-diagonal ones $12R$ and $12I$. For spin-$\frac{3}{2}$ particle, the $\xi\omega H_q$ argument takes four diagonal states 11, 22, 33, and 44 and twelve off-diagonal ones $12R$, $13R$, $14R$, $23R$, $24R$, and $34R$ and six analogous ones with I replacing R.

Without parity constraints, the number of bicoms is $32^2 = 1024$, which is also the number of observables. The 1024x1024 matrix connecting observables and bicoms may be reduced as a string of small submatrices along the diagonal, all other matrix elements vanishing. These small submatrices are 1x1, 2x2, 4x4, and 8x8 matrices. It is easy to describe the characteristics of the submatrices.

The 32 observables referring to diagonal arguments only are simply related to one bicom and more particularly to the magnitude of one amplitude by

$$\mathcal{L}(uu, UU; \xi\xi, \Xi\Xi) = 16|D(\{\xi\}, \{u\}; \{\Xi\}, \{U\})|^2. \quad (2.4)$$

The 192 observables referring to three diagonal arguments are also related to one bicom. For instance, we have

$$\mathcal{L}(uvH_p, UU; \xi\xi, \Xi\Xi) = 16H_p[D(\{\xi\}, \{u\}; \{\Xi\}, \{U\})D^*(\{\xi\}, \{v\}; \{\Xi\}, \{U\})]. \quad (2.5)$$

Eqs. (2.4) and (2.5) correspond to 1×1 submatrices.

The 384 observables referring to two diagonal arguments are related to two bicoms and correspond to 2x2 submatrices. For instance, we have

$$\begin{aligned}\mathcal{L}(uvH_p, UVH_P; \xi\xi, \Xi\Xi) = 8pPSH_{pP}[\ & D(\{\xi\}, \{u\}; \{\Xi\}, \{U\})D^*(\{\xi\}, \{v\}; \{\Xi\}, \{V\}) \\ & + pD(\{\xi\}, \{v\}; \{\Xi\}, \{U\})D^*(\{\xi\}, \{u\}; \{\Xi\}, \{V\})]. \quad (2.6)\end{aligned}$$

The 320 observables referring to one diagonal argument are related to four bicoms and correspond to 4x4 submatrices. For instance, we have

$$\begin{aligned}\mathcal{L}(uvH_p, UVH_P; \xi\omega H_q, \Xi\Xi) = 4pPSH_{pPq}[\ & D(\{\xi\}, \{u\}; \{\Xi\}, \{U\})D^*(\{\omega\}, \{v\}; \{\Xi\}, \{V\}) \\ & + pD(\{\xi\}, \{v\}; \{\Xi\}, \{U\})D^*(\{\omega\}, \{u\}; \{\Xi\}, \{V\}) \\ & + PD(\{\xi\}, \{u\}; \{\Xi\}, \{V\})D^*(\{\omega\}, \{v\}; \{\Xi\}, \{U\}) \\ & + pPD(\{\xi\}, \{v\}; \{\Xi\}, \{V\})D^*(\{\omega\}, \{u\}; \{\Xi\}, \{U\})]. \quad (2.7)\end{aligned}$$

Finally, the 96 observables without diagonal argument are related to eight bicoms [Eq. (2.3)] and correspond to 8x8 submatrices. All these submatrices, except these corresponding to Eq. (2.4), connect observables either with the real part of the bicoms or with the imaginary part.

In terms of these so-called "primary observables," the systematic determination of the amplitudes $D(\lambda, l; \Lambda, L)$ is very simple. In Ref. [12], Moravcsik gives the following theorem : "The amplitudes of any arbitrary reaction can be determined completely, except for a set of discrete ambiguities, through the one-by-one bicom-observable submatrices alone, and hence by exactly $2n-1$ measurements where n is the number of amplitudes."

Indeed, each observable with diagonal arguments only [Eq. (2.4)] directly gives the magnitude of the corresponding amplitude. Each observable with only one off-diagonal argument [Eq. (2.5)] gives the relative phase between the corresponding amplitudes. If only one set of observables leads to the determination of the magnitudes, it is clear that the determination of $n - 1$ independent relative phases can be obtained by many sets.

The determination of the amplitudes is performed in two stages [15]. First, the magnitudes are determined, and second, step by step, independent relative phases are obtained. The first stage consists of measuring the n observables with diagonal arguments, only [Eq. (2.4)]. The magnitudes being determined, the second stage consists of finding a set of, at least, $n - 1$ experiments for the determination of independent relative phases. The selection of such a set is closely linked with the analysis of the quadratic relationships [16], which occur between observables.

For a reaction described by n amplitudes, there are n^2 linearly independent observables, each being a linear combination of some of the n^2 bicoms [Eq. (2.3)]. Therefore, the n^2 observables (or bicoms) are dependent on each other in a nonlinear way, and consequently, the selection of $n - 1$ observables needs attention for leading to phases which are independent.

Quadratic relationships may be derived from the following complex identity

$$\begin{aligned}&(D(\lambda, l; \Lambda, L)D^*(\lambda', l'; \Lambda', L'))(D(\lambda', l'; \Lambda', L')D^*(\lambda'', l''; \Lambda'', L'')) \\ &\qquad = |D(\lambda', l'; \Lambda', L')|^2(D(\lambda, l; \Lambda, L)D^*(\lambda'', l''; \Lambda'', L'')). \quad (2.8)\end{aligned}$$

From this equation, $n(n-1)/2$ independent relations are written

$$[\mathrm{Re}(D(\lambda, l; \Lambda, L)D^*(\lambda', l'; \Lambda', L'))]^2 + [\mathrm{Im}(D(\lambda, l; \Lambda, L)D^*(\lambda', l'; \Lambda', L'))]^2 = |D(\lambda, l; \Lambda, L)|^2|D(\lambda', l'; \Lambda', L')|^2. \quad (2.9)$$

The magnitudes being known, the real and imaginary parts of a bicom are related by Eq. (2.9). Consequently, the determination of a relative phase is performed either by the real part or by the imaginary part of the corresponding bicom, with an ambiguity of sign. The total knowledge of the phases, without ambiguities, needs more than $n - 1$ observables.

From Eq. (2.8), a second type of relations between bicoms may be derived

$$\begin{aligned}&\mathrm{Re}(D(\lambda,l;\Lambda,L)D^*(\lambda',l';\Lambda',L'))\mathrm{Re}(D(\lambda',l';\Lambda',L')D^*(\lambda'',l'';\Lambda'',L''))\\&\quad-\mathrm{Im}(D(\lambda,l;\Lambda,L)D^*(\lambda',l';\Lambda',L'))\mathrm{Im}(D(\lambda',l';\Lambda',L')D^*(\lambda'',l'';\Lambda'',L''))\\&\qquad=|D(\lambda',l';\Lambda',L')|^2\mathrm{Re}(D(\lambda,l;\Lambda,L)D^*(\lambda'',l'';\Lambda'',L'')),\end{aligned}\tag{2.10a}$$

and

$$\begin{aligned}&\mathrm{Re}(D(\lambda,l;\Lambda,L)D^*(\lambda',l';\Lambda',L'))\mathrm{Im}(D(\lambda',l';\Lambda',L')D^*(\lambda'',l'';\Lambda'',L''))\\&\quad+\mathrm{Im}(D(\lambda,l;\Lambda,L)D^*(\lambda',l';\Lambda',L'))\mathrm{Re}(D(\lambda',l';\Lambda',L')D^*(\lambda'',l'';\Lambda'',L''))\\&\qquad=|D(\lambda',l';\Lambda',L')|^2\mathrm{Im}(D(\lambda,l;\Lambda,L)D^*(\lambda'',l'';\Lambda'',L'')).\end{aligned}\tag{2.10b}$$

Using Eq. (2.9) into Eq. (2.10a) yields

$$\begin{aligned}&|D(\lambda',l';\Lambda',L')|^2\mathrm{Re}(D(\lambda,l;\Lambda,L)D^*(\lambda'',l'';\Lambda'',L''))\\&\quad=\mathrm{Re}(D(\lambda,l;\Lambda,L)D^*(\lambda',l';\Lambda',L'))\mathrm{Re}(D(\lambda',l';\Lambda',L')D^*(\lambda'',l'';\Lambda'',L''))\\&\qquad\pm\{|D(\lambda,l;\Lambda,L)|^2|D(\lambda',l';\Lambda',L')|^2-[\mathrm{Re}(D(\lambda,l;\Lambda,L)D^*(\lambda',l';\Lambda',L'))]^2\}^{1/2}\\&\qquad\times\{|D(\lambda',l';\Lambda',L')|^2|D(\lambda'',l'';\Lambda'',L'')|^2-[\mathrm{Re}(D(\lambda',l';\Lambda',L')D^*(\lambda'',l'';\Lambda'',L''))]^2\}^{1/2}.\end{aligned}\tag{2.11}$$

In terms of phases, the magnitudes being known, Eq. (2.11) expresses the fact that the knowledge of the relative phases $(\varphi-\varphi')$ and $(\varphi'-\varphi'')$ determines $(\varphi-\varphi'')$ with some ambiguities. There are $n(n-1)(n-2)/2$ such pair of Eqs. (2.10), but only $(n-1)(n-2)/2$ are independent. Taking into account quadratic relations between the n^2 linearly independent observables, the number of independent observables is given by $n^2-n(n-1)/2-(n-1)(n-2)/2=2n-1$.

Equations (2.9) and (2.10), expressed in terms of bicoms, may be translated in terms of observables. In general, quadratic relationships between observables are lengthy. For performing this translation, the inversion of Eq. (2.3) is needed. One obtains

$$16H_{pPqQ}[D(\{\xi\},\{u\};\{\Xi\},\{U\})D^*(\{\omega\},\{v\};\{\Omega\},\{V\})]=\sum_{pPqQ\text{ fixed}}pPS\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q).\tag{2.12}$$

The summation runs over the four indices p,P,q,Q, each taking the values ±1 for off-diagonal argument, but only the value $+1$ for diagonal argument, the product $pPqQ$ being fixed to $+1$ for the real part of the bicom and to -1 for the imaginary part. Consequently, for observables referring to three diagonal arguments [Eq. (2.5)], the summation gives one term, only. For observables referring to two, one, or zero diagonal arguments [Eqs. (2.6), (2.7), and (2.3)], the summation gives two, four, or eight terms, respectively.

From Eq. (2.12), the quadratic relation of Eq. (2.9) is expressed in terms of observables as

$$\begin{aligned}\mathcal{L}(uu,UU;\xi\xi,\Xi\Xi)\mathcal{L}(vv,VV;\omega\omega,\Omega\Omega)&=\left(\sum_{pPqQ=+1}pPS\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q)\right)^2\\&\quad+\left(\sum_{pPqQ=-1}pPS\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q)\right)^2.\end{aligned}\tag{2.13}$$

The exchange properties of $\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q)$, obtained from Eq. (2.3), are written

$$\begin{aligned}\mathcal{L}(vuH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q)&=p\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q),\\\mathcal{L}(uvH_p,VUH_P;\xi\omega H_q,\Xi\Omega H_Q)&=P\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q),\\\mathcal{L}(uvH_p,UVH_P;\omega\xi H_q,\Xi\Omega H_Q)&=q\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q),\\\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Omega\Xi H_Q)&=Q\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q).\end{aligned}\tag{2.14}$$

Taking account of these exchange properties into Eq. (2.12) gives the expression of the seven other bicoms appearing in Eq. (2.3); furthermore, it generalizes Eq. (2.13) and yields all the possible cases of quadratic relations of this type.

Similarly, Eqs. (2.10) may be expressed in terms of observables as

$$\mathcal{L}(u'u',U'U';\xi'\xi',\Xi'\Xi')\sum_{pPqQ=+1} pPS\mathcal{L}(uu''H_p,UU''H_P;\xi\xi''H_q,\Xi\Xi''H_Q)$$
$$= \sum_{p'P'q'Q'=+1} p'P'S'\mathcal{L}(uu'H_{p'},UU'H_{P'};\xi\xi'H_{q'},\Xi\Xi'H_{Q'})$$
$$\times \sum_{p''P''q''Q''=+1} p''P''S''\mathcal{L}(u'u''H_{p''},U'U''H_{P''};\xi'\xi''H_{q''},\Xi'\Xi''H_{Q''})$$
$$- \sum_{p'P'q'Q'=-1} p'P'S'\mathcal{L}(uu'H_{p'},UU'H_{P'};\xi\xi'H_{q'},\Xi\Xi'H_{Q'})$$
$$\times \sum_{p''P''q''Q''=-1} p''P''S''\mathcal{L}(u'u''H_{p''},U'U''H_{P''};\xi'\xi''H_{q''},\Xi'\Xi''H_{Q''}), \quad (2.15a)$$

and

$$\mathcal{L}(u'u',U'U';\xi'\xi',\Xi'\Xi')\sum_{pPqQ=-1} pPS\mathcal{L}(uu''H_p,UU''H_P;\xi\xi''H_q,\Xi\Xi''H_Q)$$
$$= \sum_{p'P'q'Q'=+1} p'P'S'\mathcal{L}(uu'H_{p'},UU'H_{P'};\xi\xi'H_{q'},\Xi\Xi'H_{Q'})$$
$$\times \sum_{p''P''q''Q''=-1} p''P''S''\mathcal{L}(u'u''H_{p''},U'U''H_{P''};\xi'\xi''H_{q''},\Xi'\Xi''H_{Q''})$$
$$+ \sum_{p'P'q'Q'=-1} p'P'S'\mathcal{L}(uu'H_{p'},UU'H_{P'};\xi\xi'H_{q'},\Xi\Xi'H_{Q'})$$
$$\times \sum_{p''P''q''Q''=+1} p''P''S''\mathcal{L}(u'u''H_{p''},U'U''H_{P''};\xi'\xi''H_{q''},\Xi'\Xi''H_{Q''}). \quad (2.15b)$$

The use of the exchange properties [Eqs. (2.14)] of $\mathcal{L}(uvH_p,UVH_P;\xi\omega H_q,\Xi\Omega H_Q)$ generalizes Eqs. (2.15) and yields all the possible cases of quadratic relations of this type.

So far, primary observables are considered, in which the polarizations of the four particles are specified. For experimental reasons, instead of primary observables defined by Eq. (2.3), linear combinations of them are preferred, which correspond to averaged polarization state of one or several particles involved into the $N_1N_2 \to \Delta_1 N_2'$ reaction. In these so-called "secondary observables" [14], the polarization state of each particle is either averaged (i.e., the particle is unpolarized or its polarization is not measured) or satisfies "null-sum" criterion, which requires that the sum of the coefficients of primary arguments vanishes. Secondary observables are labeled with secondary diagonal arguments A and Ψ for nucleons and with A, Ψ_1, Ψ_2, Ψ_3 for Δ, the off-diagonal arguments being unchanged.

The definition of secondary arguments in terms of primary ones is chosen as

$$A = \tfrac{1}{2}[(11) + (22)], \quad \Psi = \tfrac{1}{2}[(11) - (22)], \qquad (2.16)$$

for nucleon. For Δ-particle, we choose

$$\begin{aligned} A &= \tfrac{1}{2}[(11) + (22) + (33) + (44)], \\ \Psi_1 &= \tfrac{1}{2}[(11) - (22) - (33) + (44)], \\ \Psi_2 &= \tfrac{1}{2}[-(11) + (22) - (33) + (44)], \\ \Psi_3 &= \tfrac{1}{2}[-(11) - (22) + (33) + (44)]. \end{aligned} \qquad (2.17)$$

The argument A, standing for averaged, is obtained by summing over all diagonal states of the particle. For spin-$\frac{1}{2}$ particle, the argument Ψ corresponds to the polarization along the quantization direction. For spin-$\frac{3}{2}$ particle, the choice of the three Ψ_i arguments is not unique, since the "null-sum" criterion is not sufficient to fix coefficients. The choice proposed in Eq. (2.17) is adapted for taking into account Bohr's rules, and more details concerning this choice can be found in Ref. [8]. Unfortunately, the choice of secondary observables, more adapted to experiments, increases the complexity of their relations with bicoms. The size of the submatrices connecting secondary observables and bicoms is different of the size of primary observable-bicom submatrices. The averaged observables (partially or completely) are related to more bicoms than the unaveraged ones. In particular, the determination of each magnitude $|D(\lambda,l;\Lambda,L)|$ requires the knowledge of the 32 secondary observables in which the three nucleon arguments take the two possibilities A and Ψ and the Δ

argument the four possibilities A, Ψ_1, Ψ_2, Ψ_3.

In terms of secondary observables, the systematic determination of the amplitudes is not so easy than in terms of primary ones. The number of one-by-one bicom-observable submatrices is not sufficient and the theorem given before cannot be applied directly.

Parity invariance reduces the number of independent amplitudes by a factor 2 and limits the choices for the quantization direction of each particle. The transversity formalism is obtained for the quantization direction for all particles along the normal to the reaction plane, and the helicity formalism for the quantization direction of each particle along its own momentum. Invariance under reflection with respect to the scattering plane leads to the Bohr's rules, which are extensively discussed in Ref. [8]. Taking into account parity conservation with Bohr's rules, the number of linearly independent observables is 256, and the whole matrix connecting the observables and the bicoms is a 256×256 matrix, instead of 1024×1024 one. Note that this reduction of size appears in a quite different way for transversity or helicity frames; that leads to studying each case separately.

III. TRANSVERSITY AMPLITUDE DETERMINATION

The transversity formalism presented by Kotanski [17] is obtained with directions for each particle as in Fig. 1, where the quantization direction for all particles is the normal to the reaction plane. The transversity amplitude determination presented in this section, applied to the frame of Fig. 1, may be also applied to any frame where the quantization direction for all particles is the normal to the reaction plane. This is the case for the formalism defined with a same fixed basis for all particles, more precisely with the fixed unit vectors $(\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})$ of the right-handed orthonormal basis defined by [2, 7, 8]

$$\hat{\mathbf{l}} = \hat{\mathbf{k}}, \qquad \hat{\mathbf{n}} = \frac{\mathbf{k} \times \mathbf{k}_\Delta}{|\mathbf{k} \times \mathbf{k}_\Delta|}, \quad \hat{\mathbf{m}} = \hat{\mathbf{n}} \times \hat{\mathbf{l}}, \tag{3.1}$$

where $\mathbf{k}$ and $\mathbf{k}_\Delta$ are the initial-beam-nucleon and final-Δ center-of-mass three-momenta, respectively. The transversity frame of Eq. (3.1) may be convenient to determine the amplitudes $f_i(\theta_\Delta)$ and $g_i(\theta_\Delta)$ of the spin-space decomposition [2, 7, 8] of the Δ production matrix

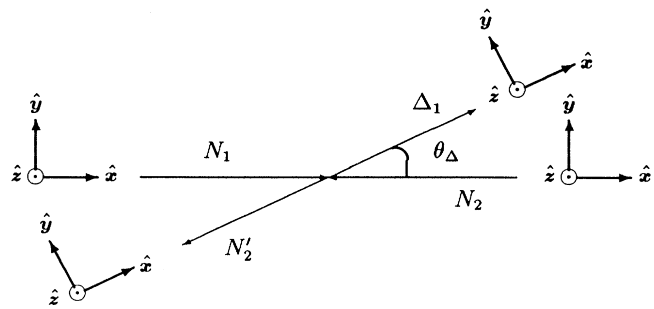

FIG. 1. Transversity frame.

analogous to the Wolfenstein representation in NN elastic scattering.

Parity invariance gives for transversity amplitudes

$$D^t(\lambda, l; \Lambda, L) = (-)^{\lambda+l+\Lambda+L} D^t(\lambda, l; \Lambda, L), \tag{3.2}$$

and set equal to zero half of these. In some cases, for which explicit notation is not necessary, the 16 remaining nonzero amplitudes are denoted T_i for $i = 1, \ldots, 16$, the correspondence being given in Table I.

Parity conservation gives, for primary observables the relation

$$\mathcal{L}^t(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = (-)^W \mathcal{L}^t(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q), \tag{3.3}$$

with $W = (u+v+U+V+\xi+\omega+\Xi+\Omega)$, which set equal to zero half, or 512, of these observables. In addition, among the remaining 512 observables, 256 simple relations exist which may be written as

$$S\mathcal{L}^t(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = (-)^{W_B} S'\mathcal{L}^t(uvH_{p'}, UVH_{P'}; \xi\omega H_{q'}, \Xi\Omega H_{Q'}), \tag{3.4}$$

with $W_B = (v + V + \xi + \Xi + 1)$, and with $p' = (-)^{u+v}p$, $P' = (-)^{U+V}P$, $Q' = (-)^{\Xi+\Omega}Q$ and $q' = (-)^{\xi+\omega}q$, the product $p'P'q'Q'$ being equal to $pPqQ$.

The classification of the linearly independent transver-

TABLE I. Abbreviated notation for transversity amplitudes $D^t(\lambda, l; \Lambda, L)$.

	$l = +1/2$, $L = +1/2$	$l = +1/2$, $L = -1/2$	$l = -1/2$, $L = +1/2$	$l = -1/2$, $L = -1/2$
$\lambda = +3/2, \Lambda = +1/2$	0	T_1	T_2	0
$\lambda = +3/2, \Lambda = -1/2$	T_3	0	0	T_4
$\lambda = +1/2, \Lambda = +1/2$	T_5	0	0	T_6
$\lambda = +1/2, \Lambda = -1/2$	0	T_7	T_8	0
$\lambda = -1/2, \Lambda = +1/2$	0	T_9	T_{10}	0
$\lambda = -1/2, \Lambda = -1/2$	T_{11}	0	0	T_{12}
$\lambda = -3/2, \Lambda = +1/2$	T_{13}	0	0	T_{14}
$\lambda = -3/2, \Lambda = -1/2$	0	T_{15}	T_{16}	0

TABLE II. $n(d\times d)$: number n of linearly independent transversity observables, connected with bicoms by $(d\times d)$ submatrices.

Number of diagonal arguments	Primary without parity	Primary with parity	Secondary without parity	Secondary with parity
$4(3N+\Delta)$	$32(1\times1)$	$16(1\times1)$	$32(32\times32)$	$16(16\times16)$
$3(3N)$	$96(1\times1)$	$16(1\times1)$	$96(8\times8)$	$16(4\times4)$
$3(2N+\Delta)$	$96(1\times1)$	0	$96(16\times16)$	0
$2(2N)$	$288(2\times2)$	$96(1\times1)$	$288(8\times8)$	$96(4\times4)$
$2(1N+\Delta)$	$96(2\times2)$	$48(1\times1)$	$96(16\times16)$	$48(8\times8)$
$1(1N)$	$288(4\times4)$	$48(2\times2)$	$288(8\times8)$	$48(4\times4)$
$1(\Delta)$	$32(4\times4)$	0	$32(16\times16)$	0
0	$96(8\times8)$	$32(4\times4)$	$96(8\times8)$	$32(4\times4)$

sity observables with respect to the number of diagonal arguments is given in Table II. In the second column, the parity conservation is not taken into account and the result is not specific of the transversity frame; the number of primary observables and the size of the submatrices, which are reported in this column, reflect exactly Eqs. (2.3) to (2.7). Under parity conservation, the classification of the 256 remaining linearly independent primary observables [Eqs. (3.3) and (3.4)] is given in the third column. Note that the parity constraints reduce the dimension of the submatrices. It is easy to see that, by inserting Eq. (3.2) in Eqs. (2.3), (2.6), and (2.7), we are left with 1×1, 2×2, and 4×4 submatrices, only.

Tables III and IV permit one to visualize from which primary observables the real and imaginary parts of a bicom may be extracted. For simplifying the notation, 1×1, 2×2, and 4×4 submatrices are denoted $1_i^{\xi\omega}$, $2_i^{\xi\omega}$ and $4^{\xi\omega}$, respectively, where the i index is simply a serial number designing on which subspaces of observables the submatrices act. Note that the subspaces of observables are different for calculating real or imaginary parts of the bicom (see Table IV). The manner of using the Tables III and IV is the following: as an example, on the crossing lines of the bicom $T_1^*T_2$, in Table III, we read 1_5^{11} which means that this bicom is related by a 1×1 submatrix to observable. From Table IV, we know that $\mathrm{Re}(T_1^*T_2)$ is related to one observable of the type $\mathcal{L}^t(12R, 12R; 11, \Xi\Xi)$. For more precisions, i.e., the value of Ξ and the proportionality coefficient, the return to the explicit notation for T_1 and T_2 is useful and the calculation of Eq. (2.12) can be performed. As another example, $\mathrm{Re}(T_1^*T_{16})$ is a combination of the four following observ-

TABLE III. Dimension and type of matrices $M_i^{\xi\omega}$ connecting bicoms and primary transversity observables.

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}
T_1^*	1^{11}	1_5^{11}	1_7^{11}	1_6^{11}	1_3^{12}	1_4^{12}	1_2^{12}	4^{12}	1_1^{13}	2_3^{13}	2_1^{13}	2_2^{13}	1_3^{14}	1_4^{14}	1_2^{14}	4^{14}
T_2^*		1^{11}	1_6^{11}	1_7^{11}	1_4^{12}	1_3^{12}	4^{12}	1_2^{12}	2_3^{13}	1_1^{13}	2_2^{13}	2_1^{13}	1_4^{14}	1_3^{14}	4^{14}	1_2^{14}
T_3^*			1^{11}	1_5^{11}	1_2^{12}	4^{12}	1_3^{12}	1_4^{12}	2_1^{13}	2_2^{13}	1_1^{13}	2_3^{13}	1_2^{14}	4^{14}	1_3^{14}	1_4^{14}
T_4^*				1^{11}	4^{12}	1_2^{12}	1_4^{12}	1_3^{12}	2_2^{13}	2_1^{13}	2_3^{13}	1_1^{13}	4^{14}	1_2^{14}	1_4^{14}	1_3^{14}
T_5^*					1^{22}	1_5^{22}	1_7^{22}	1_6^{22}	1_3^{23}	1_4^{23}	1_2^{23}	4^{23}	1_1^{24}	2_3^{24}	2_1^{24}	2_2^{24}
T_6^*						1^{22}	1_6^{22}	1_7^{22}	1_4^{23}	1_3^{23}	4^{23}	1_2^{23}	2_3^{24}	1_1^{24}	2_2^{24}	2_1^{24}
T_7^*							1^{22}	1_5^{22}	1_2^{23}	4^{23}	1_3^{23}	1_4^{23}	2_1^{24}	2_2^{24}	1_1^{24}	2_3^{24}
T_8^*								1^{22}	4^{23}	1_2^{23}	1_4^{23}	1_3^{23}	2_2^{24}	2_1^{24}	2_3^{24}	1_1^{24}
T_9^*									1^{33}	1_5^{33}	1_7^{33}	1_6^{33}	1_3^{34}	1_4^{34}	1_2^{34}	4^{34}
T_{10}^*										1^{33}	1_6^{33}	1_7^{33}	1_4^{34}	1_3^{34}	4^{34}	1_2^{34}
T_{11}^*											1^{33}	1_5^{33}	1_2^{34}	4^{34}	1_3^{34}	1_4^{34}
T_{12}^*												1^{33}	4^{34}	1_2^{34}	1_4^{34}	1_3^{34}
T_{13}^*													1^{44}	1_5^{44}	1_7^{44}	1_6^{44}
T_{14}^*														1^{44}	1_6^{44}	1_7^{44}
T_{15}^*															1^{44}	1_5^{44}
T_{16}^*																1^{44}

TABLE IV. Primary transversity observables on which act the matrices defined in Table III.

Diagonal arguments	Matrices (Table III)	$\mathrm{Re}(T_i^* T_j)$	$\mathrm{Im}(T_i^* T_j)$
$(3N+\Delta)$	$1^{\xi\xi}$	$\mathcal{L}^t(uu, UU; \xi\xi, \Xi\Xi)$	
$(3N)$	$1_1^{\xi\omega}$	$\mathcal{L}^t(uu, UU; \xi\omega R, \Xi\Xi)$	$\mathcal{L}^t(uu, UU; \xi\omega I, \Xi\Xi)$
$(2N)$	$1_2^{\xi\omega}$	$\mathcal{L}^t(uu, UU; \xi\omega R, 12R)$	$\mathcal{L}^t(uu, UU; \xi\omega R, 12I)$
	$1_3^{\xi\omega}$	$\mathcal{L}^t(uu, 12R; \xi\omega R, \Xi\Xi)$	$\mathcal{L}^t(uu, 12I; \xi\omega R, \Xi\Xi)$
	$1_4^{\xi\omega}$	$\mathcal{L}^t(12R, UU; \xi\omega R, \Xi\Xi)$	$\mathcal{L}^t(12I, UU; \xi\omega R, \Xi\Xi)$
$(1N+\Delta)$	$1_5^{\xi\xi}$	$\mathcal{L}^t(12R, 12R; \xi\xi, \Xi\Xi)$	$\mathcal{L}^t(12I, 12R; \xi\xi, \Xi\Xi)$
	$1_6^{\xi\xi}$	$\mathcal{L}^t(12R, UU; \xi\xi, 12R)$	$\mathcal{L}^t(12I, UU; \xi\xi, 12R)$
	$1_7^{\xi\xi}$	$\mathcal{L}^t(uu, 12R; \xi\xi, 12R)$	$\mathcal{L}^t(uu, 12I; \xi\xi, 12R)$
$(1N)$	$2_1^{\xi\omega}$	$\mathcal{L}^t(uu, 12H_P; \xi\omega H_q, 12R)$ for $Pq=+1$	$\mathcal{L}^t(uu, 12H_P; \xi\omega H_q, 12R)$ for $Pq=-1$
	$2_2^{\xi\omega}$	$\mathcal{L}^t(12H_p, UU; \xi\omega H_q, 12R)$ for $pq=+1$	$\mathcal{L}^t(12H_p, UU; \xi\omega H_q, 12R)$ for $pq=-1$
	$2_3^{\xi\omega}$	$\mathcal{L}^t(12H_p, 12H_P; \xi\omega R, \Xi\Xi)$ for $pP=+1$	$\mathcal{L}^t(12H_p, 12H_P; \xi\omega R, \Xi\Xi)$ for $pP=-1$
(0)	$4^{\xi\omega}$	$\mathcal{L}^t(12H_p, 12H_P; \xi\omega R, 12H_Q)$ for $pPQ=+1$	$\mathcal{L}^t(12H_p, 12H_P; \xi\omega R, 12H_Q)$ for $pPQ=-1$

ables: $\mathcal{L}^t(12R, 12R; 14R, 12R)$, $\mathcal{L}^t(12I, 12I; 14R, 12R)$, $\mathcal{L}^t(12R, 12I; 14R, 12I)$, and $\mathcal{L}^t(12I, 12R; 14R, 12I)$. The set of Tables III and IV does not give the coefficients of the combinations of bicoms, but it is not necessary for the purpose of this paper, which is to present a methodology for determining the $N_1 N_2 \rightarrow \Delta_1 N_2'$ reaction amplitudes.

By application of the theorem given in Sec. II, the question of how to design a set of 31 observables which determine the 16 complex amplitudes is very easy, a sufficient number of 1x1 submatrices being at our disposal.

First, each of the 16 magnitudes is obtained from each of the 16 nonzero observables $\mathcal{L}^t(uu, UU; \xi\xi, \Xi\Xi)$. Second, through the 80 submatrices of the type 1_i in Table III, i.e., among the 160 corresponding observables given in Table IV, a set of 15 observables is chosen, step by step, which determine 15 independent relative phases between the 16 complex amplitudes. If the set of 16 experiments giving the magnitudes is unique, many sets of 15 experiments lead to the determination of the phases. As said in Sec. II, all the combinations taking 15 observables among 160 do not have to be retained; because of quadratic relations between observables, they lead to phases which are not independent. In particular, the choice of an observable giving the real part of a bicom excludes the choice of the observable corresponding to the imaginary part of the same bicom; evidently, an ambiguity on the sign of the phase appears. For releasing these ambiguities, more than 31 observables are needed. General criteria for resolving the ambiguities can be found in Refs. [18, 19].

Concerning the transversity secondary observables, the constraints of parity conservation give

$$\mathcal{L}^t(\alpha, \beta; \gamma, \delta) = (-)^{W'} \mathcal{L}^t(\alpha, \beta; \gamma, \delta), \tag{3.5}$$

where $W' = [\alpha] + [\beta] + [\gamma] + [\delta]$ with

$$[\alpha] = \begin{cases} 0 & \text{if } \alpha = A, \Psi, \\ 1 & \text{if } \alpha = 12R, 12I, \end{cases} \tag{3.6}$$

and similarly for $[\beta]$ and $[\delta]$ nucleon arguments and for the Δ-argument

$$[\gamma] = \begin{cases} 0 & \text{if } \gamma = A, \Psi_1, \Psi_2, \Psi_3, \\ \xi + \omega & \text{if } \gamma = \xi\omega R, \xi\omega I, \end{cases} \tag{3.7}$$

Equation (3.5) sets equal to zero half of secondary observables.

The choice of secondary arguments in terms of primary ones, advocated in Eqs. (2.16) and (2.17), is adapted for taking into account Bohr's rules, which lead to the relation

$$S\mathcal{L}^t(\alpha, \beta; \gamma, \delta) = (-)^{W_B'} S' \mathcal{L}^t(\alpha', \beta'; \gamma', \delta'), \tag{3.8}$$

where α, β, δ and α', β', δ' nucleon secondary arguments are interchanged as follows

$$A \leftrightarrow \Psi, \qquad 12R \leftrightarrow 12I, \tag{3.9}$$

and where γ and γ' secondary Δ arguments are interchanged as

$$\begin{aligned} A \leftrightarrow \Psi_2, \qquad \Psi_1 \leftrightarrow \Psi_3, \\ 13R(I) \leftrightarrow 13R(I), \qquad 24R(I) \leftrightarrow 24R(I), \\ 12R \leftrightarrow 12I, \qquad 14R \leftrightarrow 14I, \\ 23R \leftrightarrow 23I, \qquad 34R \leftrightarrow 34I. \end{aligned} \tag{3.10}$$

The sign in Eq. (3.8) is obtained with $W'_B = [\alpha]+[\beta]+[\xi]$ where

$$[\xi] = \begin{cases} 0 & \text{if } \gamma = A, \Psi_1, \Psi_2, \Psi_3, \\ \xi & \text{if } \gamma = \xi\omega R, \xi\omega I. \end{cases} \tag{3.11}$$

We recall that the sign S, defined in Eq. (2.3), depends on the number of I indices among α, β, γ, and δ, and, similarly, S' on the number of I indices among α', β', γ', and δ'. Combining Eqs. (3.5) and (3.8), we are left with 256 linearly independent observables.

In Table II, the number of linearly independent secondary observables is given without parity constraints in the 4th column, and with the constraints [Eqs. (3.5) and (3.8)] in the 5th column. Finally, the submatrices which connect the bicoms and the secondary observables are of size 4×4, 8×8, and 16×16. The 16×16 submatrix connects secondary observables with the magnitude squares of the amplitudes.

Tables V and VI permit one to determine from which secondary observables the real and imaginary parts of a bicom may be extracted. The use of the set of Tables V and VI, concerning secondary observables, is exactly the same as the use of the set of Tables III and IV, which concerns primary observables. Here, for simplifying the notation, 4×4, 8×8, and 16×16 submatrices are denoted $4_i^{\xi\omega}$, 8_i, and 16, respectively, where the i index is simply a serial number designing on which subspaces of secondary observables the submatrices act. As an example, from Table IV, the bicom $T_1^*T_2$ is related by a 8×8 submatrix, denoted 8_1, to secondary observables. From Table VI, the real part of $T_1^*T_2$ is related to eight secondary observables of the type $\mathcal{L}^t(12R,12R;\gamma,\delta)$, the value of the secondary Δ argument γ being A, Ψ_1, Ψ_2, and Ψ_3, and the value of the secondary nucleon argument δ being A and Ψ.

The question of how to design a set of 31 secondary observables for determining the 16 complex amplitudes is not so easy than in the case of the primary observables, because no 1×1 submatrix is at our disposal.

As said earlier, the determination of the amplitudes is performed in two stages. The first stage consists of measuring the 16 independent secondary observables $\mathcal{L}^t(\alpha,\beta;\gamma,\delta)$ with α, β, and δ equal to A or Ψ and γ equal to A, Ψ_1, Ψ_2 and Ψ_3, the magnitude squares of the amplitudes being given in terms of linear combinations of these observables. Table VII shows that the magnitudes can be determined from experiments which involve, at most, two polarized particles at a time.

The magnitudes being determined, the second stage consists of finding a set of, at least, 15 experiments for the determination of independent relative phases. Moravcsik, Sinky, and Goldstein claim [15] that the easiest way to ensure that a set determines completely the amplitudes is to measure all the observables appearing in the subspace on which acts a submatrix. The submatrices at disposal being of size 4×4 and 8×8, it is clear that the Moravcsik's claim leads to a total of more than 15 experiments. A

TABLE V. Dimension and type of matrices $M_i^{\xi\omega}$ connecting bicoms and secondary transversity observables.

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}
T_1^*	16	8_1	8_3	8_2	4_3^{12}	4_4^{12}	4_2^{12}	4_8^{12}	4_1^{13}	4_7^{13}	4_5^{13}	4_6^{13}	4_3^{14}	4_4^{14}	4_2^{14}	4_8^{14}
T_2^*		16	8_2	8_3	4_4^{12}	4_3^{12}	4_8^{12}	4_2^{12}	4_7^{13}	4_1^{13}	4_6^{13}	4_5^{13}	4_4^{14}	4_3^{14}	4_8^{14}	4_2^{14}
T_3^*			16	8_1	4_2^{12}	4_8^{12}	4_3^{12}	4_4^{12}	4_5^{13}	4_6^{13}	4_1^{13}	4_7^{13}	4_2^{14}	4_8^{14}	4_3^{14}	4_4^{14}
T_4^*				16	4_8^{12}	4_2^{12}	4_4^{12}	4_3^{12}	4_6^{13}	4_5^{13}	4_7^{13}	4_1^{13}	4_8^{14}	4_2^{14}	4_4^{14}	4_3^{14}
T_5^*					16	8_1	8_3	8_2	4_3^{23}	4_4^{23}	4_2^{23}	4_8^{23}	4_1^{24}	4_7^{24}	4_5^{24}	4_6^{24}
T_6^*						16	8_2	8_3	4_4^{23}	4_3^{23}	4_8^{23}	4_2^{23}	4_7^{24}	4_1^{24}	4_6^{24}	4_5^{24}
T_7^*							16	8_1	4_2^{23}	4_8^{23}	4_3^{23}	4_4^{23}	4_5^{24}	4_6^{24}	4_1^{24}	4_7^{24}
T_8^*								16	4_8^{23}	4_2^{23}	4_4^{23}	4_3^{23}	4_6^{24}	4_5^{24}	4_7^{24}	4_1^{24}
T_9^*									16	8_1	8_3	8_2	4_3^{34}	4_4^{34}	4_2^{34}	4_8^{34}
T_{10}^*										16	8_2	8_3	4_4^{34}	4_3^{34}	4_8^{34}	4_2^{34}
T_{11}^*											16	8_1	4_2^{34}	4_8^{34}	4_3^{34}	4_4^{34}
T_{12}^*												16	4_8^{34}	4_2^{34}	4_4^{34}	4_3^{34}
T_{13}^*													16	8_1	8_3	8_2
T_{14}^*														16	8_2	8_3
T_{15}^*															16	8_1
T_{16}^*																16

TABLE VI. Secondary transversity observables on which act the matrices defined in Table V. Here α, β, and δ are equal to A or Ψ for the nucleons and γ equal to A, Ψ_1, Ψ_2, or Ψ_3 for the Δ.

Diagonal arguments	Matrices (Table V)	$\mathrm{Re}(T_i^* T_j)$	$\mathrm{Im}(T_i^* T_j)$
$(3N+\Delta)$	16	$\mathcal{L}^t(\alpha,\beta;\gamma,\delta)$	
$(3N)$	$4_1^{\xi\omega}$	$\mathcal{L}^t(\alpha,\beta;\xi\omega R,\delta)$	$\mathcal{L}^t(\alpha,\beta;\xi\omega I,\delta)$
$(2N)$	$4_2^{\xi\omega}$	$\mathcal{L}^t(\alpha,\beta;\xi\omega R,12R)$	$\mathcal{L}^t(\alpha,\beta;\xi\omega R,12I)$
	$4_3^{\xi\omega}$	$\mathcal{L}^t(\alpha,12R;\xi\omega R,\delta)$	$\mathcal{L}^t(\alpha,12I;\xi\omega R,\delta)$
	$4_4^{\xi\omega}$	$\mathcal{L}^t(12R,\beta;\xi\omega R,\delta)$	$\mathcal{L}^t(12I,\beta;\xi\omega R,\delta)$
$(1N+\Delta)$	8_1	$\mathcal{L}^t(12R,12R;\gamma,\delta)$	$\mathcal{L}^t(12I,12R;\gamma,\delta)$
	8_2	$\mathcal{L}^t(12R,\beta;\gamma,12R)$	$\mathcal{L}^t(12I,\beta;\gamma,12R)$
	8_3	$\mathcal{L}^t(\alpha,12R;\gamma,12R)$	$\mathcal{L}^t(\alpha,12I;\gamma,12R)$
$(1N)$	$4_5^{\xi\omega}$	$\mathcal{L}^t(\alpha,12H_P;\xi\omega H_q,12R)$ for $Pq=+1$	$\mathcal{L}^t(\alpha,12H_P;\xi\omega H_q,12R)$ for $Pq=-1$
	$4_6^{\xi\omega}$	$\mathcal{L}^t(12H_p,\beta;\xi\omega H_q,12R)$ for $pq=+1$	$\mathcal{L}^t(12H_p,\beta;\xi\omega H_q,12R)$ for $pq=-1$
	$4_7^{\xi\omega}$	$\mathcal{L}^t(12H_p,12R;\xi\omega H_q,\delta)$ for $pq=+1$	$\mathcal{L}^t(12H_p,12R;\xi\omega H_q,\delta)$ for $pq=-1$
(0)	$4_8^{\xi\omega}$	$\mathcal{L}^t(12H_p,12H_P;\xi\omega R,12H_Q)$ for $pPQ=+1$	$\mathcal{L}^t(12H_p,12H_P;\xi\omega R,12H_Q)$ for $pPQ=-1$

total of 16 experiments can be obtained from two 8×8 matrices, or from four 4×4 ones, or from one 8×8 and two 4×4 ones. A check of these possibilities show that they cannot determine completely the phases. With a total of 20 experiments, possibilities obtained from five 4×4 matrices, only, lead to a complete determination of the phases.

Yet, relaxing the preceeding claim, the problem can be solved completely with 15 experiments only, from five 4×4 submatrices. The method is to measure all the observables appearing in three subspaces, two observables appearing in a 4th subspace and one observable appearing in a 5th subspace. The inconvenient of such a method lies in the increasing of the number of ambiguities.

The choice of the five 4×4 submatrices obey to precise rules. The three subspaces of observables entirely determined must necessarily correspond to three different values of the Δ argument among $\xi\omega = 12, 13, 14, 23, 24, 34$. In addition, if these three different Δ arguments are denoted $\xi\omega$, $\xi'\omega'$, and $\xi''\omega''$, it is necessary that the indices 1,2,3,4 appear, once at least, among the six indices ξ, ξ', ξ'', ω, ω', and ω''. These three chosen submatrices belong to a set of six matrices, called "first exclusion set," which is written

$$(4_i^{12},\ 4_j^{14},\ 4_k^{23},\ 4_l^{34},\ 4_m^{13},\ 4_n^{24}), \tag{3.12}$$

with $i,j,k,l = 2,3,4,8$ and $m,n = 1,5,6,7$. In the triad of indices (i,j,n), the n serial index is given with respect to i and j by

$$n = \begin{cases} i+j & \text{if } i \neq j \text{ and } i,j = 2,3,4, \\ |i-j|+1 & \text{otherwise,} \end{cases} \tag{3.13}$$

and similarly, for the triads (i,k,m), (j,l,m), and (k,l,n). The relation between the indices of a triad (i_1,i_2,i_3) is explicitly given in Table VIII. The 4th submatrix is chosen without constraint among the 18 available submatrices obtained after subtracting the first exclusion set from the 24 initial matrices. The 4th submatrix belongs to a set of six matrices, called "second exclusion set," which is written

$$(4_{i'}^{12},\ 4_{j'}^{14},\ 4_{k'}^{23},\ 4_{l'}^{34},\ 4_{m'}^{13},\ 4_{n'}^{24}), \tag{3.14}$$

TABLE VII. A set of 31 secondary observables determining the 16 complex transversity amplitudes $D^t(\lambda,l;\Lambda,L)$.

	Magnitudes		
$\mathcal{L}^t(A,A;A,A)$	$\mathcal{L}^t(\Psi,A;A,A)$	$\mathcal{L}^t(A,\Psi;A,A)$	$\mathcal{L}^t(A,A;A,\Psi)$
$\mathcal{L}^t(A,A;\Psi_1,A)$	$\mathcal{L}^t(\Psi,A;\Psi_1,A)$	$\mathcal{L}^t(A,\Psi;\Psi_1,A)$	$\mathcal{L}^t(A,A;\Psi_1,\Psi)$
$\mathcal{L}^t(A,A;\Psi_2,A)$	$\mathcal{L}^t(\Psi,A;\Psi_2,A)$	$\mathcal{L}^t(A,\Psi;\Psi_2,A)$	$\mathcal{L}^t(A,A;\Psi_2,\Psi)$
$\mathcal{L}^t(A,A;\Psi_3,A)$	$\mathcal{L}^t(\Psi,A;\Psi_3,A)$	$\mathcal{L}^t(A,\Psi;\Psi_3,A)$	$\mathcal{L}^t(A,A;\Psi_3,\Psi)$
	Phases		
$\mathcal{L}^t(A,A;13R,A)$	$\mathcal{L}^t(\Psi,A;13R,A)$	$\mathcal{L}^t(A,\Psi;13R,A)$	$\mathcal{L}^t(A,A;13R,\Psi)$
$\mathcal{L}^t(A,A;24R,A)$	$\mathcal{L}^t(\Psi,A;24R,A)$	$\mathcal{L}^t(A,\Psi;24R,A)$	$\mathcal{L}^t(A,A;24R,\Psi)$
$\mathcal{L}^t(A,A;12R,12R)$	$\mathcal{L}^t(\Psi,A;12R,12R)$	$\mathcal{L}^t(A,\Psi;12R,12R)$	$\mathcal{L}^t(A,A;12I,12I)$
$\mathcal{L}^t(A,12R;12R,A)$	$\mathcal{L}^t(A,12I;12I,A)$	$\mathcal{L}^t(12R,A;12R,A)$	

TABLE VIII. The i_3 serial index of the triad (i_1, i_2, i_3).

	$i_1 = 2$	$i_1 = 3$	$i_1 = 4$	$i_1 = 8$
$i_2 = 2$	1	5	6	7
$i_2 = 3$	5	1	7	6
$i_2 = 4$	6	7	1	5
$i_2 = 8$	7	6	5	1

with $i', j', k', l' = 2, 3, 4, 8$ and $m', n' = 1, 5, 6, 7$, and $i' \neq i$, $j' \neq j$, $k' \neq k$, $l' \neq l$, $m' \neq m$, $n' \neq n$. The triads of indices (i', j', n), (i', j, n'), (i', k', m), (i', k, m'), (j', l', m), (j, l', m'), (k', l', n), and (k, l', n') obey the rule of Table VIII. The 5th submatrix is chosen without constraint among the 12 available submatrices obtained after subtracting the first and second exclusion sets from the 24 initial matrices.

For giving an explicit example, we can choose to measure experiments with as much as possible diagonal nucleon arguments (i.e., A and Ψ arguments), the Δ argument being necessarily nondiagonal ($\xi \neq \omega$). This can be done from 4_1^{13} and 4_1^{24} submatrices, plus one submatrix $4_i^{\xi\omega}$ chosen among the twelve submatrices $4_{2,3,4}^{\xi\omega}$ for $\xi\omega = 12, 14, 23, 34$. To these twelve experiments, two can be added, which correspond to a $4_j^{\xi\omega}$ chosen among the eight submatrices $4_{2,3,4}^{\xi\omega}$ for $j \neq i$ and $\xi\omega = 12, 14, 23, 34$. The 15th experiment appears in the subspace of a $4_k^{\xi\omega}$ submatrix chosen among four submatrices $4_{2,3,4}^{\xi\omega}$ for $k \neq j$, $k \neq i$ and $\xi\omega = 12, 14, 23, 34$. Finally, a set of 31 secondary observables which determine completely (up to a global phase and a discrete number of ambiguities) the 16 complex transversity amplitudes is given in Table VII, as an explicit example, by taking the sequence 4_1^{13}, 4_1^{24}, 4_2^{12}, 4_3^{12}, 4_4^{12} submatrices to determine the phases.

IV. HELICITY AMPLITUDE DETERMINATION

The helicity formalism developed by Jacob and Wick [20] is obtained with directions for each particle as in Fig. 2, where the quantization direction of each particle is its own momentum. For invariance under parity, helicity amplitudes satisfy

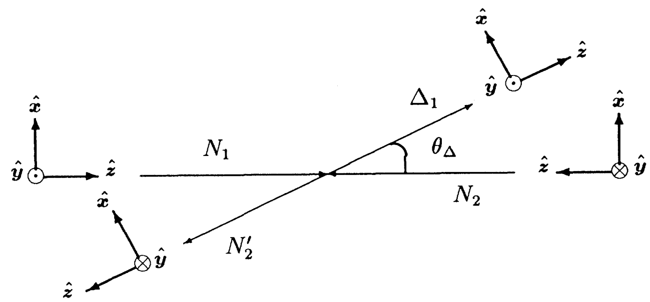

FIG. 2. Helicity frame.

$$D^h(\lambda, l; \Lambda, L) = (-)^{\lambda+l+\Lambda+L+1} D^h(-\lambda, -l; -\Lambda, -L). \tag{4.1}$$

In some cases, for which explicit notation is not necessary, the 16 linearly independent remaining amplitudes are denoted F_i and G_i for $i = 1, \ldots, 8$, the correspondence being given in Table IX.

Parity conservation leads to the following relation between primary observables

$$\mathcal{L}^h(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = pPqQ(-)^W \mathcal{L}^h(\tilde{v}\tilde{u}H_p, \tilde{V}\tilde{U}H_P; \tilde{\omega}\tilde{\xi}H_q, \tilde{\Omega}\tilde{\Xi}H_Q), \tag{4.2}$$

with $W = (u + v + U + V + \xi + \omega + \Xi + \Omega)$, and where the "mirror" index $\tilde{u}$ is related to the index u by $\tilde{u} = 2, 1$ for $u = 1, 2$, respectively, and similarly for the nucleon indices v, U, V, Ξ, and Ω. For the Δ, the "mirror" index $\tilde{\xi}$ is related to the index ξ by $\tilde{\xi} = 4, 3, 2, 1$ for $\xi = 1, 2, 3, 4$, respectively, and similarly for the index ω. Remark that equality between $\tilde{v}\tilde{u}$ and uv for 12 and between $\tilde{\omega}\tilde{\xi}$ and $\xi\omega$ for 14 and 23 yields 16 vanishing primary observables. Equation (4.2) leads to 512 linearly independent observables. In addition, 256 more complicated linear relations exist, which are given by

$$\begin{aligned} pPS\mathcal{L}^h(uvH_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = \frac{1}{8}(-)^W \sum_{p'P'q'Q'=pPqQ} p'P'S' \, [& (-)^{u+U+\xi+\Xi}\mathcal{L}^h(u\tilde{v}H_{p'}, U\tilde{V}H_{P'}; \xi\tilde{\omega}H_{q'}, \Xi\tilde{\Omega}H_{Q'}) \\ & + p(-)^{v+U+\xi+\Xi}\mathcal{L}^h(v\tilde{u}H_{p'}, U\tilde{V}H_{P'}; \xi\tilde{\omega}H_{q'}, \Xi\tilde{\Omega}H_{Q'}) \\ & + P(-)^{u+V+\xi+\Xi}\mathcal{L}^h(u\tilde{v}H_{p'}, V\tilde{U}H_{P'}; \xi\tilde{\omega}H_{q'}, \Xi\tilde{\Omega}H_{Q'}) \\ & + pP(-)^{v+V+\xi+\Xi}\mathcal{L}^h(v\tilde{u}H_{p'}, V\tilde{U}H_{P'}; \xi\tilde{\omega}H_{q'}, \Xi\tilde{\Omega}H_{Q'}) \\ & + Q(-)^{u+U+\xi+\Omega}\mathcal{L}^h(u\tilde{v}H_{p'}, U\tilde{V}H_{P'}; \xi\tilde{\omega}H_{q'}, \Omega\tilde{\Xi}H_{Q'}) \\ & + pQ(-)^{v+U+\xi+\Omega}\mathcal{L}^h(v\tilde{u}H_{p'}, U\tilde{V}H_{P'}; \xi\tilde{\omega}H_{q'}, \Omega\tilde{\Xi}H_{Q'}) \\ & + PQ(-)^{u+V+\xi+\Omega}\mathcal{L}^h(u\tilde{v}H_{p'}, V\tilde{U}H_{P'}; \xi\tilde{\omega}H_{q'}, \Omega\tilde{\Xi}H_{Q'}) \\ & + pPQ(-)^{v+V+\xi+\Omega}\mathcal{L}^h(v\tilde{u}H_{p'}, V\tilde{U}H_{P'}; \xi\tilde{\omega}H_{q'}, \Omega\tilde{\Xi}H_{Q'})]. \end{aligned} \tag{4.3}$$

TABLE IX. Abbreviated notation for helicity amplitudes $D^h(\lambda, l; \Lambda, L)$.

	$l=+1/2$ $L=+1/2$	$l=+1/2$ $L=-1/2$	$l=-1/2$ $L=+1/2$	$l=-1/2$ $L=-1/2$
$\lambda=+3/2, \Lambda=+1/2$	G_1	F_1	F_2	G_2
$\lambda=+3/2, \Lambda=-1/2$	F_3	G_3	G_4	F_4
$\lambda=+1/2, \Lambda=+1/2$	F_5	G_5	G_6	F_6
$\lambda=+1/2, \Lambda=-1/2$	G_7	F_7	F_8	G_8
$\lambda=-1/2, \Lambda=+1/2$	G_8	$-F_8$	$-F_7$	G_7
$\lambda=-1/2, \Lambda=-1/2$	$-F_6$	G_6	G_5	$-F_5$
$\lambda=-3/2, \Lambda=+1/2$	$-F_4$	G_4	G_3	$-F_3$
$\lambda=-3/2, \Lambda=-1/2$	G_2	$-F_2$	$-F_1$	G_1

The summation runs over the four indices p', P', q', Q', each taking the values ±1 for off-diagonal arguments, but only the value $+1$ for diagonal arguments, the product $p'P'q'Q'$ being equal to $pPqQ$.

The classification of the linearly independent helicity observables with respect to the number of diagonal arguments is given in Table X. In the second column, the parity conservation is not taken into account and the result, being not specific of the helicity frame, is the same as in Table II. Under parity conservation, the classification of a set of 256 remaining linearly independent primary observables is given in the third column. Equation (4.2) links observables which have the same number of diagonal arguments for the nucleons and for the Δ, and simply divides by a factor 2 the number of observables of the second column. Equation (4.3) links five or nine observables with different numbers of diagonal arguments for the nucleons and the Δ.

From Eq. (4.3), there are 16 relations which link an observable, with zero diagonal argument and with $\xi\omega = 14, 23$, as linear combination of eight observables, with $(3N + \Delta)$ diagonal arguments. There are 32 relations which link an observable, with zero diagonal argument and with $\xi\omega = 12, 13$, as linear combination of eight observables, with $(3N)$ diagonal arguments. There are also 16 relations which link an observable, with the (Δ) diagonal argument, as linear combination of four observables, with $(3N)$ diagonal arguments. There are 48 relations which link an observable, with $(1N)$ diagonal argument and with $\xi\omega = 14, 23$, as linear combination of four observables, with $(2N + \Delta)$ diagonal arguments. There are 96 relations which link an observable, with $(1N)$ diagonal argument and with $\xi\omega = 12, 13$, as linear combination of eight observables, with $(2N)$ diagonal arguments. Finally, there are 48 relations which link an observable, with $(1N + \Delta)$ diagonal arguments, as linear combination of four observables, with $(2N)$ diagonal arguments.

Among observables linked by Eq. (4.3), we choose to retain in the third column of the Table X a set of 256 linearly independent primary observables related to the bicoms by 1×1 submatrices, at most as possible. Contrary to the transversity case, the parity constraints do not reduce the dimension of the submatrices. In the third column of Table X, the retained set corresponds to $(3N + \Delta)$, $(3N)$, $(2N + \Delta)$, and $(2N)$ diagonal arguments; the item $144(2\times2)$ for $(2N)$ diagonal argument line may be replaced, partially or completely, by $48(2\times2)$ for $(1N + \Delta)$ one and by $96(4\times4)$ for $(1N)$ one.

Tables XI and XII permit one to determine from which primary observables the real and imaginary parts of a bicom may be extracted. For simplifying the notation, 1×1 and 2×2 submatrices are denoted $1_i^{\xi\omega}$ and $2_i^{\xi\omega}$, respectively, where the i index is simply a serial number designing on which subspace of observables the submatrices act. The manner of using the set of Tables XI and XII is the same as that for the set of Tables III and IV.

As for the transversity case, by application of the the-

TABLE X. $n(d \times d)$: number n of linearly independent helicity observables, connected with bicoms by $(d \times d)$ submatrices.

Number of diagonal arguments	Primary without parity	Primary with parity	Secondary without parity	Secondary with parity
$4(3N+\Delta)$	$32(1\times1)$	$16(1\times1)$	$32(32\times32)$	$16(16\times16)$
$3(3N)$	$96(1\times1)$	$48(1\times1)$	$96(8\times8)$	$32(8\times8)+16(4\times4)$
$3(2N+\Delta)$	$96(1\times1)$	$48(1\times1)$	$96(16\times16)$	$48(8\times8)$
$2(2N)$	$288(2\times2)$	$144(2\times2)$	$288(8\times8)$	$96(8\times8)+48(4\times4)$
$2(1N+\Delta)$	$96(2\times2)$	0	$96(16\times16)$	0
$1(1N)$	$288(4\times4)$	0	$288(8\times8)$	0
$1(\Delta)$	$32(4\times4)$	0	$32(16\times16)$	0
0	$96(8\times8)$	0	$96(8\times8)$	0

TABLE XI. Dimension and type of matrices $M_i^{\xi\omega}$ connecting bicoms and primary helicity observables.

	G_1	G_2	G_3	G_4	F_1	F_2	F_3	F_4	G_5	G_6	G_7	G_8	F_5	F_6	F_7	F_8
G_1^*	1^{11}	2_1^{14}	2_3^{14}	2_2^{14}	1_3^{11}	1_4^{11}	1_2^{11}	1_1^{14}	2_2^{12}	2_3^{12}	2_1^{12}	1_1^{13}	1_1^{12}	2_1^{13}	2_3^{13}	2_2^{13}
G_2^*		1^{11}	2_2^{14}	2_3^{14}	1_4^{11}	1_3^{11}	1_1^{14}	1_2^{11}	2_3^{12}	2_2^{12}	1_1^{13}	2_1^{12}	2_1^{13}	1_1^{12}	2_2^{13}	2_3^{13}
G_3^*			1^{11}	2_1^{14}	1_2^{11}	1_1^{14}	1_3^{11}	1_4^{11}	2_1^{12}	1_1^{13}	2_2^{12}	2_3^{12}	2_3^{13}	2_2^{13}	1_1^{12}	2_1^{13}
G_4^*				1^{11}	1_1^{14}	1_2^{11}	1_4^{11}	1_3^{11}	1_1^{13}	2_1^{12}	2_3^{12}	2_2^{12}	2_2^{13}	2_3^{13}	2_1^{13}	1_1^{12}
F_1^*					1^{11}	2_1^{14}	2_3^{14}	2_2^{14}	1_1^{12}	2_1^{13}	2_3^{13}	2_2^{13}	2_2^{12}	2_3^{12}	2_1^{12}	1_1^{13}
F_2^*						1^{11}	2_2^{14}	2_3^{14}	2_1^{13}	1_1^{12}	2_2^{13}	2_3^{13}	2_3^{12}	2_2^{12}	1_1^{13}	2_1^{12}
F_3^*							1^{11}	2_1^{14}	2_3^{13}	2_2^{13}	1_1^{12}	2_1^{13}	2_1^{12}	1_1^{13}	2_2^{12}	2_3^{12}
F_4^*								1^{11}	2_2^{13}	2_3^{13}	2_1^{13}	1_1^{12}	1_1^{13}	2_1^{12}	2_3^{12}	2_2^{12}
G_5^*									1^{22}	2_1^{23}	2_3^{23}	2_2^{23}	1_3^{22}	1_4^{22}	1_2^{22}	1_1^{23}
G_6^*										1^{22}	2_2^{23}	2_3^{23}	1_4^{22}	1_3^{22}	1_1^{23}	1_2^{22}
G_7^*											1^{22}	2_1^{23}	1_2^{22}	1_1^{23}	1_3^{34}	1_4^{22}
G_8^*												1^{22}	1_1^{23}	1_2^{22}	1_4^{34}	1_3^{22}
F_5^*													1^{22}	2_1^{23}	2_3^{23}	2_2^{23}
F_6^*														1^{22}	2_2^{23}	2_3^{23}
F_7^*															1^{22}	2_1^{23}
F_8^*																1^{22}

orem given in Sec. II, the question of how to design a set of 31 observables which determine the 16 complex amplitudes is very easy, a sufficient number of 1×1 submatrices being at our disposal.

First, each of the 16 magnitudes is obtained from each of the 16 independent observables $\mathcal{L}^h(uu,UU;\xi\xi,\Xi\Xi)$. Second, through the 48 submatrices of the type 1_i in Table XI, i.e., among the 96 corresponding observables given in Table XII, a set of 15 observables is chosen, step by step, which determines 15 independent relative phases between the 16 complex amplitudes. For instance, we can choose to measure observables with as much as possible diagonal nucleon arguments. This can be done from twelve submatrices of the type $1_1^{\xi\omega}$ for $\xi\omega = 12, 13, 14, 23$, plus three submatrices $1_i^{\xi\xi}$ chosen

TABLE XII. Primary helicity observables on which act the matrices defined in Table XI.

Diagonal arguments	Matrices (Table XI)	Real part of bicom	Imaginary part of bicom
$(3N+\Delta)$	$1^{\xi\xi}$	$\mathcal{L}^h(uu,UU;\xi\xi,\Xi\Xi)$	
$(3N)$	$1_1^{\xi\omega}$	$\mathcal{L}^h(uu,UU;\xi\omega R,\Xi\Xi)$	$\mathcal{L}^h(uu,UU;\xi\omega I,\Xi\Xi)$
$(2N+\Delta)$	$1_2^{\xi\xi}$	$\mathcal{L}^h(uu,UU;\xi\xi,12R)$	$\mathcal{L}^h(uu,UU;\xi\xi,12I)$
	$1_3^{\xi\xi}$	$\mathcal{L}^h(uu,12R;\xi\xi,\Xi\Xi)$	$\mathcal{L}^h(uu,12I;\xi\xi,\Xi\Xi)$
	$1_4^{\xi\xi}$	$\mathcal{L}^h(12R,UU;\xi\xi,\Xi\Xi)$	$\mathcal{L}^h(12I,UU;\xi\xi,\Xi\Xi)$
$(2N)$	$2_1^{\xi\omega}$	$\mathcal{L}^h(uu,UU;\xi\omega H_q,12H_Q)$	$\mathcal{L}^h(uu,UU;\xi\omega H_q,12H_Q)$
		for $qQ=+1$	for $qQ=-1$
	$2_2^{\xi\omega}$	$\mathcal{L}^h(uu,12H_P;\xi\omega H_q,\Xi\Xi)$	$\mathcal{L}^h(uu,12H_P;\xi\omega H_q,\Xi\Xi)$
		for $Pq=+1$	for $Pq=-1$
	$2_3^{\xi\omega}$	$\mathcal{L}^h(12H_p,UU;\xi\omega H_q,\Xi\Xi)$	$\mathcal{L}^h(12H_p,UU;\xi\omega H_q,\Xi\Xi)$
		for $pq=+1$	for $pq=-1$

among $1^{\xi\xi}_{2,3,4}$ for $\xi\xi = 11, 22$.

When secondary observables are concerned, parity conservation for helicity observables yields to

$$\mathcal{L}^h(\alpha,\beta;\gamma,\delta) = (-)^{W'}\mathcal{L}^h(\alpha,\beta;\widetilde{\gamma},\delta), \tag{4.4}$$

where W' is equal to $[\alpha]+[\beta]+[\gamma]+[\delta]$ but with, for the nucleon argument

$$[\alpha] = \begin{cases} 0 & \text{if } \alpha = A, 12I, \\ 1 & \text{if } \alpha = \Psi, 12R, \end{cases} \tag{4.5}$$

and similarly for $[\beta]$ and $[\delta]$ nucleon arguments and for the Δ argument

$$[\gamma] = \begin{cases} 0 & \text{if } \gamma = A, \Psi_1, \\ 1 & \text{if } \gamma = \Psi_2, \Psi_3, \\ \xi+\omega & \text{if } \gamma = \xi\omega R, \\ \xi+\omega+1 & \text{if } \gamma = \xi\omega I. \end{cases} \tag{4.6}$$

Secondary Δ arguments γ and $\widetilde{\gamma}$ are equal for A, Ψ_1, Ψ_2, Ψ_3, $14R$, $14I$, $23R$, $23I$ or are related as $\widetilde{\gamma} = 34R$, $34I$, $24R$, $24I$ for $\gamma = 12R$, $12I$, $13R$, $13I$, respectively. Equation (4.4) makes secondary observables vanish for which γ is equal to $\widetilde{\gamma}$ and W' is odd.

The choice of secondary arguments in terms of primary ones, advocated in Eqs. (2.16) and (2.17), is adapted for taking into account Bohr's rules, which lead to the relations

$$\begin{aligned} S\mathcal{L}^h(\alpha,\beta;A,\delta) &= (-)^{W''_B}S'(\mathcal{L}^h(\alpha',\beta';14I,\delta') - \mathcal{L}^h(\alpha',\beta';23I,\delta')), \\ S\mathcal{L}^h(\alpha,\beta;\Psi_1,\delta) &= (-)^{W''_B}S'(\mathcal{L}^h(\alpha',\beta';14I,\delta') + \mathcal{L}^h(\alpha',\beta';23I,\delta')), \\ S\mathcal{L}^h(\alpha,\beta;\Psi_2,\delta) &= (-)^{W''_B+1}S'(\mathcal{L}^h(\alpha',\beta';14R,\delta') + \mathcal{L}^h(\alpha',\beta';23R,\delta')), \\ S\mathcal{L}^h(\alpha,\beta;\Psi_3,\delta) &= (-)^{W''_B+1}S'(\mathcal{L}^h(\alpha',\beta';14R,\delta') - \mathcal{L}^h(\alpha',\beta';23R,\delta')), \end{aligned} \tag{4.7}$$

and

$$S\mathcal{L}^h(\alpha,\beta;12H_q,\delta) = (-)^{W''_B+1}S'\mathcal{L}^h(\alpha',\beta';13H_{q'},\delta'),$$

$$\text{with } qq' = (-)^{[\alpha]+[\beta]+[\delta]+1}, \tag{4.8}$$

where α,β,δ and α',β',δ' interchange as follows

$$A \leftrightarrow 12I, \qquad \Psi \leftrightarrow 12R, \tag{4.9}$$

and where $[\alpha]$, $[\beta]$, $[\delta]$ are given by Eq. (4.5). The sign in Eqs. (4.7) and (4.8) is obtained with W''_B equal to $[\alpha_B]+[\beta_B]+[\delta_B]+[\delta]$, where

$$[\alpha_B] = \begin{cases} 0 & \text{if } \alpha = A, 12R, \\ 1 & \text{if } \alpha = \Psi, 12I, \end{cases} \tag{4.10}$$

and similarly for $[\beta_B]$ and $[\delta_B]$ nucleon arguments.

Then, we are left with 256 linearly independent secondary helicity observables.

In Table X, the number of linearly independent secondary observables is given without parity constraints in the 4th column (which is the same as in Table II), and with the constraints in the 5th column. Equation (4.4) links observables which have the same number of diagonal arguments for the nucleons and for the Δ. Equations (4.7) and (4.8) link observables with different numbers of diagonal arguments for the nucleons and the Δ. More precisely, Eq. (4.7) connects $(3N)$ to (Δ), $(2N)$ to $(1N+\Delta)$, $(1N)$ to $(2N+\Delta)$, and (0) to $(3N+\Delta)$. Equation (4.8) connects $(3N)$ to (0) and $(2N)$ to $(1N)$. In the 5th column of Table X, the retained set of linearly independent secondary observables corresponds to $(3N+\Delta)$, $(3N)$, $(2N+\Delta)$, and $(2N)$ diagonal arguments. Finally, the submatrices which connect the bicoms and the secondary observables are of size 4×4, 8×8, and 16×16. The 16×16 submatrix connects secondary observables with the magnitude squares of the amplitudes.

Tables XIII and XIV permit one to determine from which secondary observables the real and imaginary parts of a bicom may be extracted. The use of the set of Tables XIII and XIV, concerning secondary observables, is exactly the same as the use of the set of Tables XI and XII, which concerns primary observables. Here, for simplifying the notation, 4×4, 8×8, and 16×16 submatrices are denoted $4^{\xi\omega}_i$, $8^{\xi\omega}_i$ or simply 8_i and 16, respectively, where the i index is a serial number designing on which subspace of secondary observables the submatrices act. Note that the superindex $\xi\omega$ appears when the Δ argument is nondiagonal. The $8_{2,3,4}$ and 16 submatrices correspond to diagonal Δ arguments A, Ψ_1, Ψ_2, and Ψ_3. As an example, combining Tables XIII and XIV, the real part of the bicom $G_1^*G_2$ is related to the four following secondary observables $\mathcal{L}^h(A,A;14R,12R)$, $\mathcal{L}^h(\Psi,\Psi;14R,12R)$, $\mathcal{L}^h(A,A;14I,12I)$, and $\mathcal{L}^h(\Psi,\Psi;14I,12I)$, the imaginary part of the same bicom being related to $\mathcal{L}^h(A,\Psi;14R,12I)$, $\mathcal{L}^h(\Psi,A;14R,12I)$, $\mathcal{L}^h(A,\Psi;14I,12R)$, and $\mathcal{L}^h(\Psi,A;14I,12R)$.

As for the transversity frame, the determination of the helicity amplitudes is performed in two stages. The first stage consists of measuring the 16 independent secondary observables $\mathcal{L}^h(\alpha,\beta;\gamma,\delta)$ with α, β, and δ equal to A or Ψ and γ equal to A, Ψ_1, Ψ_2, and Ψ_3, the magnitude squares of the amplitudes being given in terms of linear combinations of these observables. Table XV shows, in contrast to the transversity case, that the determination of the magnitudes needs experiments which involve four polarized particles at a time.

The magnitudes being determined, the second stage consists of finding a set of, at least, 15 experiments for the determination of independent relative phases. The submatrices at disposal being of size 4×4 and 8×8, a total of more than 15 experiments is needed, if all the observables appearing in the subspaces on which act the matrices are measured, following the Moravcsik's claim.

TABLE XIII. Dimension and type of matrices $M_i^{\xi\omega}$ connecting bicoms and secondary helicity observables.

	G_1	G_2	G_3	G_4	F_1	F_2	F_3	F_4	G_5	G_6	G_7	G_8	F_5	F_6	F_7	F_8
G_1^*	16	4_2^{14}	4_4^{14}	4_3^{14}	8_3	8_4	8_2	4_1^{14}	8_6^{12}	8_7^{12}	8_5^{12}	8_1^{13}	8_1^{12}	8_5^{13}	8_7^{13}	8_6^{13}
G_2^*		16	4_3^{14}	4_4^{14}	8_4	8_3	4_1^{14}	8_2	8_7^{12}	8_6^{12}	8_1^{13}	8_5^{12}	8_5^{13}	8_1^{12}	8_6^{13}	8_7^{13}
G_3^*			16	4_2^{14}	8_2	4_1^{14}	8_3	8_4	8_5^{12}	8_1^{13}	8_6^{12}	8_7^{12}	8_7^{13}	8_6^{13}	8_1^{12}	8_5^{13}
G_4^*				16	4_1^{14}	8_2	8_4	8_3	8_1^{13}	8_5^{12}	8_7^{12}	8_6^{12}	8_6^{13}	8_7^{13}	8_5^{13}	8_1^{12}
F_1^*					16	4_2^{14}	4_4^{14}	4_3^{14}	8_1^{12}	8_5^{13}	8_7^{13}	8_6^{13}	8_6^{12}	8_7^{12}	8_5^{12}	8_1^{13}
F_2^*						16	4_3^{14}	4_4^{14}	8_5^{13}	8_1^{12}	8_6^{13}	8_7^{13}	8_7^{12}	8_6^{12}	8_1^{13}	8_5^{12}
F_3^*							16	4_2^{14}	8_7^{13}	8_6^{13}	8_1^{12}	8_5^{13}	8_5^{12}	8_1^{13}	8_6^{12}	8_7^{12}
F_4^*								16	8_6^{13}	8_7^{13}	8_5^{13}	8_1^{12}	8_1^{13}	8_5^{12}	8_7^{12}	8_6^{12}
G_5^*									16	4_2^{23}	4_4^{23}	4_3^{23}	8_3	8_4	8_2	4_1^{23}
G_6^*										16	4_3^{23}	4_4^{23}	8_4	8_3	4_1^{23}	8_2
G_7^*											16	4_2^{23}	8_2	4_1^{23}	8_3	8_4
G_8^*												16	4_1^{23}	8_2	8_4	8_3
F_5^*													16	4_2^{23}	4_4^{23}	4_3^{23}
F_6^*														16	4_3^{23}	4_4^{23}
F_7^*															16	4_2^{23}
F_8^*																16

With a total of 16 experiments, obtained from two 8×8, or from one 8×8 and two 4×4, or from four 4×4 submatrices, the phases cannot be completely determined. With a total of 20 experiments, the phases can be completely determined from one 8×8 plus three 4×4 submatrices, only.

For solving the problem completely with 15 experiments, use must be made of one of the 8×8 submatrices among the $8_i^{\xi\omega}$ ones for $i = 1, 5, 6, 7$ and $\xi\omega = 12, 13$. The method must necessary measure all the eight observables appearing in the corresponding subspace. In addition, the method is to measure the four observables appearing in a subspace on which acts a 4×4 submatrix, two observables corresponding to a second 4×4 subma-

TABLE XIV. Secondary helicity observables on which act the matrices defined in Table XIII. Here α, β, and δ are equal to A or Ψ for the nucleons and γ equal to A, Ψ_1, Ψ_2, or Ψ_3 for the Δ.

Diagonal arguments	Matrices (Table XIII)	Real part of bicom	Imaginary part of bicom
$(3N+\Delta)$	16	$\mathcal{L}^h(\alpha,\beta;\gamma,\delta)$	
$(3N)$	$4_1^{\xi\omega}$, $8_1^{\xi\omega}$	$\mathcal{L}^h(\alpha,\beta;\xi\omega R,\delta)$	$\mathcal{L}^h(\alpha,\beta;\xi\omega I,\delta)$
$(2N+\Delta)$	8_2	$\mathcal{L}^h(\alpha,\beta;\gamma,12R)$	$\mathcal{L}^h(\alpha,\beta;\gamma,12I)$
	8_3	$\mathcal{L}^h(\alpha,12R;\gamma,\delta)$	$\mathcal{L}^h(\alpha,12I;\gamma,\delta)$
	8_4	$\mathcal{L}^h(12R,\beta;\gamma,\delta)$	$\mathcal{L}^h(12I,\beta;\gamma,\delta)$
$(2N)$	$4_2^{\xi\omega}$, $8_5^{\xi\omega}$	$\mathcal{L}^h(\alpha,\beta;\xi\omega H_q,12H_Q)$	$\mathcal{L}^h(\alpha,\beta;\xi\omega H_q,12H_Q)$
		for $qQ=+1$	for $qQ=-1$
	$4_3^{\xi\omega}$, $8_6^{\xi\omega}$	$\mathcal{L}^h(\alpha,12H_P;\xi\omega H_q,\delta)$	$\mathcal{L}^h(\alpha,12H_P;\xi\omega H_q,\delta)$
		for $Pq=+1$	for $Pq=-1$
	$4_4^{\xi\omega}$, $8_7^{\xi\omega}$	$\mathcal{L}^h(12H_p,\beta;\xi\omega H_q,\delta)$	$\mathcal{L}^h(12H_p,\beta;\xi\omega H_q,\delta)$
		for $pq=+1$	for $pq=-1$

TABLE XV. A set of 31 secondary observables determining the 16 complex helicity amplitudes $D^h(\lambda, l; \Lambda, L)$.

	Magnitudes		
$\mathcal{L}^h(A, A; A, A)$	$\mathcal{L}^h(\Psi, \Psi; A, A)$	$\mathcal{L}^h(\Psi, A; A, \Psi)$	$\mathcal{L}^h(A, \Psi; A, \Psi)$
$\mathcal{L}^h(A, A; \Psi_1, A)$	$\mathcal{L}^h(\Psi, \Psi; \Psi_1, A)$	$\mathcal{L}^h(\Psi, A; \Psi_1, \Psi)$	$\mathcal{L}^h(A, \Psi; \Psi_1, \Psi)$
$\mathcal{L}^h(\Psi, A; \Psi_2, A)$	$\mathcal{L}^h(A, \Psi; \Psi_2, A)$	$\mathcal{L}^h(A, A; \Psi_2, \Psi)$	$\mathcal{L}^h(\Psi, \Psi; \Psi_2, \Psi)$
$\mathcal{L}^h(\Psi, A; \Psi_3, A)$	$\mathcal{L}^h(A, \Psi; \Psi_3, A)$	$\mathcal{L}^h(A, A; \Psi_3, \Psi)$	$\mathcal{L}^h(\Psi, \Psi; \Psi_3, \Psi)$
	Phases		
$\mathcal{L}^h(A, A; 12R, A)$	$\mathcal{L}^h(\Psi, \Psi; 12R, A)$	$\mathcal{L}^h(\Psi, A; 12R, \Psi)$	$\mathcal{L}^h(A, \Psi; 12R, \Psi)$
$\mathcal{L}^h(\Psi, A; 12R, A)$	$\mathcal{L}^h(A, \Psi; 12R, A)$	$\mathcal{L}^h(A, A; 12R, \Psi)$	$\mathcal{L}^h(\Psi, \Psi; 12R, \Psi)$
$\mathcal{L}^h(\Psi, A; 14R, A)$	$\mathcal{L}^h(A, \Psi; 14R, A)$	$\mathcal{L}^h(A, A; 14R, \Psi)$	$\mathcal{L}^h(\Psi, \Psi; 14R, \Psi)$
$\mathcal{L}^h(12R, A; 14R, A)$	$\mathcal{L}^h(12I, A; 14I, A)$	$\mathcal{L}^h(A, 12R; 14R, A)$	

trix and finally, one observable corresponding to a third 4×4 one. The rule of choice for the three 4×4 submatrices is to retain one $4_1^{\xi\omega}$ submatrix for $\xi\omega = 14$ or 23, plus one $4_i^{\xi\omega}$ and one $4_j^{\xi\omega}$ for $i, j = 2, 3, 4$ and $i \neq j$, and for $\xi\omega = 14, 23$.

For giving an explicit example, we can choose to measure experiments with as much as possible diagonal nucleon arguments (i.e., A and Ψ arguments), the Δ argument being necessarily nondiagonal ($\xi \neq \omega$). This can be done using the sequence of submatrices 8_1^{12} or 8_1^{13}, 4_1^{14} or 4_1^{23}, plus two submatrices $4_i^{\xi\omega}$ and $4_j^{\xi\omega}$ with $i \neq j$, among $4_{2,3,4}^{14,23}$. In Table XV, the corresponding set of 31 secondary observables which determine completely (up to a global phase and a discrete number of ambiguities) the 16 complex helicity amplitudes can be found, as an explicit example, by taking the sequence 8_1^{12}, 4_1^{14}, 4_4^{14}, 4_3^{14}, for the determination of the phases.

V. CONCLUSION

The present work is devoted to the presentation of an amplitude analysis method, for the $NN \to \Delta N$ transition. The study is performed in the optimal formalism, which optimally diagonalizes the matrix connecting observables and bilinear combinations of amplitudes (bicoms), and, consequently, is well adapted to phenomenological amplitude determination. The quadratic relations existing between the spin observables of the $NN \to \Delta N$ transition are discussed.

In optimal formalism, as far as "primary observables" are concerned, the systematic determination of the amplitudes is very simple, a sufficient number of 1×1 submatrices connecting observables and bicoms being at our disposal.

However, it is much simpler to perform experiments in which some particles are unpolarized, which leads us to redefine observables in terms of "secondary observables." Unfortunately, the choice of secondary observables, more adapted to experiments, increases the complexity of their relations with bicoms. The determination of the amplitudes is performed in two stages. First, the magnitudes are determined by means of a specific set of 16 observables. Second, taking account of the quadratic relations, a methodology is developed to find all the sets of 15 observables giving the relative phases, with the least possible ambiguities. Transversity and helicity cases are specifically examined.

An explicit application of the method to the experimental data, for which it is convenient to use the density matrix formalism, will be presented in a following paper.

Turning back to the spin-space decomposition of the transition matrix [2,7,8], the knowledge of the 16 complex spin amplitudes $f_i(\theta_\Delta)$ and $g_i(\theta_\Delta)$ may be obtained by inverting the amplitude transformation given in Ref. [8], from optimal transversity or helicity amplitudes. Note, however, that the amplitude transformation can only be made once the optimal amplitudes are fully determined, the problem of ambiguities being solved. It could be interesting to study the conditions for realizing a partial phenomenological analysis.

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APPENDIX

The ρ and Q operators of Eq. (2.2), describing initial polarizations and measured final polarizations, denote all the spin-space operators required to generate spin observables of the reaction. In the optimal formalism, the ρ and Q operators are chosen to be "minimally Hermitian," so that the corresponding matrices have minimal number of nonzero elements compatible with the Hermiticity requirement. For each particle, ρ and Q associated matrix elements are defined by

$$(\rho^{uvH_p})_{l'l} = \tfrac{1}{2}[(1+p) + i(1-p)] \times (\delta_{\{u\}l}\delta_{\{v\}l'} + p\delta_{\{u\}l'}\delta_{\{v\}l}), \qquad \text{(A1)}$$

where, for a spin s particle, l and l' correspond to $+s, \cdots, -s$ magnetic components along its quantization axis. The symbol $\{u\}$ is related to the index u by

$$\{u\} = s(2u-1) \qquad \text{modulo } (2s+1). \qquad \text{(A2)}$$

For spin-$\frac{1}{2}$ particles, one obtains two ρ^{uu} diagonal matrices

$$\rho^{11} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \tag{A3}$$

and two ρ^{uvH_p} off-diagonal matrices

$$\rho^{12R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^{12I} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{A4}$$

For spin-$\frac{3}{2}$ particle, one obtains four $Q^{\xi\xi}$ diagonal matrices

$$Q^{11} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$Q^{33} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{44} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \tag{A5}$$

six $Q^{\xi\omega R}$ off-diagonal matrices

$$Q^{12R} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q^{13R} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{14R} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$Q^{23R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{24R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Q^{34R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{A6}$$

and six $Q^{\xi\omega I}$ off-diagonal matrices

$$Q^{12I} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{13I} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{14I} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$Q^{23I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^{24I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad Q^{34I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{A7}$$

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