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Fluctuations around the Wheeler-DeWitt trajectories in third-quantized cosmology

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The effect of third quantization on Wheeler-DeWitt quantum cosmology is discussed by employing flat and open simple minisuperspace models. The wave packets are constructed based on the field-theoretical extension of the method of Hermitian invariant. These wave packets are the most "classical" states and evolve along the Wheeler-DeWitt trajectories, the solutions to the Wheeler-DeWitt equation, in superspace. The "observables" are defined by considering the universe field measurements. It is shown that the fluctuation around each Wheeler-DeWitt trajectory measured by the Heisenberg uncertainty converges on its minimum rapidly in the course of the universe expansion, while it becomes large as the scale factor approaches zero. The result suggests the importance of the description by third-quantized theory in the earliest stage of the universe.

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The metric of superspace [1], which is the collection of spatial three-metrics of spacetime, has a hyperbolic signature. Correspondingly, the Wheeler-DeWitt equation, which is the basic equation in quantum cosmology [2,3], is of the Klein-Gordon type. There exists, therefore, a problem in the naive probabilistic interpretation of the wave function as a solution of the Wheeler-DeWitt equation. Undoubtedly, the probabilistic interpretation is of central importance and should be respected in all quantized theories.

Several proposals have been made in the literature to incorporate the probabilistic interpretation in quantum cosmology. Among others, an attractive idea is to perform the so-called third quantization [4,5] by analogy with Klein-Gordon field theory. This approach enables us to discuss the creation and annihilation of (multi)universes in superspace. The necessity of third quantization has been suggested also in the context of the cosmological-constant problem and baby universes [6].

In a recent paper [7], however, it has been pointed out that, in a closed-minisuperspace model, fluctuations of the quantized universe field itself are still dominant even in the region where spacetime is expected to be classical. Clearly, there are some important points to be clarified. First of all, the closed model is tachyonic (i.e., of imaginary mass) with an identification of the cosmic-scale factor with time. In addition, quantum fields themselves are not measurable quantities in field theory.

The purpose of this Brief Report is to study the behavior of fluctuations around the solutions to the Wheeler-DeWitt equation (called here the *Wheeler−DeWitt trajectories*) due to the effect of third quantization by employing a simple and analytically tractable minisuperspace model. We construct the wave packets of the system, which are the most classical states within third-quantized theory and evolve along the Wheeler-DeWitt trajectories. Then we define the observables as the average values of the universe field over certain measurement regions in its superspace domain and evaluate the fluctuations of the observables around the Wheeler-DeWitt trajectories. We show that the Heisenberg uncertainty converges on its minimum value rapidly in the course of the Universe expansion, while it becomes large as the cosmic-scale factor approaches zero.

What is theoretically characteristic in the present discussion is that the Hamiltonian describing quantum dynamics of the universe field depends explicitly on the cosmic-scale factor as time. This brings difficulties to defining the ground state as the state of nothing. To treat such a nonstationary field theory, here we examine a new method, which is a field-theoretical extension of the method of Hermitian invariant. We shall see how this method provides a time-dependent ground state in a peculiar manner.

The Wheeler-DeWitt equation for the Friedmann-Robertson-Walker universe filled with a homogeneous massless scalar field ϕ is given by

$$\left[\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}+\lambda e^{6\alpha}-ke^{4\alpha}\right]\psi(\alpha,\phi)=0 \ , \tag{1}$$

where λ and k are a cosmological constant and a curvature parameter, respectively. The cosmic-scale factor has been replaced by α as $a=e^{\alpha}$. Possible potential terms of the scalar field have been neglected for the sake of simplicity. [As explained below, a closed-universe model ($k=+1$) will not be treated in the main context of this paper.]

Equation (1) is a hyperbolic equation. It is apparently not unique to identify one of variables with internal time. However, it is quite essential to recognize that the hyperbolicity of Eq. (1) reflects the generic structure of superspace. The signature of superspace [1] is always $(-,+,\cdots,+)$, and the timelike signature comes from the volume factors of spatial three-geometries in the canonical formulation of general relativity. Therefore we identify α (ϕ) with a time (position) coordinate in Eq. (1). (In fact, this identification is most natural and becomes more plausible when a possible anisotropy is taken into account in the model. Another identification of the matter field ϕ with time, then, leads to the wrong signatures in the kinetic terms, and so some additional hypothetical manipulations are needed.)

To proceed to third quantization, we regard Eq. (1) as the Euler-Lagrange equation derived from the action

$$S=\tfrac{1}{2}\int d\alpha d\phi\left[\left[\frac{\partial\psi}{\partial\alpha}\right]^2-\left[\frac{\partial\psi}{\partial\phi}\right]^2-(\lambda e^{6\alpha}-ke^{4\alpha})\psi^2\right]. \tag{2}$$

ψ is no longer a wave function, but a basic dynamical variable to be quantized. The canonical momentum Π conjugate to ψ and the Hamiltonian are constructed in the usual way. Third quantization is then performed by imposing the equal-time commutation relation

$$[\psi(\alpha,\phi),\Pi(\alpha,\phi')]=i\delta(\phi-\phi') . \tag{3}$$

Working in the Schrödinger picture with a substitution $\Pi(\phi)\rightarrow -i\delta/\delta\psi(\phi)$, we obtain the functional Schrödinger equation in the ψ representation:

$$i\frac{\partial}{\partial\alpha}\Psi[\psi;\alpha]=H(\alpha)\Psi[\psi;\alpha] , \tag{4}$$

where the Hamiltonian is given by

$$H(\alpha)=\tfrac{1}{2}\int d\phi\left[-\frac{\delta^2}{\delta\psi(\phi)^2}+\left[\frac{\partial\psi}{\partial\phi}\right]^2+(\lambda e^{6\alpha}-ke^{4\alpha})\psi^2\right] . \tag{5}$$

The model is, thus, formally equivalent to the neutral Klein-Gordon field theory with a time-dependent mass

$$m^2(\alpha)=\lambda e^{6\alpha}-ke^{4\alpha} . \tag{6}$$

It should be noted that, in order to avoid instantaneous tachyonic states in the whole range of time α, we have to assume both λ and $-k$ to be non-negative. Otherwise, we cannot expect any physically meaningful solutions of Eq. (4). (Therefore the closed Friedmann-Robertson-Walker model with a vanishing cosmological constant does not admit third quantization, at least within the present discussion. See the final comment.) Henceforth, we restrict ourselves to the cases $\lambda\geq 0$ and $k=-1,0$.

Now, because of the explicit time dependence, there is no well-defined notion of a Fock vacuum relative to the Hamiltonian, in general. This makes it nontrivial to construct the state of nothing. One possible approach to this problem is to solve the functional Schrödinger equation (4) explicitly by using the Gaussian ansatz [8]. Here, instead, we examine another approach, which is the field-theoretical extension of the method of Hermitian invariant [9].

The time-dependent Hermitian invariant is defined by the solution of the operator equation

$$\frac{\partial I(\alpha)}{\partial\alpha}+i[H(\alpha),I(\alpha)]=0 . \tag{7}$$

To simplify the analysis, we employ the "momentum representation" for the operators:

$$\psi(\phi)=\frac{1}{\sqrt{2\pi}}\int dp\, e^{ip\phi}\eta(p) ,\qquad \frac{\delta}{\delta\psi(\phi)}=\frac{1}{\sqrt{2\pi}}\int dp\, e^{-ip\phi}\frac{\delta}{\delta\eta(p)} , \tag{8}$$

$$H(\alpha)=\tfrac{1}{2}\int dp\left\{-\frac{\delta^2}{\delta\eta(p)\delta\eta(-p)}+[p^2+m^2(\alpha)]\eta(p)\eta(-p)\right\} , \tag{9}$$

provided that $\eta^*(p)=\eta(-p)$ due to the reality of ψ.

The Hamiltonian (9) is found to admit the invariant

$$I(\alpha)=\tfrac{1}{2}\int dp\left\{\left[i\rho\frac{\delta}{\delta\eta(p)}+\dot\rho\eta(-p)\right]\times\left[i\rho\frac{\delta}{\delta\eta(-p)}+\dot\rho\eta(p)\right]+\rho^{-2}\eta(p)\eta(-p)\right\} , \tag{10}$$

where $\rho=\rho(\alpha,p)$ is a real solution of the nonlinear auxiliary equation

$$\ddot\rho+[p^2+m^2(\alpha)]\rho=\rho^{-3} . \tag{11}$$

ρ can be taken as an even function of p. The overdots in these equations stand for differentiation with respect to α. The quantity $I(\alpha)$ is not unique since it depends on the choice of initial conditions for ρ; but this arbitrariness does not bring any new additional problems to the following discussion.

By definition, the eigenvalues of $I(\alpha)$ remain constant under time evolution in contrast with those of the Hamiltonian. This property enables us to define a time-dependent ground state in a peculiar manner. Let us introduce the operators

$$A(\alpha,p)=\frac{1}{\sqrt{2}}\left[(\rho^{-1}-i\dot\rho)\eta(-p)+\rho\frac{\delta}{\delta\eta(p)}\right] , \tag{12}$$

$$A^\dagger(\alpha,p)=\frac{1}{\sqrt{2}}\left[(\rho^{-1}+i\dot\rho)\eta(p)-\rho\frac{\delta}{\delta\eta(-p)}\right] , \tag{13}$$

which satisfy the equal-time commutation relation

$$[A(\alpha,p),A^\dagger(\alpha,p')]=\delta(p-p') . \tag{14}$$

The other equal-time commutators vanish. This is of the oscillator type, and therefore it is natural to regard $A^\dagger(\alpha,p)$ and $A(\alpha,p)$ as the creation and annihilation operators.

In terms of these operators, $I(\alpha)$ is expressed as

$$I(\alpha)=\int dp\left[A^\dagger(\alpha,p)A(\alpha,p)+\frac{V}{4\pi}\right] . \tag{15}$$

Here $V=2\pi\delta(p=0)$ is the volume of the ϕ space and is an isolated divergent quantity.

The time independence of the divergence in Eq. (15) suggests the existence of a well-defined normal-ordering procedure with respect to a ground state, which should satisfy the condition

$$A(\alpha,p)|\Omega;\alpha)=0\ , \tag{16}$$

where we have used a representation-free notation.

An important point is that condition (16) is preserved in time, because the generalized annihilation operator evolves like

$$\frac{\partial A(\alpha,p)}{\partial\alpha}+i[H(\alpha),A(\alpha,p)]=-i\rho^{-2}A(\alpha,p)\ . \tag{17}$$

In the η representation, the normalized solution of Eq. (16) is given by

$$\Omega[\eta;\alpha]=(\det D)^{1/4}\times\exp\left[-\tfrac{1}{2}\int dp\,\eta(p)(\rho^{-2}-i\rho^{-1}\dot{\rho})\eta(-p)\right]\ , \tag{18}$$

with $D(p,p';\alpha)=\rho^{-2}(\alpha,p)\delta(p-p')$. This is the lowest eigenstate of $I(\alpha)$, and plays the role of the ground state in the instantaneous Fock space associated with the operators (12) and (13).

The state $|\Omega;\alpha)$ is not a solution of the Schrödinger equation (4). However, the corresponding Schrödinger state is easily constructed by using the phase degree of freedom, that is,

$$|\Omega;\alpha\rangle=e^{i\theta(\alpha)}|\Omega;\alpha)\ , \tag{19}$$

with the auxiliary equation

$$\frac{d\theta}{d\alpha}=(\Omega;\alpha|i\frac{\partial}{\partial\alpha}|\Omega;\alpha)-(\Omega;\alpha|H(\alpha)|\Omega;\alpha)=-\frac{V}{4\pi}\int dp\,\rho^{-2}\ . \tag{20}$$

Equation (19) means that the difference between two states is just a gauge transformation and, therefore, the time-dependent Schrödinger vacuum is also well defined. The state (19) is regarded as the state of nothing.

We now construct the wave packet at certain time α_0 as

$$|\chi;\alpha_0)=\exp\left[\int dp[\chi(p)A^\dagger(\alpha_0,p)-\chi^*(p)A(\alpha_0,p)]\right]|\Omega;\alpha_0)\ . \tag{21}$$

This is an eigenstate of the operator $A(\alpha_0,p)$ with the complex eigenvalue $\chi(p)$, which is assumed to be square integrable. So it is a coherent state and is most classical. It is further assumed that this state coincides with the Schrödinger state at α_0: $|\chi;\alpha_0\rangle\equiv|\chi;\alpha_0)$. The subsequent time evolution is given by

$$|\chi;\alpha\rangle=U(\alpha,\alpha_0)|\chi;\alpha_0\rangle\ , \tag{22}$$

$$U(\alpha,\alpha_0)=T\exp\left[-i\int_{\alpha_0}^{\alpha}d\alpha'\,H(\alpha')\right]\ , \tag{23}$$

where the symbol T stands for the chronological ordering of the exponential operator.

The expectation value of the universe field $\psi(\phi)$ with respect to the classical state (22) is calculated as

$$\langle\chi;\alpha|\psi(\phi)|\chi;\alpha\rangle=\frac{1}{\sqrt{4\pi}}\int dp\,e^{ip\phi}\rho(\alpha,p)\left[\chi^*(p)\exp\left[i\int_{\alpha_0}^{\alpha}d\alpha'\,\rho^{-2}(\alpha',p)\right]+\chi(-p)\exp\left[-i\int_{\alpha_0}^{\alpha}d\alpha'\,\rho^{-2}(\alpha',p)\right]\right]\ . \tag{24}$$

As can be seen, this represents precisely a set of solutions of the Wheeler-DeWitt equation (1) associated with the various solutions to Eq. (11). Thus the wave packets of the classical states evolve along the Wheeler-DeWitt trajectories in superspace.

Now, we discuss the effects of fluctuation on the Wheeler-DeWitt trajectories by third quantization. It can be done best by evaluating the Heisenberg uncertainty in the measurement of the universe field.

The problem of measurements in quantum field theory was investigated by Bohr and Rosenfeld [10]. One of the most important points is that the quantities which can be measured are not quantum fields themselves, but their average values over certain measurement regions of the domain. In the present case, these regions are laid on the ϕ axis in superspace since we are working in the Schrödinger picture. Let us take a region of the volume $2\phi_1$ centered at the position $\phi=\phi_0$ and define the operators

$$\psi_0=\frac{1}{2\phi_1}\int_{\phi_0-\phi_1}^{\phi_0+\phi_1}d\phi\,\psi(\phi)\ ,\qquad \frac{\delta}{\delta\psi_0}=\frac{1}{2\phi_1}\int_{\phi_0-\phi_1}^{\phi_0+\phi_1}d\phi\,\frac{\delta}{\delta\psi(\phi)}\ . \tag{25}$$

Then the rescaled operators $X\equiv\sqrt{2\phi_1}\psi_0$, $P\equiv-i\sqrt{2\phi_1}\delta/\delta\psi_0$ satisfy the quantum-mechanical commutation relation $[X,P]=i$. The minimum uncertainty associated with the measurements of X and P is equal to one-half.

The variance of X in the classical state (22) is calculated as

$$(\Delta X)^2=\frac{1}{2\pi\phi_1}\int dp\left[\frac{\sin p\phi_1}{p}\right]^2\rho^2(\alpha,p)\cong\tfrac{1}{2}\rho^2(\alpha,0)\ . \tag{26}$$

The above approximation of ρ by its zero mode becomes exact when $\phi_1 \to \infty$. Similarly, the variance of P is

$$(\Delta P)^2 \cong \tfrac{1}{2}[\rho^{-2}(\alpha,0)+\dot{\rho}^2(\alpha,0)] \; . \tag{27}$$

Thus we obtain the uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \cong \tfrac{1}{4}[1+\rho^2(\alpha,0)\dot{\rho}^2(\alpha,0)] \; . \tag{28}$$

This formula shows how the Wheeler-DeWitt trajectories (24) are subjected to fluctuations by third quantization.

We evaluate the right-hand side of this equation by using the asymptotic solutions to Eq. (11) at $\alpha \to \pm\infty$ with $\alpha_0 \to -\infty$ in Eq. (22). In this case the precise form of Eq. (6) is no longer essential. We make a simple choice of parameters: $\lambda=0$, $k=-1$. (The other choices do not affect the subsequent conclusion as long as the no-tachyon condition is satisfied.) Asymptotic solutions to Eq. (11) are then found to be $\rho(\alpha,0) \sim -c\alpha$ $(\alpha \to -\infty)$, $\exp(-\alpha)$ $(\alpha \to \infty)$, where c is a positive integration constant. Thus we arrive at the main result:

$$(\Delta X)^2(\Delta P)^2 \sim \begin{cases} C^4\alpha^2 \quad (\alpha \to -\infty) \; , \\ \tfrac{1}{4}(1+e^{-4\alpha}) \quad (\alpha \to \infty) \; . \end{cases} \tag{29}$$

This result has clear physical meanings. The third-quantized universe can rapidly reduce its fluctuations in the course of expansion. The fluctuations of the ψ component become squeezed on the Wheeler-DeWitt trajectories. Therefore the later adiabatic stage of evolution can be described well by the Wheeler-DeWitt equation. (The large fluctuations in the momentum component may not lead to any conceptual difficulties since, in this classical region, α should lose its role of a time parameter and should be treated as a dynamical variable.) On the other hand, the large fluctuations around the Wheeler-DeWitt-trajectories at the small scale factor [11] suggest that the description by third-quantized theory might be important in the earliest stage of the evolution.

So far, we have discussed flat- and open-minisuperspace models. Closed models could not be treated in a unified way within the present framework. A main difficulty is their tachyonic nature. In this point it is quite interesting to introduce a self-interaction for the universe field. The introduction of a quartic interaction can remove the tachyon states from the closed models, for example. In that case breakdown of symmetry would occur and the creation of the universe might be discussed from the viewpoint of the instability of the ground state regarded as the state of nothing.

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[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973); M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmology* (Princeton University Press, Princeton, NJ, 1975).

[2] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, Nucl. Phys. **B239**, 257 (1984); J. J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).

[3] A. Vilenkin, Phys. Lett. **117B**, 25 (1982); Phys. Rev. D **30**, 509 (1984).

[4] A. Hosoya and M. Morikawa, Phys. Rev. D **39**, 1123 (1989).

[5] M. McGuigan, Phys. Rev. D **38**, 3031 (1988); **39**, 2229 (1989).

[6] V. A. Rubakov, Phys. Lett. B **214**, 503 (1988); T. Banks, Nucl. Phys. **B309**, 493 (1988); S. B. Giddings and A. Strominger, *ibid.* **B321**, 481 (1989).

[7] H.-J. Pohle, Phys. Lett. B **261**, 257 (1991).

[8] R. Floreanini, C. T. Hill, and R. Jackiw, Ann. Phys. (N.Y.) **175**, 345 (1987); R. Floreanini and L. Vinet, Phys. Rev. D **36**, 1731 (1987); J. Guven, B. Lieberman, and C. T. Hill, *ibid.* **39**, 438 (1989).

[9] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969); J. G. Hartley and J. R. Ray, Phys. Rev. D **25**, 382 (1982); X.-C. Gao, J.-B. Xu, and T.-Z. Qian, Phys. Rev. A **44**, 7016 (1991).

[10] N. Bohr and L. Rosenfeld, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. **12** (8) (1933) [English translation reprinted in *Quantum Theory and Measurement,* edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983)]. See also W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, Oxford, 1954).

[11] The result (29) is an initial-value-problem-like representation in the $\phi_1 \to \infty$ limit. Strictly speaking, for a finite ϕ_1, the result in the limit $\alpha \to -\infty$ is linked with the treatment of zero mode ($p=0$) in Eqs. (26) and (27). If the limit $\alpha \to -\infty$ is taken in Eq. (11) with keeping p finite, then the solution $\rho^2(\alpha,p) \sim |p|^{-1}\{C_1+\sqrt{C^2-1}\sin[2|p|(\alpha-\alpha_1)]\}$ is obtained, provided that α_1 and C_1 (≥ 1) are integration constants. Also, in this case, the integral (26) is divergent.