ϕ^4 model near the critical point: The Gaussian variational approximation

Arthur K. Kerman and Cécile Martin*
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Dominique Vautherin
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406, Orsay CEDEX, France
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Using the Gaussian variational approximation, we describe the approach to the continuum limit in ϕ^4 theory in $3+1$ space-time dimensions. We study the solutions of the variational equations and their stability and compare our results with those of Monte Carlo calculations on a lattice. The importance of lattice effects is investigated by putting the Gaussian wave functional on a lattice. We find an abrupt decrease of the fourth derivative of the effective potential in the vicinity of the critical point, which is consistent with Monte Carlo calculations. In the continuum limit $b \to 0^+$, the renormalized theory shows a broken phase which is degenerate with the symmetric phase. In the asymmetric phase the theory is found to be asymptotically free, in agreement with the conclusions of Branchina *et al.*

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I. INTRODUCTION

Using the Gaussian approximation, which is equivalent to the Hartree-Bogoliubov approximation, we describe the approach to the continuum limit in ϕ^4 theory in $3+1$ space-time dimensions. We study the solutions of the variational equations and their stability and compare our results with those of Monte Carlo calculations on a lattice [1]. These results can be useful in the understanding of mass generation through the Higgs-boson mechanism. It is also important to know if it is judicious to use Gaussian trial wave functionals to describe the vacuum state of a quantum field theory in three space dimensions [2,3]. Our approach to the critical point allows one to compare this approximation directly with Monte Carlo simulations, the accuracy of which depends strongly on the size of the lattice (see Ref. [4] for a comparison between the Gaussian effective potential and lattice results in $1+1$ space-time dimensions). Moreover, one may hope that this comparison between the two nonperturbative methods will be useful in the case of QCD [5].

The Hamiltonian density we consider is

$$\mathcal{H}=\tfrac{1}{2}\pi^2(\mathbf{x})+\tfrac{1}{2}[\nabla\phi(\mathbf{x})]^2+\frac{a}{2}\phi^2(\mathbf{x})+\frac{b}{24}\phi^4(\mathbf{x}) . \quad (1)$$

The expectation value V of $\mathcal{H}$ for a Gaussian wave functional

$$\Psi[\phi(\mathbf{x})]=\mathcal{N}\exp\left[-\tfrac{1}{4}\int d\mathbf{x}\,d\mathbf{y} \times[\phi(\mathbf{x})-\varphi]G^{-1}(\mathbf{x},\mathbf{y})[\phi(\mathbf{y})-\varphi]\right], \quad (2)$$

where the kernel G is parametrized as $\frac{1}{4}G^{-2}(\mathbf{x},\mathbf{y})=\langle\mathbf{x}|(-\Delta+m^2)|\mathbf{y}\rangle$, is a function of the two variational parameters φ and m, which we take to be independent of $\mathbf{x}$. No new infinities are introduced when φ and m are functions of $\mathbf{x}$. So we restrict ourselves to this case for our comparison. With this assumption one obtains, up to an additive infinite constant,

$$V(\varphi,m)=\frac{1}{4}G^{-1}(m^2)-\frac{m^2}{2}G(m^2) +\frac{1}{2b}\left[a+\frac{b}{2}G(m^2)\right]^2 +\frac{1}{2}\left[a+\frac{b}{2}G(m^2)\right]\varphi^2+\frac{b}{24}\varphi^4 . \quad (3)$$

In this equation, $G(m^2)$ and $G^{-1}(m^2)$ are defined as $\langle\mathbf{x}|G|\mathbf{x}\rangle$ and $\langle\mathbf{x}|G^{-1}|\mathbf{x}\rangle$, i.e.,

$$G(m^2)=\left\langle\mathbf{x}\left|\frac{1}{2\sqrt{-\Delta+m^2}}\right|\mathbf{x}\right\rangle ,$$

$$G^{-1}(m^2)=\langle\mathbf{x}|2\sqrt{-\Delta+m^2}|\mathbf{x}\rangle .$$

These quantities are independent of position because we have assumed φ and m to have the same property. Minimization with respect to φ and m yields

$$\frac{\partial V}{\partial\varphi}=\left[a+\frac{b}{2}G(m^2)+\frac{b}{6}\varphi^2\right]\varphi=0 , \quad (4)$$

*Permanent address: Division de Physique Théorique, Institut de Physique Nucléaire, F-91406, Orsay CEDEX, France.

$$\frac{\partial V}{\partial m}=mG'(m^2)\left[a+\frac{b}{2}G(m^2)+\frac{b}{2}\varphi^2-m^2\right]=0\ , \tag{5}$$

where $G'(m^2)=dG(m^2)/dm^2$. By calculating the stability matrix one can show that the solution of Eq. (5) which corresponds to $m\equiv 0$ is always unstable. By combining (4) and (5) when $m\neq 0$ and $\varphi\neq 0$, we can show that $m^2=b\varphi^2/3$ which we define as the broken phase.

In the first part of this paper we consider the Gaussian approximation with a lattice regularization. For the Gaussian wave functional on a lattice, analytical formulas are derived which allow an evaluation of the uncertainties arising from the finite number N of mesh points used. In the second part of the paper we investigate the case where the theory is regularized by means of a cutoff in momentum space. In this case, analytic formulas suitable to discuss the approach to the continuum limit are presented. A comparison with earlier work is made.

II. THE GAUSSIAN WAVE FUNCTIONAL ON A LATTICE

We first consider the Gaussian wave functional on a lattice of N^3 sites with a mesh size $\Delta x=1/\Lambda$ and periodic boundary conditions. As in Ref. [1], let us introduce the dimensionless quantities $\varphi_{\rm latt}$, $m_{\rm latt}$, and r_0: $\varphi=\Lambda\varphi_{\rm latt}$, $m=\Lambda m_{\rm latt}$, and $a=\Lambda^2 r_0$. Using the eigenvalues of the operator $-\Delta$ on a lattice [6],

$$-\Delta(k_x,k_y,k_z)=4\Lambda^2\left[\sin^2\frac{k_x\pi}{N}+\sin^2\frac{k_y\pi}{N}+\sin^2\frac{k_z\pi}{N}\right],$$

where k_x, k_y, and $k_z=1,\ldots,N$, the gap equation, defined as

$$a+\frac{b}{2}\left\langle \mathbf{x}\left|\frac{1}{2\sqrt{-\Delta+m^2}}\right|\mathbf{x}\right\rangle+\frac{b}{2}\varphi^2-m^2=0\ , \tag{6}$$

becomes

$$r_0=m_{\rm latt}^2-\frac{b}{2}\varphi_{\rm latt}^2-\frac{b}{8N^3}\sum_{k_x,k_y,k_z}\left[\sin^2\frac{k_x\pi}{N}+\sin^2\frac{k_y\pi}{N}+\sin^2\frac{k_z\pi}{N}+\frac{m_{\rm latt}^2}{4}\right]^{-1/2}. \tag{7}$$

In the broken phase we have $m_{\rm latt}^2=(b/3)\varphi_{\rm latt}^2$: i.e.,

$$r_0=-\frac{m_{\rm latt}^2}{2}-\frac{b}{8N^3}\sum_k\frac{1}{\left[\sin^2\frac{k_x\pi}{N}+\sin^2\frac{k_y\pi}{N}+\sin^2\frac{k_z\pi}{N}+\frac{m_{\rm latt}^2}{4}\right]^{1/2}}\ . \tag{8}$$

The function $r_0(m_{\rm latt})$ defined by Eq. (8) has a maximum r_c which is attained when $m_{\rm latt}$ is equal to a critical value m_c such that

$$1=\frac{b}{32N^3}\sum_k\left[\sin^2\frac{k_x\pi}{N}+\sin^2\frac{k_y\pi}{N}+\sin^2\frac{k_z\pi}{N}+\frac{m_c^2}{4}\right]^{-3/2}. \tag{9}$$

The corresponding critical value r_c of the maximum is given by

$$r_c=-\frac{m_c^2}{2}-\frac{b}{8N^3}\sum_{k_x,k_y,k_z}\left[\sin^2\frac{k_x\pi}{N}+\sin^2\frac{k_y\pi}{N}+\sin^2\frac{k_z\pi}{N}+\frac{m_c^2}{4}\right]^{-1/2}. \tag{10}$$

For $r_0<r_c$, there is a solution with broken symmetry. For $r_0>r_c$, Eq. (8) has no solution and there is thus no solution with broken symmetry. The symmetric solution satisfies

$$r_0=m_{\rm latt}^2-\frac{b}{8N^3}\sum_k\frac{1}{\left[\sin^2\frac{k_x\pi}{N}+\sin^2\frac{k_y\pi}{N}+\sin^2\frac{k_z\pi}{N}+\frac{m_{\rm latt}^2}{4}\right]^{1/2}}\ . \tag{11}$$

Figure 1 shows the solution $m_{\rm latt}$ of the gap equation (7) [i.e., Eq. (8) for $r_0<r_c$ and Eq. (11) for $r_0>r_c$] as a function of r_0 for $N=10$ and a bare coupling constant $b=6$. We also show the results of the Monte Carlo calculations of Ref. [1] with the same value of b. The Gaussian approximation on a $10\times 10\times 10$ lattice and the Monte Carlo lattice calculations give the same qualitative behavior with two branches corresponding to a symmetric phase on the right of the critical point and to a phase with broken symmetry on the left of the critical point. The Gaussian approximation is qualitatively more satisfactory than first-order perturbation theory (dashed line in Fig.

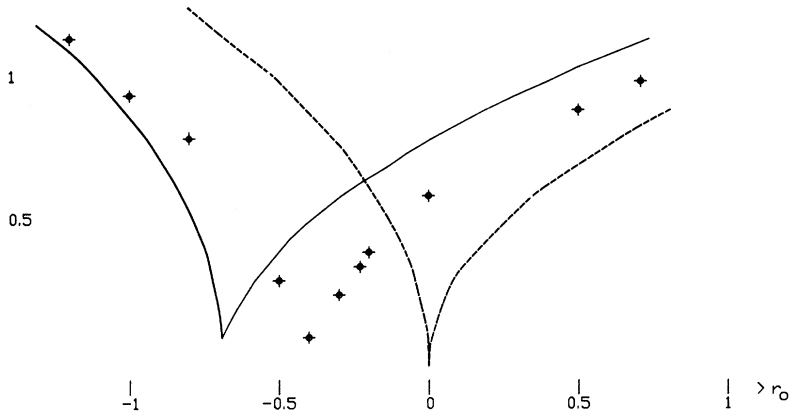

FIG. 1. The solution m_{latt} of the gap equation (7) as a function of r_0 for $b=6$. The solid line shows the results of the Gaussian approximation on a $10\times10\times10$ lattice, the points are the result of Ref. [1], and the dashed line is the prediction of perturbation theory.

1), which gives $m^2_{\text{latt}}=-2r_0$ for $r_0<r_c$ from Eq. (8) and $m^2_{\text{latt}}=r_0$ for $r_0>r_c$ from Eq. (11).

We point out that in Monte Carlo calculations, because of the finite size of the lattice, m_{latt} never actually goes to zero. The same occurs for the Gaussian on a lattice. For instance, the minimum of m_{latt} for $b=6$ is 0.075, for $N=10$, and 0.045 for $N=20$. These values are attained on the symmetric branch at $r_0=r_c$.

Using the Gaussian approximation on a bigger lattice does not significantly modify the curve in Fig. 1 but it allows one to determine the critical value r_c more accurately. For $N=10$ we obtain a value for r_c ($r_c=-0.695$ at $m_c=0.125$) lower than the Monte Carlo result ($r_c=-0.4$). The perturbative value is $r_c=0$. For $N=20$ we obtain $r_c=-0.686$ at $m_c=0.075$. When the number of lattice points is large ($N\gg1/m_{\text{latt}}$), the semiconvergent Riemann sum (9) occurring in the definition of the critical point can be replaced by an integral so that Eq. (9) becomes

$$1=\frac{b}{32}\int_0^1 dx\int_0^1 dy\int_0^1 dz\left[\frac{1}{\sin^2\pi x+\sin^2\pi y+\sin^2\pi z+\frac{m_c^2}{4}}\right]^{3/2}. \tag{12}$$

For small b, m_c is small and the dominant contribution in the integrand arises near the origin. We obtain

$$\ln m_c^2=-\frac{16\pi^2}{b}+\cdots, \tag{13}$$

in agreement with the results of Consoli and Ciancitto [7].

It is important to note that expression (13) for m_c^2 is intrinsically nonperturbative. Indeed m_c^2 vanishes as well as all its derivatives at $b=0$. Therefore the result (13) cannot be obtained by a naive perturbation expansion.

The corresponding value of r_c is

$$r_c=-\frac{m_c^2}{2}-\frac{b}{4\pi^3}\int_0^{\pi/2}dx\int_0^{\pi/2}dy\int_0^{\pi/2}dz\frac{1}{\left[\sin^2x+\sin^2y+\sin^2z+\frac{m_c^2}{4}\right]^{1/2}}. \tag{14}$$

Approximating $\sin x$ by x, limiting the integration volume to a sphere of radius $\pi/2$, and setting m_c to zero in the argument of the square root gives the following (reasonable) estimate of r_c:

$$r_c\simeq-\frac{m_c^2}{2}-\frac{b}{32}.$$

For $b=6$, the previous formulas give $m_c=10^{-6}$ and $r_c=-0.3$. The critical point is thus located in a region where the number of mesh points necessary to get a good accuracy is very large. Indeed this condition requires $N\gg N_0=1/m_c=\exp(8\pi^2/b)$. For $b=6$ one has $N_0=10^6$ while for $b=60$, a value also considered in Ref. [1], $N_0=5$.

It is worthwhile emphasizing that, from Eq. (13), we learn that m_c does not vanish in the limit of a large number of mesh points. This has interesting consequences. Indeed it means that, in order to have after renormalization a finite value of the expectation value of the field φ (determined by $\varphi^2=3m^2/b=3\Lambda^2 m_{\text{latt}}^2/b$), the bare coupling constant b must be sent to zero with the mesh size $\Delta x=1/\Lambda$ according to the relation

$$\frac{1}{b}=\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{M^2}+\frac{1}{16\pi^2}\ln\frac{1}{b}\ , \tag{15}$$

where M is an arbitrary mass scale. To lowest order this gives

$$\frac{1}{b}=\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{M^2}+\frac{1}{16\pi^2}\ln\left[\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{M^2}\right]+\cdots\ . \tag{16}$$

This formula allows one to calculate the β function which can be constructed as

$$\beta(b)=-\frac{db}{d\ \ln M}\ .$$

To lowest order we obtain

$$\beta(b)=-\frac{b^2}{8\pi^2}$$

so that the theory in the symmetry-broken phase is found to be asymptotically free in agreement with the results of Branchina *et al.* [8]. We shall return later in Sec. II to a more detailed discussion of this formula.

Another quantity for which useful comparisons can be made between the Gaussian approximation and Monte Carlo calculations is the renormalized coupling constant defined as

$$\lambda_R=\frac{d^4V(m(\varphi),\varphi)}{d\varphi^4}\ ,$$

where $m(\varphi)$ is the solution of $\partial V(m,\varphi)/\partial m=0$ and the fourth derivative is calculated at the minimum of $V(m,\varphi)$.

In the phase with broken asymmetry, λ_R is found to be (after a tedious calculation)

$$\lambda_R=b\frac{1+bG'}{1-bG'/2}+3b^3\frac{G''\varphi^2}{(1-bG'/2)^3}+\frac{1}{2}G'''\frac{b^4\varphi^4}{(1-bG'/2)^4}+\frac{3}{4}\frac{b^5G''^2\varphi^4}{(1-bG'/2)^5}\ , \tag{17}$$

where primes indicate differentiations with respect to m^2. In the symmetric phase the expression of λ_R is

$$\lambda_R=b\frac{1+bG'}{1-bG'/2}\ . \tag{18}$$

The variation of the renormalized coupling constant as a function of the bare constant r_0 is shown for $b=6$ in Fig. 2 for the Gaussian approximation using a $20\times20\times20$ lattice and the Monte Carlo method. Similar qualitative trends are seen on both curves, in particular the abrupt decrease of λ_R in the vicinity of the critical point.

Thus our variational result agrees with the Monte Carlo calculations of Ref. [1].

With formulas (17) and (18) for the renormalized coupling constant λ_R, it is also possible to obtain the perturbative results for the ϕ^4 theory in the infrared domain. For this purpose, we use the definition of Callan and Symanzik for the β function (which is not the definition we have used previously when we work with the bare theory):

$$\beta(\lambda_R)=m\frac{d\lambda_R}{dm}$$

for fixed b and Λ. m is the solution of the gap equation and the infrared region corresponds to the limit $m\to0$. In the symmetric phase we find

$$\beta(\lambda_R)=\frac{3\lambda_R^2}{16\pi^2}\frac{1}{(1+bG')^2}\ .$$

For small b, this gives $\beta(\lambda_R)=3\lambda_R^2/16\pi^2$, which is the usual perturbative result. In the asymmetric phase, the relation between λ_R and b is more complicated. Howev-

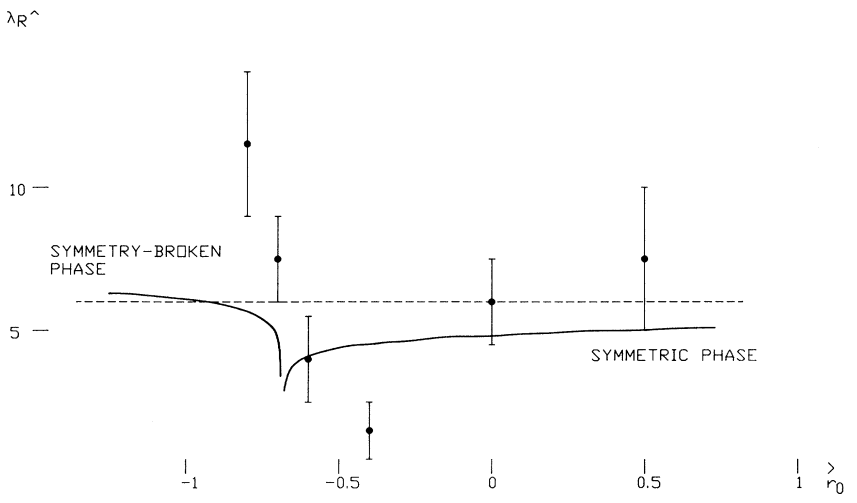

FIG. 2. The renormalized coupling constant as a function of r_0 for $b=6$. The solid line shows the results of the Gaussian approximation on a $20\times20\times20$ lattice. The points are the results of Ref. [1].

er, for small b, we obtain the same result for the β function as in the symmetric phase. The results obtained with the Gaussian variational approximation are therefore consistent with the fact that the ϕ^4 theory is perturbatively infrared-free.

One question left out from the previous discussion is a check of the stability of the (position-independent) solutions we have obtained. Are these solutions real minima or are they sometimes maxima? These questions are easily analyzed in the framework of a momentum space regularization, which we now examine.

III. GAUSSIAN APPROXIMATION WITH A MOMENTUM CUTOFF

For large enough quantization volume,

$$G(m^2)=\left[\frac{1}{2\pi}\right]^3\frac{1}{2}\int d\mathbf{k}\frac{1}{\sqrt{k^2+m^2}}\ ,$$

$$G^{-1}(m^2)=\left[\frac{1}{2\pi}\right]^3 2\int d\mathbf{k}\sqrt{k^2+m^2}\ .$$

Denoting the cutoff by Λ we find

$$G(m^2)=\frac{1}{8\pi^2}\left\{\Lambda^2-m^2\ln\left[\frac{2\Lambda}{\sqrt{e}\ m}\right]\right\}, \tag{19}$$

$$G^{-1}(m^2)=\frac{1}{8\pi^2}\left\{2\Lambda^4+2m^2\Lambda^2-m^4\ln\left[\frac{2\Lambda}{\sqrt{e}\ m}\right]-\frac{m^4}{4}\right\}, \tag{20}$$

up to terms of the order of m^2/Λ^2. To parametrize the approach to the critical point, it is convenient to use instead of the constants r_0 and b a dimensionless mass scale $\mu_{\rm latt}=\mu/\Lambda$ and a new coupling constant λ:

$$\epsilon\mu_{\rm latt}^2=\frac{r_0+\dfrac{b}{16\pi^2}}{1+\dfrac{b}{16\pi^2}\ln\left[\dfrac{2}{\sqrt{e}\ \mu_{\rm latt}}\right]}\ ,\quad \epsilon=\pm 1\ , \tag{21}$$

$$\lambda=\frac{\dfrac{b}{2}}{1+\dfrac{b}{16\pi^2}\ln\left[\dfrac{2}{\sqrt{e}\ \mu_{\rm latt}}\right]}\ . \tag{22}$$

The case $\epsilon=+1$ (-1) corresponds to the right (left) of the critical point as can be seen from the relation

$$\epsilon\mu_{\rm latt}^2=\frac{2\lambda}{b}(r_0-r_c)\ , \tag{23}$$

where $r_c=-b/16\pi^2$ for the regularization scheme considered in the present section. As in Sec. I we will work in the dimensionless quantities $m_{\rm latt}$ and $\varphi_{\rm latt}$. Keeping the ultraviolet cutoff Λ fixed we will now send r_0 to r_c, i.e., $\mu_{\rm latt}^2$ to zero. This corresponds to the renormalization condition $a+b\Lambda^2/16\pi^2=0$. The gap equation (6) for $m_{\rm latt}$ is now written as

$$m_{\rm latt}^2=\epsilon\mu_{\rm latt}^2+\frac{\lambda}{8\pi^2}\ln\left[\frac{m_{\rm latt}}{\mu_{\rm latt}}\right]+\lambda\varphi_{\rm latt}^2\ , \tag{24}$$

where we have neglected terms such as m^2/Λ^2. On both sides of the critical point we have studied the stability of the solutions of Eqs. (4) and (5) by considering the stability matrix defined as

$$A=\begin{bmatrix}\dfrac{\partial^2 V}{\partial\varphi\partial\varphi} & \dfrac{\partial^2 V}{\partial\varphi\partial m}\\ \dfrac{\partial^2 V}{\partial m\partial\varphi} & \dfrac{\partial^2 V}{\partial m\partial m}\end{bmatrix}. \tag{25}$$

We have also compared (in units of Λ^4) the shapes of the potentials $V(\varphi_{\rm latt},m_{\rm latt}=0)$, $V(\varphi_{\rm latt},\overline{m}_{\rm latt})$ as functions of $\varphi_{\rm latt}$ and $V(\varphi_{\rm latt}=0,m_{\rm latt})$, $V(\overline{\varphi}_{\rm latt},m_{\rm latt})$ as functions of $m_{\rm latt}$, where $\overline{\varphi}(m)$ and $\overline{m}(\varphi)$ are the nonzero solutions of Eqs. (4) and (5), respectively.

On the right of the critical point (i.e., $\epsilon=+1$ or $r_0>r_c$), the minima of $V(\varphi_{\rm latt},m_{\rm latt}=0)$ and $V(\varphi_{\rm latt},\overline{m}_{\rm latt})$ are at $\varphi_{\rm latt}=0$. For r_0 not far from r_c, $V(\varphi_{\rm latt},\overline{m}_{\rm latt})$ is lower than $V(\varphi_{\rm latt},m_{\rm latt}=0)$ near $\varphi_{\rm latt}=0$. The stability matrix at the point $\varphi_{\rm latt}=0$, $\overline{m}_{\rm latt}$ is given by
given by

$$A=\begin{bmatrix}m^2 & 0\\ 0 & 2m^2G'(m^2)\left[\dfrac{b}{2}G'(m^2)-1\right]\end{bmatrix}, \tag{26}$$

and its determinant is equal to

$$\det A=2\overline{m}_{\rm latt}^4G'(\overline{m}_{\rm latt}^2)\left[\frac{b}{2}G'(\overline{m}_{\rm latt}^2)-1\right]. \tag{27}$$

When we approach the critical point from the right and neglecting terms of order $1/\ln(\mu_{\rm latt})$, $\overline{m}_{\rm latt}$ is a solution of the equation

$$\overline{m}_{\rm latt}^2\left[\ln\overline{m}_{\rm latt}-\frac{16\pi^2}{b}-\ln\left[\frac{2}{\sqrt{e}}\right]\right]=\mu_{\rm latt}^2\ln(\mu_{\rm latt})=0\ , \tag{28}$$

that is, $\overline{m}_{\rm latt}=0$ or $\overline{m}_{\rm latt}^2=(4/e)\exp(32\pi^2/b)$. The second solution has to be eliminated because it corresponds to a mass higher than the cutoff.

These results obtained in the case of $\epsilon=+1$ or $r_0>r_c$ confirm the analytical results obtained previously by Bardeen and Moshe [9] in the large-N approximation and the Monte Carlo calculations of Ref. [1].

For $r_0<r_c$ or $\epsilon=-1$, the absolute minimum occurs at nonzero values $\overline{\varphi}_{\rm latt}$ and $\overline{m}_{\rm latt}$ which are related by $\overline{m}_{\rm latt}^2=(b/3)\overline{\varphi}_{\rm latt}^2$. The stability matrix at this point is given by

$$A=\begin{bmatrix}m^2 & mG'(m^2)b\varphi\\ mG'(m^2)b\varphi & 2m^2G'(m^2)\left[\dfrac{b}{2}G'(m^2)-1\right]\end{bmatrix}, \tag{29}$$

and its determinant is equal to

$$\det A = -2\bar{m}_{\text{latt}}^4 G'(m_{\text{latt}}^2)[bG'(\bar{m}_{\text{latt}}^2)+1] . \tag{30}$$

We have a minimum if

$$\frac{4}{e^2}\exp\left[-\frac{16\pi^2}{b}\right] < \bar{m}_{\text{latt}}^2 < \frac{4}{e^2} . \tag{31}$$

Approaching the critical point from the left, $\bar{m}_{\text{latt}}$ is now a solution of the equation

$$-\bar{m}_{\text{latt}}^2\left[\ln\bar{m}_{\text{latt}}+\frac{8\pi^2}{b}-\ln\left[\frac{2}{\sqrt{e}}\right]\right]=\mu_{\text{latt}}^2\ln(\mu_{\text{latt}})=0 , \tag{32}$$

that is, $\bar{m}_{\text{latt}}=0$ or

$$\bar{m}_{\text{latt}}^2=\frac{4}{e}\exp\left[-\frac{16\pi^2}{b}\right] . \tag{33}$$

The nonzero solution verifies the inequality (31) provided that $b<16\pi^2$, and in this case it corresponds to the absolute minimum. This nonzero value for the mass obtained in the continuum limit on one side of the critical point cannot be excluded by the Monte Carlo calculations of Ref. [1], which have rather large error bars.

As we already found in the previous section, the attractive feature of expression (33) is that it shows directly the prescription we have to use for the renormalization procedure. If spontaneous symmetry breaking is required to arise at a finite scale $M^2=\bar{\varphi}^2/3$, then the dependence of the coupling constant on the cutoff must be

$$\frac{b(\Lambda)}{16\pi^2}\ln\left[\frac{4\Lambda^2}{eb(\Lambda)M^2}\right]=1 , \tag{34}$$

or, keeping only the two first-order terms,

$$\frac{1}{b(\Lambda)}=\frac{1}{16\pi^2}\ln\left[\frac{4\Lambda^2}{eM^2}\right]+\frac{1}{16\pi^2}\ln\left[\frac{1}{16\pi^2}\ln\left[\frac{4\Lambda^2}{eM^2}\right]\right] . \tag{35}$$

It is worthwhile noting that Eq. (34) is also encountered when examining the stability of the solutions of the gap equation. Indeed, the determinant of the stability matrix given by Eq. (30) can be written as

$$\det\tilde{A}=b^{4\alpha-2}2M^4b\mathcal{L}(1-b\mathcal{L}) , \tag{36}$$

where $\mathcal{L}=(1/16\pi^2)\ln(4\Lambda^2/e^2b^\alpha M^2)$ and $\tilde{A}$ is the stability matrix obtained after the rescaling of the variational parameters $m^2\to b^\alpha m^2$ and $\varphi^2\to b^{-1+\alpha}\varphi^2$. In (36) we have taken the rescaled mass of the broken solution to be the mass scale M. The determinant (36) can be written as

$$\det\tilde{A}=b^{4\alpha-2}2M^4c(1-c) ,$$

with

$$c\equiv b\mathcal{L}=\frac{b}{16\pi^2}\ln\left[\frac{4\Lambda^2}{e^2b^\alpha M^2}\right] , \tag{37}$$

and thus it is positive for $0<c<1$. The values $c=0$ and $c=1$ correspond to $\det\tilde{A}=0$.

Equation (37) can be viewed as a generalization of the renormalization condition (34). With an adequate choice of α, it provides a finite value for the energy in the (m,φ) plane if the bare mass a is endowed the renormalization prescription

$$a+\frac{b\Lambda^2}{16\pi^2}=\epsilon\frac{b^\alpha M^2}{2}\left[1-\frac{b}{16\pi^2}\ln\left[\frac{4\Lambda^2}{eb^\alpha M^2}\right]\right] , \tag{38}$$

i.e.,

$$a+\frac{b\Lambda^2}{16\pi^2}=\epsilon\frac{b^\alpha M^2}{2}(1-x) , \tag{39}$$

where $\epsilon=\pm1$ and $x=c+b/16\pi^2$. From this equation the interpretation of the constant c (or x) is straightforward. We start with two bare parameters a and b and, as expected, we end up with two parameters: a numerical constant c or x and the dimensionless bare coupling constant b. We can also choose c (or x) and the scale mass M as independent parameters. The value $x=1$ corresponds to the massless theory (which is the case studied by the authors of Ref. [8]).

From Eq. (37) we obtain a nonperturbative estimation of the β function [defined as $\beta(b)=-M\,db/dM$]:

$$\beta(b)=\frac{2b}{\alpha-\frac{16\pi^2c}{b}} . \tag{40}$$

For small b ($\alpha b/16\pi^2c\ll1$),

$$\beta(b)=-\frac{b^2}{8\pi^2c} . \tag{41}$$

This expression has been obtained by the authors of Ref. [8] from the requirement of renormalization-group invariance. These authors also introduced a numerical constant $0<c<1$. The value $c=\frac{2}{3}$ corresponds to the β function calculated from the one-loop effective potential.

With the renormalization conditions (37) and (38), the expression for the energy as a function of the rescaled variational parameters m and φ becomes

$$\begin{aligned}8b^{1-2\alpha}V=&\left[2x(1-x)(m^2+\epsilon M^2)^2+\tfrac{1}{3}[\varphi^2+3\epsilon(1-x)M^2-3xm^2]^2\right]\\&+\left[\frac{b}{16\pi^2}\right]\left[2l(1+x)m^4+(1+2x)m^4+2\epsilon l(1-x)M^2m^2-2\epsilon(1-2x)M^2m^2+2lm^2\varphi^2-2m^2\varphi^2-2\epsilon M^2\varphi^2\right]\\&+\left[\frac{b}{16\pi^2}\right]^2[l^2m^4+m^4-2lm^4-2\epsilon lM^2m^2+2\epsilon M^2m^2] ,\end{aligned} \tag{42}$$

where $l=\ln(m^2/M^2)$ and we have subtracted a constant term. In Eq. (42), the energy is written explicitly as an expansion in the small parameter $b/16\pi^2$. Only for $\alpha=\frac{1}{2}$ do we get an expression for the energy for which the dominant first term is finite in the (m,φ) plane. For $x=1$, the first brackets reduce to $3[m^2-(\varphi^2/3)]^2$ and vanish in the broken phase. In order to find the minimum of V in this case we have thus to consider the next term in $b/16\pi^2$ since the first term does not determine both m and φ. For $b\to 0^-$, the expression for the energy given in Ref. [5] can be obtained from Eq. (42) with the choice $\alpha=0$, $x=-2-b/g$, $\epsilon=\mu^2/M^2(2+x)/(1-x)$, and $\mu^2=e^2M^2$, g being the renormalized coupling constant and μ the mass at the symmetric minimum.

IV. CONCLUSIONS

We are led to the following conclusions for the phase structure of the ϕ^4 theory in $3+1$ space-time dimensions in the Gaussian approximation.

(i) If the bare coupling constant b and the bare mass a satisfy the renormalization conditions (37) and (38) with $c>0$. For $\epsilon(1-x)>0$, the minimum occurs at

$$\varphi_R=0,\quad m_R^2=\frac{\epsilon(1-x)}{2+x}M^2\ . \tag{43}$$

(The subscript R is for the rescaled fields.) The energy and the determinant of the stability matrix at the minimum are given by

$$V=-b^{2\alpha-1}\frac{c(1-c)^2}{(c+2)}\frac{M^4}{8}+O(b^{2\alpha})\ , \tag{44}$$

$$\det\tilde{A}=b^{4\alpha-2}\frac{c(1-c)^2}{c+2}M^4+O(b^{4\alpha-1})\ . \tag{45}$$

For $\epsilon=-1$ and $0<c<1$, the minimum occurs at

$$\varphi_R^2=3M^2,\quad m_R^2=M^2\ . \tag{46}$$

The energy and the determinant at the minimum are given by

$$V=-b^{2\alpha-1}(1-c)(3-c)\frac{M^4}{8}+O(b^{2\alpha})\ , \tag{47}$$

$$\det\tilde{A}=b^{4\alpha-2}c(1-c)2M^4\ . \tag{48}$$

(ii) If $c=1$ and $\epsilon=-1$, i.e.,

$$a+\frac{b\lambda^2}{16\pi^2}=\frac{b}{16\pi^2}\frac{b^\alpha M^2}{2},\quad \frac{b}{16\pi^2}\ln\left[\frac{4\Lambda^2}{e^2b^\alpha M^2}\right]=1\ , \tag{49}$$

we obtain one minimum and one saddle point. For the symmetric minimum,

$$\varphi_R=0,\quad m_R^2=\frac{b}{48\pi^2}M^2\ , \tag{50}$$

the energy and the determinant are given by

$$V=-b^{2\alpha-1}\left[\frac{b}{16\pi^2}\right]^2\frac{M^4}{24}\ ,$$
$$\det\tilde{A}=b^{4\alpha-2}\left[\frac{b}{16\pi^2}\right]^2\frac{M^4}{3}\ . \tag{51}$$

For the asymmetric saddle point,

$$\varphi_R^2=3M^2,\quad m_R^2=M^2\ , \tag{52}$$

the corresponding quantities are

$$V=b^{2\alpha-1}\frac{b}{16\pi^2}\frac{M^4}{8},\quad \det\tilde{A}=0\ . \tag{53}$$

(iii) If $x=1$ and $\epsilon=-1$, i.e.,

$$a+\frac{b\Lambda^2}{16\pi^2}=0,\quad \frac{b}{16\pi^2}\ln\left[\frac{4\Lambda^2}{eb^\alpha M^2}\right]=1\ , \tag{54}$$

we obtain again one saddle point and one minimum. The symmetric solution $\varphi_R=0$, $m_R=0$ is a saddle point: $V=0$ and $\det\tilde{A}=0$. For the asymmetric minimum, $\varphi_R^2=3M^2$, $m_R^2=M^2$, the energy and the determinant are given by

$$V=-b^{2\alpha-1}\frac{b}{16\pi^2}\frac{M^4}{8},\quad \det\tilde{A}=b^{4\alpha-2}\frac{b}{16\pi^2}M^4\ . \tag{55}$$

By choosing the rescaling $\alpha=\frac{1}{2}$, the energy remains finite in the limit $\Lambda\to\infty$. For the renormalization choices (ii) and (iii) we have found two phases $\varphi_R=0$ and $\varphi_R=\sqrt{3}M$ which happened to be degenerate in the continuum limit $b\to 0^+$. On the line $m^2=b\varphi^2/3$, the potential becomes flat in the limit $b\to 0^+$. However, interesting dynamical evolution with a momentum dependence can be studied.

The main result emerging from the Gaussian variational approximation is that the theory is found to be asymptotically free in the broken phase. This conclusion differs drastically from the predictions of perturbation theory. We want to emphasize that the exponential form (33) for the solution m_{latt} at the critical point shows that the occurrence of spontaneous symmetry breaking cannot be seen in perturbation theory. The differences between the perturbative and the variational results can be understood by considering the diagrammatic interpretation of both methods. We recall that the Hartree-Bogoliubov approximation for the propagator is an infinite resummation of a certain type of Feynman diagrams. Indeed, it has been shown [2] that the Gaussian approximation for the ϕ propagator sums all diagrams without overlapping divergences. In addition to the geometrical series generated by the one-loop diagrams with the free propagator, it contains also all the so-called "cactus" diagrams. Therefore, in contrast with the usual perturbative diagrammatic expansion, the one-loop diagram which contributes to the n-point functions in the Gaussian variational approximation contains the propagator with the self-consistent mass. The perturbative calculation and the variational Gaussian approximation agree in the symmetric phase. However, the variational method contains a stability analysis. When the expansions are performed

around two different vacua, the predictions are different.

The question of the stability of the broken phase in the case of a more general set of trial states is still an open problem. The Gaussian approximation does not take into account all the overlapping divergences. A systematic post-Gaussian expansion which converges near $b \to 0^+, \overline{m}^2 = bM^2$ is necessary to have quantitative control on the variational approximation (see Ref. [10] for post-Gaussian corrections in the precarious phase and in the autonomous phase).

In conclusion, we have presented a comparison between the Monte Carlo lattice calculations and the Gaussian approximation for ϕ^4 theory in $3+1$ dimensions. In this latter case we have been able to work out the continuum limit and to evaluate finite-size effects by putting the Gaussian wave functional on a lattice. We have found that in the continuum limit $b \to 0^+$, $a + b\Lambda^2/16\pi^2 = 0$, the renormalized theory shows a broken phase which is degenerate with the symmetric phase.

Finally, we want to reemphasize that the Gaussian approximation is not consistent to all orders of perturbation theory. The perturbative expansion, which leads to the so-called true effective potential, is itself understood only as an asymptotic expansion. However, the agreement we have found with lattice calculations encourages us to think that the Gaussian approximation contains the essential physical properties of the theory.

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