Two-fermion bound-state equation using light-front Tamm-Dancoff field theory in 3+1 dimensions

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Attention has recently been paid to light-front Tamm-Dancoff field theory in 1+1 dimensions, including the proposition of a sector-dependent renormalization procedure. We extend this formalism to 3+1 dimensions, and examine a two-fermion system using light-front field theory in the lowest nontrivial Tamm-Dancoff approximation. It is found that the extension to 3+1 dimensions results in an integral equation with a kernel which does not decrease rapidly enough to admit solutions for $L_3=0$ states.

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I. INTRODUCTION

There has recently been a renewal of interest in Tamm-Dancoff field theory as a tool for the investigation of relativistic bound states. This approach, named for Tamm [1] and Dancoff [2] who proposed it, involves an approximation based on a truncation of Fock space. In particular, a subspace of the space of all Fock states is chosen (comprising, for example, those states which would seem to be reasonable intermediate states in a given problem), and the Hamiltonian is diagonalized in this subspace (see Ref. [3] for more details). This approach was investigated in the 1950s with some success (see, for example, Ref. [4]) but was ultimately abandoned, essentially due to difficulties involving the construction of the physical vacuum (in practice, disconnected and divergent vacuum terms were encountered which it was not found possible to renormalize consistently).

The renewal of interest stems from the observation [5] that since the vacuum in light-front field theory is trivial [6] (for massive particles), the application of the Tamm-Dancoff procedure within this framework should not be subject to the same difficulties as were found in the original "equal time" approach. Again to be pragmatic, the parts of the Hamiltonian which would be responsible for the generation of disconnected vacuum terms do not appear due to conservation of light-front momentum.

In this paper we discuss the application of light-front Tamm-Dancoff field theory to the case of fermions and mesons interacting through a Yukawa coupling, in 3+1 dimensions. This work is intended to be seen as an extension of that performed by Perry, Harindranath and Wilson [5,7] who considered the same problem in 1+1 dimensions. In particular, a nonperturbative, sector-dependent renormalization scheme was proposed by these authors, and the primary motivation for the work presented here is to determine whether or not this scheme is applicable in 3+1 dimensions.

As a vehicle for this examination, a two-fermion system is considered in the lowest nontrivial Tamm-Dancoff approximation, and a bound-state equation for this system obtained. In a two-fermion system there are two basic processes occurring in parallel. One is the exchange of mesons between the two fermions, and the other is the dressing of the individual fermions. The renormalization scheme used requires that the self-energy correction which arises from each individual fermion dressing itself be regulated by introducing the mass counterterm from the one-fermion system (where this dressing is the sole process). The goal of this procedure is to ensure that above the two-particle production threshold the individual fermions propagate with their physical masses. Once this has been accomplished attention can be focused on the exchange interactions responsible for producing the bound state.

Thus the first section of this paper examines the first Tamm-Dancoff approximation for a single fermion. This examination is taken to the point at which a mass counterterm is available, and we then move directly to the two-fermion case. In contrast with the 1+1 case we find that it is necessary to renormalize the coupling constant in addition to the masses and the last part of the two-fermion section is devoted to this.

We then examine the asymptotic behavior of the equation which results from the above procedure, and show that it is ill defined in the sense that it admits no solutions for states related to $L_3=0$. In the last section we attempt to identify the reason for this bad behavior.

Details of the conventions used, and the quantization procedure (commutation relations, etc.) are relegated to an appendix.

II. FIRST TAMM-DANCOFF APPROXIMATION FOR A SINGLE FERMION

As a prelude to the examination of the two-fermion problem (and because we need some of the results from the single-fermion case for that problem) we consider the case of a single fermion in the first Tamm-Dancoff approximation. This is an approximation to the case of a physical fermion, in the sense that the Tamm-Dancoff truncation limits the "dressing" of the fermion to a maximum of one meson.

Our basic equation is the Einstein equation

$$(2P^+P^- - P_\perp^2)|\Psi\rangle = M^2|\Psi\rangle \ . \qquad (1)$$

We split the Hamiltonian P^- into $P_M^- + P_I^-$ (i.e., into terms diagonal and nondiagonal, respectively, in the creation and annihilation operators), after which our equation reads

$$(M^2-2P^+P_M^-+P_\perp^2)|\Psi\rangle=2P^+P_I^-|\Psi\rangle \ . \qquad (2)$$

The required truncation of Fock space is effected by the expansion for $|\Psi\rangle$:

$$|\Psi(P)\rangle=\sum_\alpha C_0(P;\alpha)b_\alpha^\dagger(P)|0\rangle + \sum_\alpha \int da^+d^2a_\perp db^+d^2b_\perp \Delta_\delta(\Lambda,a,b) \times C_1(a,b;\alpha)b_\alpha^\dagger(a)a^\dagger(b)|0\rangle \ , \qquad (3)$$

where

$$\Delta_\delta(\Lambda,a,b)=\theta(a^+)\theta(b^+)\delta(P^+-a^+-b^+) \times\delta^2(\mathbf{P}_\perp-\mathbf{a}_\perp-\mathbf{b}_\perp) \times\theta\left[\Lambda-\frac{m_a^2+a_\perp^2}{2a^+}-\frac{m_b^2+b_\perp^2}{2b^+}\right] , \qquad (4)$$

which ensures momentum conservation, the positivity of a^+ and b^+, and regulates the integral via a cutoff Λ. For the sake of convenience we sometimes include the momentum-conserving δ functions in the definition of the cutoff functions, and when this is the case a subscript δ is appended. Upon substitution, we project the equation onto noninteracting one-fermion and one-fermion–one-boson states

$$|p_b;r\rangle=b_r^\dagger(p_b)|0\rangle \ , \qquad (5)$$

$$|p_b,p_a;r\rangle=b_r^\dagger(p_b)a^\dagger(p_a)|0\rangle \ , \qquad (6)$$

which, via orthogonality of the basis states, will produce coupled equations for the expansion coefficients C_0 and C_1.

Using the Hamiltonian in the appendix we obtain

$$\left[M^2-\left[m_{b_0}^2+\frac{\lambda^2\beta(1)}{2(2\pi)^3}\right]\right]C_0(P;r)=\frac{\lambda m_b\sqrt{2P^+}}{(2\pi)^{3/2}}\sum_\alpha\int d^2q\int_0^1 dz\frac{\Delta(\Lambda,P-q,q)}{[(1-z)z]^{1/2}}\bar{u}_r(P)\Gamma u_\alpha(P-q)C_1(P-q,q;\alpha) \qquad (7)$$

and

$$\left[M^2-\frac{1}{x}\left[m_{b_1}^2+\frac{\lambda^2\beta(x)}{2(2\pi)^3}\right]-\frac{1}{1-x}\left[m_{a_1}^2+\frac{\lambda^2\alpha(1-x)}{2(2\pi)^3}\right]-\frac{1-x}{x}p_{b\perp}^2-\frac{x}{1-x}p_{a\perp}^2+2\mathbf{p}_{b\perp}\cdot\mathbf{p}_{a\perp}\right]\Delta(\Lambda,p_b,p_a)C_1(p_b,p_a;r)$$
$$=\frac{\lambda m_b}{(2\pi)^{3/2}}\left[\frac{2}{P^+}\right]^{1/2}\sum_\alpha\frac{1}{[(1-x)x]^{1/2}}\bar{u}_r(p_b)\Gamma u_\alpha(P)C_0(P;\alpha) \ , \qquad (8)$$

respectively, where we have expressed the "+" components of the momenta as fractions of P^+; explicitly

$$p_b^+=xP^+ \ ,$$
$$p_a^+=(1-x)P^+ \ ,$$
$$q^+=zP^+ \ .$$

In deriving the above equations the sector-dependent renormalization scheme proposed by Perry and Harindranath [7] was employed. This involves assuming that the bare masses that appear in these coupled equations depend on the Fock-space sector from which they originate. Thus, in the first equation (which arises from the contraction with $|p_b;r\rangle$) we use m_{b_0}, and in the second (contracting with $|p_b,p_a;r\rangle$) we use m_{b_1}. For masses which appear due in interaction terms we use the physical masses, since we are adopting the convention that ". . . a fermion which cannot dress itself propagates with the physical mass, and the proper mass for other cases is dependent on the degree to which the fermion can dress itself" [5]. If we were using a coupling-constant expansion we could justify this by arguing that any corrections to these masses must be of a higher order in λ. Ostensibly we cannot use this argument here, since the Tamm-Dancoff truncation is not equivalent to an expansion in the coupling constant. We can note, however, that our choice of the truncated Fock-state basis would be highly inappropriate if indeed the coupling constant were large, since we would have no reason in that case to expect only the lowest Fock states to be physically significant. While this observation lends support to the use of physical masses in the interaction terms in our case, we must essentially include this "ansatz" in the description of the renormalization procedure, as an assumption.

Another assumption made is that we can omit the contribution of the "instantaneous" terms in the Hamiltonian to the equation which arises from the highest sector of Fock space under consideration. If we were performing calculations in the Tamm-Dancoff limit we would find that all instantaneous terms arising from particular sectors were needed to cancel terms appearing in higher sectors. However, since we are truncating Fock space we rule that these terms are artifacts of the truncation, and we omit them. The most obvious consequence of this omission is that the highest sector equation is algebraic rather than being an integral equation, since the inclusion of these "instantaneous" terms would have yielded an additional integral term in C_1 on the right-hand side of Eq. (8).

Now in the highest sector we can use a threshold condition to determine the counterterms needed to renormalize the fermion and meson masses in this sector. At threshold we expect C_1 to contain a pole corresponding to the scattering states, and for this to remain a solution of Eq. (8) its coefficient must vanish. Thus, setting $\mathbf{p}_{b\perp}=\mathbf{p}_{a\perp}=0$ at threshold, and denoting the appropriate momentum fraction $x_{\rm th}$, this condition becomes

$$M^2-\frac{1}{x_{\rm th}}\left[m_{b_1}^2+\frac{\lambda^2\beta(x_{\rm th})}{2(2\pi)^3}\right]-\frac{1}{1-x_{\rm th}}\left[m_{a_1}^2+\frac{\lambda^2\alpha(1-x_{\rm th})}{2(2\pi)^3}\right]=0\ . \quad (9)$$

The momentum fractions can be recovered from the fact that at threshold $M=m_b+m_a$ and $P^+=M/\sqrt{2}$. The use of these allows us to identify

$$m_b^2=m_{b_1}^2+\frac{\lambda^2\beta(x_{\rm th})}{2(2\pi)^3}\ ,\qquad m_a^2=m_{a_1}^2+\frac{\lambda^2\alpha(1-x_{\rm th})}{2(2\pi)^3}\ , \quad (10)$$

which we accept as the form of the highest sector mass counterterms. These counterterms (which represent an infinite renormalization) are used to eliminate the bare masses in favor of the physical masses in this sector.

The next step is to eliminate C_1 between Eqs. (7) and (8) in order to obtain an equation in C_0 only. In doing this we can drop the cutoff function Δ since its arguments are external momenta which we can choose to satisfy the cutoff condition. Denoting

$$D_1(p_b,p_b)=M^2-\frac{m_b^2}{x}-\frac{m_a^2}{1-x}-\frac{(1-x)p_{b\perp}^2}{x}-\frac{xp_{a\perp}^2}{1-x}+2\mathbf{p}_{b\perp}\cdot\mathbf{p}_{a\perp} \quad (11)$$

we find

$$C_1(p_b,p_a;r)=\frac{\lambda m_b}{(2\pi)^{3/2}}\left[\frac{2}{P^+}\right]^{1/2}\times\sum_\alpha\frac{1}{D_1(p_b,p_b)}\frac{1}{[(1-x)x]^{1/2}}\times\bar{u}_r(p_b)\Gamma u_\alpha(P)C_0(P;\alpha)\ . \quad (12)$$

Using this equation to provide an expression for $C_1(P-q,q;\alpha)$ and substituting into Eq. (7) yields

$$\left[M^2-\left[m_{b_0}^2+\frac{\lambda^2\beta(1)}{2(2\pi)^3}\right]\right]C_0(P;r)=\lambda^2\frac{2m_b^2}{2(2\pi)^3}\int d^2q\int_0^1dz\frac{\Delta(\Lambda,P-q,q)}{z(1-z)}\frac{1}{D_1(P-q,q)}\sum_{\alpha,\beta}\bar{u}_r(P)\Gamma u_\alpha(P-q)\bar{u}_\alpha(P-q)\Gamma u_\beta(P)C_0(P;\beta)\ . \quad (13)$$

The spinor sum can be evaluated using completeness relations to give

$$\sum_\alpha\bar{u}_r(P)\Gamma u_\alpha(P-q)\bar{u}_\alpha(P-q)\Gamma u_\beta(P)=\frac{1}{2}\delta_{r,\beta}\left[\frac{P\cdot(P-q)_{m_b}}{m_b^2}-1\right] \quad (14)$$

for $\Gamma=i\gamma_5$, where the subscript m_b is meant to imply that the vector $(P-q)$ is on shell, rather than either of its component vectors. Explicitly we have

$$P\cdot(P-q)_{m_b}=\frac{m_b^2+(\mathbf{P}_\perp-\mathbf{q}_\perp)^2}{2(1-z)}+\frac{(1-z)(m_b^2+\mathbf{P}_\perp^2)}{2}-\mathbf{P}_\perp\cdot(\mathbf{P}_\perp-\mathbf{q}_\perp)\ . \quad (15)$$

Thus, our equation becomes

$$\left[M^2-\left[m_{b_0}^2+\frac{\lambda^2\beta(1)}{2(2\pi)^3}\right]\right]C_0(P;r)=\lambda^2\frac{2m_b^2}{2(2\pi)^3}\int d^2q\int_0^1dz\frac{\Delta(\Lambda,P-q,q)}{D_1(P-q,q)}\frac{1}{z(1-z)}\frac{1}{2m_b^2}[P\cdot(P-q)_{m_b}-m_b^2]C_0(P;r)\ , \quad (16)$$

which is purely algebraic in $C_0(P;r)$.

Below the two-particle production threshold we would naturally interpret $M=m_b$, and so the equation for C_0 provides us with the mass counterterm for m_{b_0}, namely,

$$m_{b_0}^2=m_b^2-\frac{\lambda^2\beta(1)}{2(2\pi)^3}-\lambda^2\frac{2m_b^2}{2(2\pi)^3}\int d^2q\int_0^1dz\frac{\Delta(\Lambda,P-q,q)}{D_1(P-q,q)}\frac{1}{z(1-z)}\frac{1}{2m_b^2}[P\cdot(P-q)_{m_b}-m_b^2]\ . \quad (17)$$

This counterterm is all we require from the one-fermion equation at present, since it represents the self-energy correction due to the possibility of the fermion dressing itself with a single meson. The hope is that when dealing with more complicated configurations we will be able to use this counterterm to remove the divergence associated with this self-energy. To what extent this hope is realized will be addressed in the following section. For now we can note that, unlike the $1+1$ case, this renormalization is an infinite one, diverging logarithmically as the cutoff is removed.

III. FIRST TAMM-DANCOFF APPROXIMATION FOR TWO FERMIONS

We next consider the case of two fermions in the first Tamm-Dancoff approximation. Here we have, in addition to the self-energy effects seen in the one-fermion case, interactions between the two fermions, mediated by the single meson that is allowed in this approximation.

Our basic equation is, as before, the Einstein equation, which after the separation of the diagonal and nondiagonal terms, reads

$$(M^2-2P^+P_M^-+P_\perp^2)|\Psi\rangle=2P^+P_I^-|\Psi\rangle \ . \tag{18}$$

In this case the required truncation of Fock space is effected by the expansion for $|\Psi\rangle$:

$$|\Psi(P)\rangle=\sum_{\alpha,\beta}\int da\, db\, \Delta_\delta^0(\Lambda,a,b)C_0(a,b;\alpha,\beta)B_\alpha^\dagger(a)b_\beta^\dagger(b)|0\rangle + \sum_{\alpha,\beta}\int da\, db\, dc\, \Delta_\delta^1(\Lambda,a,b,c)C_1(a,b,c;\alpha,\beta)B_\alpha^\dagger(a)b_\beta^\dagger(b)a^\dagger(c)|0\rangle \ , \tag{19}$$

where $da\equiv da^+d^2a_\perp$, etc. and we have used $B^\dagger$ and $b^\dagger$ to denote the creation operators for the two (assumed distinct) types of fermions in the problem. The cutoff functions are defined, in exact analogy with the one-fermion case, by

$$\Delta_\delta^0(\Lambda,a,b)=\theta(a^+)\theta(b^+)\delta(P^+-a^+-b^+)\delta^2(\mathbf{P}_\perp-\mathbf{a}_\perp-\mathbf{b}_\perp)\theta\left[\Lambda-\frac{m_a^2+a_\perp^2}{2a^+}-\frac{m_b^2+b_\perp^2}{2b^+}\right] \tag{20}$$

and

$$\Delta_\delta^1(\Lambda,a,b,c)=\theta(a^+)\theta(b^+)\theta(c^+)\delta(P^+-a^+-b^+-c^+)\delta^2(\mathbf{P}_\perp-\mathbf{a}_\perp-\mathbf{b}_\perp-\mathbf{c}_\perp)\times\theta\left[\Lambda-\frac{m_a^2+a_\perp^2}{2a^+}-\frac{m_b^2+b_\perp^2}{2b^+}-\frac{m_c^2+c_\perp^2}{2c^+}\right] . \tag{21}$$

In this case, after substituting for $|\Psi\rangle$, we project the equation onto noninteracting two-fermion and two-fermion–one-boson states:

$$|p_B,p_b;r,s\rangle=B_r^+(p_B)b_s^\dagger(p_b)|0\rangle \ , \tag{22}$$

$$|p_B,p_b,p_a;r,s\rangle=B_r^\dagger(p_B)b_s^\dagger(p_b)a^\dagger(p_a)|0\rangle \ . \tag{23}$$

As in the one-fermion case, we expect the equations which arise from these projections to be coupled equations in C_0 and C_1. Following, again, the policy of discarding "instantaneous" terms in the highest sector of Fock space we obtain

$$\left[M^2-\frac{1}{x}\left[m_{B_0}^2+\frac{\lambda^2\beta(x)}{2(2\pi)^3}\right]-\frac{1}{1-x}\left[m_{b_0}^2+\frac{\lambda^2\beta(1-x)}{2(2\pi)^3}\right]-\frac{1-x}{x}p_{B\perp}^2-\frac{x}{1-x}p_{b\perp}^2+2\mathbf{p}_{B\perp}\cdot\mathbf{p}_{b\perp}\right]\Delta^0(\Lambda,p_B,p_b)C_0(p_B,p_b;r,s)$$
$$=\frac{\lambda m_B}{(2\pi)^{3/2}}\sqrt{2P^+}\sum_\alpha\int d^2q\int_0^x dz\frac{1}{[x(x-z)z]^{1/2}}\bar{u}_r(p_B)\Gamma u_\alpha(p_B-q)\Delta^1(\Lambda,p_B-q,p_b,q)C_1(p_B-q,p_b,q;\alpha,s)$$
$$+\frac{\lambda m_b}{(2\pi)^{3/2}}\sqrt{2P^+}\sum_\alpha\int d^2q\int_0^{1-x}dz\frac{1}{[(1-x)(1-x-z)z]^{1/2}}\bar{u}_s(p_b)\Gamma u_\alpha(p_b-q)\times\Delta^1(\Lambda,p_B,p_b-q,q)C_1(p_B,p_b-q,q;r,\alpha) \tag{24}$$

and

$$\left[M^2-\frac{1}{x}\left[m_{B_1}^2+\frac{\lambda^2\beta(x)}{2(2\pi)^3}\right]-\frac{1}{1-x-y}\left[m_{b_1}^2+\frac{\lambda^2\beta(1-x-y)}{2(2\pi)^3}\right]-\frac{1}{y}\left[m_{a_1}^2+\frac{\lambda^2\alpha(y)}{2(2\pi)^3}\right]-\frac{1-x}{x}p_{B\perp}^2-\frac{x+y}{1-x-y}p_{b\perp}^2\right.$$

$$\left.-\frac{1-y}{y}p_{a\perp}^2+2(\mathbf{p}_{B\perp}\cdot\mathbf{p}_{b\perp}+\mathbf{p}_{B\perp}\cdot\mathbf{p}_{a\perp}+\mathbf{p}_{b\perp}\cdot\mathbf{p}_{a\perp})\right]C_1(p_B,p_b,p_a,r,s)$$

$$=\frac{\lambda m_B}{(2\pi)^{3/2}}\left[\frac{2}{P^+}\right]^{1/2}\frac{1}{[x(x+y)y]^{1/2}}\sum_\alpha \bar{u}_r(p_B)\Gamma u_\alpha(p_B+p_a)C_0(p_B+p_a,p_b;\alpha,s)$$

$$+\frac{\lambda m_b}{(2\pi)^{3/2}}\left[\frac{2}{P^+}\right]^{1/2}\frac{1}{[(1-x)(1-x-y)y]^{1/2}}\sum_\alpha \bar{u}_s(p_b)\Gamma u_\alpha(p_b+p_a)C_0(p_B,p_b+p_a;r,\alpha)\ , \quad (25)$$

where we have again expressed the "+" components of the momenta as fractions of P^+; namely,

$$p_B^+=xP^+\ ,$$
$$p_a^+=yP^+\ ,$$
$$q^+=zP^+\ ,$$

the fraction for p_b^+ being fixed by the condition that $P^+=p_B^++p_b^++p_a^+$.

As in the one-fermion case, we have used the physical values for masses appearing due to interaction terms, but otherwise have allowed the bare masses to depend on the Fock-space sector from which they originate. Given the neglect of the "instantaneous" terms in the highest sector we see that the equation arising from this sector is algebraic (i.e., is not an integral equation), and we can again use a threshold condition to determine the mass counterterms. Thus we consider the three-particle threshold, where $M=m_B+m_b+m_a$. Immediately we can set $\mathbf{p}_B=\mathbf{p}_b=\mathbf{p}_a=0$, and the threshold momentum fractions can be determined from the fact that $P^+=M/\sqrt{2}$ and, for example, $p_B^+=m_B/\sqrt{2}$. At threshold we expect C_1 to contain a pole corresponding to the scattering states, and for this to remain a solution of Eq. (25) its coefficient must vanish. Thus we are led to identify

$$m_B^2=m_{B_1}^2+\frac{\lambda^2\beta(x_{\rm th})}{2(2\pi)^3}\ ,$$
$$m_b^2=m_{b_1}^2+\frac{\lambda^2\beta(1-x_{\rm th}-y_{\rm th})}{2(2\pi)^3}\ , \quad (26)$$
$$m_a^2=m_{a_1}^2+\frac{\lambda^2\alpha(y_{\rm th})}{2(2\pi)^3}\ ,$$

which we use to eliminate the bare masses in this sector.

As in the one-fermion case we can now solve this equation for C_1, and use the result to eliminate it from the lowest sector equation. Denoting

$$D_1(p_B,p_b,p_a)=M^2-\frac{m_B^2}{x}-\frac{m_b^2}{1-x-y}-\frac{m_a^2}{y}-\frac{1-x}{x}p_{B\perp}^2-\frac{x+y}{1-x-y}p_{b\perp}^2-\frac{1-y}{y}p_{a\perp}^2+2(\mathbf{p}_{B\perp}\cdot\mathbf{p}_{b\perp}+\mathbf{p}_{B\perp}\cdot\mathbf{p}_{a\perp}+\mathbf{p}_{b\perp}\cdot\mathbf{p}_{a\perp}) \quad (27)$$

we find

$$C_1(p_B,p_b,p_a,r,s)=\frac{\lambda}{(2\pi)^{3/2}}\left[\frac{2}{P^+}\right]^{1/2}\sum_\alpha\frac{1}{D_1(p_B,p_b,p_a)}$$
$$\times\left[\frac{m_B}{[x(x+y)y]^{1/2}}\bar{u}_r(p_B)\Gamma u_\alpha(p_B+p_a)C_0(p_B+p_a,p_b;\alpha,s)\right.$$
$$\left.+\frac{m_b}{[(1-x)(1-x-y)y]^{1/2}}\bar{u}_s(p_b)\Gamma u_\alpha(p_b+p_a)C_0(p_B,p_b+p_a;r,\alpha)\right]\ . \quad (28)$$

In the lowest sector equation we need expressions for $C_1(p_B-q,p_b,q;\alpha,s)$ and $C_1(p_B,p_b-q,q;r,\alpha)$; thus we need to modify the arguments in the above expression, in order to perform the substitutions required. This having been done we obtain

$$\left[M^2-\frac{1}{x}\left[m_{B_0}^2+\frac{\lambda^2\beta(x)}{2(2\pi)^3}\right]-\frac{1}{1-x}\left[m_{b_0}^2+\frac{\lambda^2\beta(1-x)}{2(2\pi)^3}\right]-\frac{1-x}{x}p_{B\perp}^2-\frac{x}{1-x}p_{b\perp}^2+2\mathbf{p}_{B\perp}\cdot\mathbf{p}_{b\perp}\right]$$
$$\times\Delta^0(\Lambda,p_B,p_b)C_0(p_B,p_b;r,s)$$
$$=\frac{2\lambda^2}{(2\pi)^3}\left[\frac{m_B^2}{x}\int d^2q\int_0^x\frac{dz}{(x-z)z}\frac{\Delta^1(\Lambda,p_B-q,p_b,q)}{D_1(p_B-q,p_b,q)}\frac{1}{2m_B^2}[p_B\cdot(p_B-q)_{m_B}-m_B^2]\right.$$
$$\left.+\frac{m_b^2}{1-x}\int d^2q\int_0^{1-x}\frac{dz}{(1-x-z)z}\frac{\Delta^1(\Lambda,p_B,p_b-q,q)}{D_1(p_B,p_b-q,q)}\frac{1}{2m_b^2}[p_b\cdot(p_b-q)_{m_b}-m_b^2]\right]C_0(p_B,p_b,q;r,s)$$
$$+\frac{2\lambda^2}{(2\pi)^3}\sum_{\alpha,\beta}\left[m_Bm_b\int d^2q\int_0^x dz\frac{\bar u_r(p_B)\Gamma u_\alpha(p_B-q)\bar u_s(p_b)\Gamma u_\beta(p_b+q)}{[x(x-z)(1-x)(1-x+z)]^{1/2}z}\right.$$
$$\times\frac{\Delta^1(\Lambda,p_B-q,p_b,q)}{D_1(p_B-q,p_b,q)}C_0(p_B-q,p_b+q;\alpha,\beta)$$
$$+m_bm_B\int d^2q\int_0^{1-x}dz\frac{\bar u_s(p_b)\Gamma u_\alpha(p_b-q)\bar u_r(p_B)\Gamma u_\beta(p_B+q)}{[(1-x)(1-x-z)x(x+z)]^{1/2}z}\frac{\Delta^1(\Lambda,p_B,p_b-q,q)}{D_1(p_B,p_b-q,q)}$$
$$\left.\times C_0(p_B+q,p_b-q;\beta,\alpha)\right],\tag{29}$$

with

$$p_B\cdot(p_B-q)_{m_B}=\frac{x}{2(x-z)}[m_B^2+(\mathbf{p}_{B\perp}-\mathbf{q}_\perp)^2]+\frac{x-z}{2x}(m_B^2+\mathbf{p}_{B\perp}^2)-\mathbf{p}_{B\perp}\cdot(\mathbf{p}_{B\perp}-\mathbf{q}_\perp),$$
$$p_b\cdot(p_b-q)_{m_b}=\frac{1-x}{2(1-x-z)}[m_b^2+(\mathbf{p}_{b\perp}-\mathbf{q}_\perp)^2]+\frac{1-x-z}{2(1-x)}(m_b^2+\mathbf{p}_{b\perp}^2)-\mathbf{p}_{b\perp}\cdot(\mathbf{p}_{b\perp}-\mathbf{q}_\perp),\tag{30}$$

where spinor completeness relations have been used to perform contractions wherever possible.

It can be seen that the first two terms on the right-hand side of the above equation are of the form of the mass corrections which we found in the one-fermion case. They clearly correspond to the emission and reabsorption of a meson by the same fermion. The remaining terms correspond to the two distinct x^+ orderings of meson exchange between the two fermions. The mass correction terms can be grouped with the mass terms on the left-hand side of the equation to which they correspond, and then we ostensibly have an equation whose only bare parameters are m_{B_0} and m_{b_0}. At this point, following Perry and Harindranath [7], we recall the lowest sector counterterm derived in the one-fermion case [Eq. (17)]. In order to use this counterterm in the two-fermion case we need to take account of the fact that in the one-fermion case the momentum fraction x did not appear, since it was obviously unity. Thus we need to modify the expression to take account of this fact, and having done this we obtain

$$m_{B_0}^2=m_B^2-\frac{\lambda^2\beta(x)}{2(2\pi)^3}-\frac{\lambda^2}{2(2\pi)^3}\int d^2q\int_0^1 dz\frac{\Delta(\Lambda,p_B-q,p_b,q)}{\tilde D_1(p_B,q)}\frac{1}{z(1-z)}\times[p_B\cdot(p_B-q)'_{m_B}-m_B^2]\tag{31}$$

and

$$m_{b_0}^2=m_b^2-\frac{\lambda^2\beta(1-x)}{2(2\pi)^3}-\frac{\lambda^2}{2(2\pi)^3}\int d^2q\int_0^1 dz\frac{\Delta(\Lambda,p_B,p_b-q,q)}{\tilde D_1(p_b,q)}\frac{1}{z(1-z)}\times[p_b\cdot(p_b-q)'_{m_b}-m_b^2],\tag{32}$$

where we denote

$$\tilde D_1(p_B,q)=-\frac{zm_B^2}{1-z}-\frac{m_a^2}{z}-\frac{zp_{B\perp}^2}{1-z}-\frac{q_\perp^2}{z(1-z)}+\frac{2\mathbf{p}_{B\perp}\cdot\mathbf{q}_\perp}{1-z},\tag{33}$$

with an equivalent expression for $\tilde D_1(p_b,q)$.

The term $p_B\cdot(p_B-q)'_{m_B}$ has been modified by the various changes of variables (in particular, the shift of the upper limit of the z integration from x to 1) and is now given by

$$\frac{1}{2(1-z)}[m_B^2+(\mathbf{p}_{B\perp}-\mathbf{q}_\perp)^2]+\frac{1-z}{2}(m_B^2+p_{B\perp}^2)-\mathbf{p}_{B\perp}\cdot(\mathbf{p}_{B\perp}-\mathbf{q}_\perp).\tag{34}$$

A similar expression exists for $p_b\cdot(p_b-q)'_{m_b}$. After substituting for the bare masses we perform several shifts of integration variables. First, in order to facilitate comparison of the mass correction terms and the counterterms, we alter the z integration variables such that all mass correction term momentum fraction integrations are over the range $0\rightarrow 1$. Second we shift the integration variables in the exchange terms such that we can combine them. Finally we notice that the momentum dependence of the spinors and coefficient function in the interaction terms can be simplified by a shift of $\mathbf{q}_\perp$ given that we make the additional assumption that

$$\mathbf{p}_{b\perp} = -\mathbf{p}_{B\perp} , \tag{35}$$

which selects the center-of-mass frame with respect to the transverse momenta, and should not lead to a loss of generality. These operations produce the equation

$$\Bigg\{ M^2 - \frac{m_B^2}{x}\left[1+\frac{2\lambda^2}{(2\pi)^3}\int d^2q\int_0^1 dz\frac{1}{z(1-z)}\frac{1}{m_B^2}[p_B\cdot(p_B-q)_{m_B}-m_B^2]\left[\frac{1}{D_B}-\frac{1}{D_B'}\right]\right]$$

$$-\frac{m_b^2}{1-x}\left[1+\frac{2\lambda^2}{(2\pi)^3}\int d^2q\int_0^1 dz\frac{1}{z(1-z)}\frac{1}{m_b^2}[p_b\cdot(p_b-q)_{m_b}-m_b^2]\left[\frac{1}{D_b}-\frac{1}{D_b'}\right]\right]-\frac{p_{B\perp}^2}{x(1-x)}\Bigg\} C_0(x,p_{B\perp};r,s)$$

$$=\frac{2\lambda^2}{(2\pi)^3}m_B m_b\sum_{\alpha,\beta}\int d^2q\int_0^1\frac{dz}{(x-z)}\frac{1}{[x(1-x)z(1-z)]^{1/2}}\bar{u}_r(m_B,x,p_{B\perp})$$

$$\times\Gamma u_\alpha(m_B,z,q_\perp)\bar{u}_s(m_b,1-x,-p_{B\perp})\Gamma u_\beta(m_b,1-z,-q_\perp)$$

$$\times\left[\frac{\theta(z-x)}{\tilde{D}_a}-\frac{\theta(x-z)}{\tilde{D}_b}\right]C_0(z,q_\perp;\alpha,\beta) , \tag{36}$$

where we have also dropped the cutoff functions in order to simplify the notation. We express the arguments of the coefficient functions in terms of momentum fractions and transverse components, and include the appropriate mass as an argument of the spinors for clarity. Several denominator functions have been defined in the writing of this equation. Explicitly these are

$$D_B = xM^2 - \frac{m_B^2}{1-z} - \frac{xm_b^2}{1-x} - \frac{m_a^2}{z} - \frac{(1-xz)p_{B\perp}^2}{(1-x)(1-z)} - \frac{q_\perp^2}{z(1-z)} + \frac{2\mathbf{p}_{B\perp}\cdot\mathbf{q}_\perp}{1-z} ,$$

$$D_B' = -\frac{zm_B^2}{1-z} - \frac{m_a^2}{z} - \frac{zp_{B\perp}^2}{1-z} - \frac{q_\perp^2}{z(1-z)} + \frac{2\mathbf{p}_{B\perp}\cdot\mathbf{q}_\perp}{1-z} , \tag{37}$$

$$\tilde{D}_a = M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{1-z} - \frac{m_a^2}{z-x} - \frac{zp_{B\perp}^2}{x(z-x)} - \frac{(1-x)q_\perp^2}{(z-x)(1-z)} + \frac{2\mathbf{p}_{B\perp}\cdot\mathbf{q}_\perp}{z-x} ,$$

where D_b and D_b' are related to D_B and D_B', respectively, by the transformation $x\to 1-x$, $m_B\leftrightarrow m_b$, and $\mathbf{p}_{B\perp}\to -\mathbf{p}_{B\perp}$, while $\tilde{D}_a$ and $\tilde{D}_b$ are related by $x\to 1-x$, $m_B\leftrightarrow m_b$, and $z\to 1-z$. These transformations have no fundamental significance (rather they are just a matter of definition), but they do have the consequence that the right-hand side is continuous (though not differentiable) at $z=x$.

A. Coupling constant renormalization

Up to this point we have closely followed Perry and Harindranath [7], and have found that most aspects of the procedure they outline can be repeated in $3+1$ dimensions provided only that the transverse momentum terms are included. At this point, however, we encounter a significant difference. The mass correction terms, on the left-hand side of Eq. (36), which in the $1+1$ case were rendered finite by the use of the mass counterterms from the one-fermion case, are still divergent in $3+1$ dimensions. Inspection of these mass correction terms reveals that it is the integration over the transverse momentum which diverges logarithmically. Since it is in the inclusion of the transverse momenta that the $3+1$ treatment differs from the $1+1$ case, it seems reasonable that this new divergence should be related to these momenta.

In order to examine the mass correction terms it is convenient to define a parameter

$$\delta = M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{1-x} - \frac{p_{B\perp}^2}{x(1-x)} , \tag{38}$$

which is analogous to the binding energy of the two-fermion bound state; in particular, vanishing when the fermions are on shell. This quantity arises very naturally in this analysis, and is, for example, the coefficient of C_0 on the left-hand side of Eq. (36) if the mass correction terms are ignored. It is also of use when considering the mass correction terms themselves, since it can easily be seen that

$$D_B = D_B' + x\delta ,$$

$$D_b = D_b' + (1-x)\delta .$$

We can immediately see, therefore, that the mass correction terms are zero at, or above, the two-particle threshold. This indicates that the renormalization procedure has at least been successful in fixing the fermion masses in the only regime where they can be unambiguously checked, namely, when they are propagating as free particles.

In order to try and isolate the source of the divergence, and in light of the observation that the mass correction terms are well defined (zero) at zeroth order in δ, we expand in powers of δ. Thus we find, for the m_B case,

$$\left[\frac{1}{D_B'+x\delta}-\frac{1}{D_B'}\right] =-\frac{x\delta}{D_B'^2}\left[1-\frac{x\delta}{D_B'}+\left[\frac{x\delta}{D_B'}\right]^2-\cdots\right] =-\frac{x\delta}{D_B'^2}+\frac{x^2\delta^2}{D_B'^2(D_B'+x\delta)} . \tag{39}$$

The advantage of isolating the first-order term in δ is that it is the only term which diverges, and thus in isolating it we have isolated the divergence. The same expansion for the m_b term yields

$$\left[\frac{1}{D_b}-\frac{1}{D_b'}\right]=-\frac{1-x)\delta}{D_b'^2}+\frac{(1-x)^2\delta^2}{D_b'^2[D_b'+(1-x)\delta]} , \tag{40}$$

which behaves similarly.

Denoting the contribution of the first term to the mass correction by $\lambda^2\delta I_B$ we find that

$$I_B=\frac{1}{2(2\pi)^2}\int dq_\perp q_\perp dz\, z\frac{q_\perp^2+z^2m_B^2}{[z^2m_B^2+(1-z)m_a^2+q_\perp^2]^2}>0 . \tag{41}$$

If we regulate this expression by imposing an upper limit Λ on the $q_\perp$ integration, this transverse integration can be readily performed:

$$I_B=\frac{1}{4(2\pi)^2}\int_0^1 dz\, z\left[\ln\left[1+\frac{\Lambda^2}{z^2m_B^2+(1-z)m_a^2}\right]-\frac{(1-z)m_a^2\Lambda^2}{z^2m_B^2+(1-z)m_a^2+\Lambda^2}\right] . \tag{42}$$

Note that the second term in Eq. (42) is finite as $\Lambda\to\infty$ whereas the first gives a logarithmic divergence. We denote them separately through $I_B=I_B^{\ln}+I_B^f$.

Denoting the full (finite) mass correction term by Δ we have

$$\Delta(\Lambda)=\left[\frac{-1}{(2\pi)^2x}\int dq_\perp q_\perp\int_0^1 dz\frac{1}{2z(1-z)^2}(q_\perp^2+z^2m_B^2)\times\frac{x^2\delta^2}{D_B'^2(D_B'+x\delta)}+\delta I_B^f\right] +(x\to 1-x;m_B\to m_b) . \tag{43}$$

We can then rewrite the coefficient of C_0 on the left-hand side of Eq. (36) as

$$\delta+\lambda^2\delta(I_B^{\ln}+I_b^{\ln})+\lambda^2\Delta(\Lambda) . \tag{44}$$

Now if we write the first two terms of this equation as

$$\delta(1+\lambda^2(I_B^{\ln}+I_b^{\ln}))$$

we can divide the whole of Eq. (36) by the divergent term, and recover a well-defined equation, provided we identify

$$\tilde{\lambda}^2=\frac{\lambda^2}{1+\lambda^2(I_B^{\ln}+I_b^{\ln})} \tag{45}$$

as the renormalized coupling constant.

It should be noted that this renormalization of the coupling constant arises from the self-energy divergence alone, since given the Tamm-Dancoff truncation there are no vertex divergences in this theory (vertex corrections would appear if sectors with two mesons were considered).

We thus find, in 3+1 dimensions with a $\Gamma=i\gamma_5$ coupling, the bound-state equation

$$\left[M^2-\frac{m_B^2}{x}-\frac{m_b^2}{1-x}-\frac{p_{B\perp}^2}{x(1-x)}+\tilde{\lambda}^2\Delta\right]C_0(x,p_{B\perp};r,s) =\frac{2\tilde{\lambda}^2}{(2\pi)^3}m_Bm_b\sum_{\alpha,\beta}\int d^2q\int_0^1\frac{dz}{x-z}\frac{1}{[x(1-x)z(1-z)]^{1/2}}\bar{u}_r(m_B,x,p_{B\perp})\Gamma u_\alpha(m_B,z,q_\perp) \times\bar{u}_s(m_b,1-x,-p_{B\perp})\Gamma u_\beta(m_b,1-z,-q_\perp)\left[\frac{\theta(z-x)}{\tilde{D}_a}-\frac{\theta(x-z)}{\tilde{D}_b}\right]C_0(z,q_\perp;\alpha,\beta) \tag{46}$$

in the lowest (nontrivial) Tamm-Dancoff approximation.

IV. ASYMPTOTIC BEHAVIOR

At this point we have rendered the mass correction terms finite, using a coupling-constant renormalization, and as a consequence there are no "explicit" divergences remaining in the bound-state equation. At first sight it appears that we should be able to proceed to solve the equation for the bound-state wave functions, but when this is attempted it is found that we still encounter divergence problems.

The reason that the equation is more badly behaved than it appears lies in the transverse momentum dependence of free particle spinors when expressed in light-front variables. That this dependence is different from the usual equal-time case can be demonstrated from a consideration of the Lorentz transformation properties of the spinors.

We know that

$$\bar{u}(p)\gamma^{\mu}u(p)=u^{\dagger}(p)\gamma^{0}\gamma^{\mu}u(p)\sim p^{\mu} \ . \tag{47}$$

Writing

$$\begin{aligned} u^{\dagger}(p)u(p)&=u^{\dagger}(p)(\Lambda^{+}+\Lambda^{-})u(p) \\ &=u^{(+)\dagger}(p)u^{(+)}(p)+u^{(-)\dagger}(p)u^{(-)}(p) \ , \end{aligned} \tag{48}$$

with $\Lambda^{\pm}=(1/\sqrt{2})\gamma^{0}\gamma^{\pm}$, and using the fact that

$$u^{(\pm)\dagger}(p)u^{(\pm)}(p)=(1/\sqrt{2})u^{\dagger}(p)\gamma^{0}\gamma^{\pm}u(p)\sim p^{\pm} \tag{49}$$

we can see that since $p^{-}=(m^{2}+p_{\perp}^{2})/2p^{+}$, we must have

$$u(p)\sim p_{\perp} \quad (\sim\sqrt{p_{\perp}} \text{ in equal-time case}) \tag{50}$$

for large $p_{\perp}$. Explicit forms for the spinors are given in the Appendix.

With this transverse momentum dependence in the spinors, a naive power counting analysis reveals that there may be divergences remaining. In order to establish this with more certainty, we will examine the asymptotic form of the solutions of Eq. (46) in the same spirit as, for example, Dalitz, Sundaresan, and Bethe [8].

As a first step towards obtaining an explicit representation of Eq. (46), we wish to expand the amplitude

$$C_{0}(x,\mathbf{p}_{B\perp};\alpha,\beta)\equiv C_{0}(x,p_{B\perp},\phi_{B};\alpha,\beta)$$

in terms of its azimuthal angular variable ϕ_{B} in which we expect it to be periodic. Prior to doing this, however, we will redefine

$$C_{0}(x,p_{B\perp},\phi_{B};\alpha,\beta)\rightarrow e^{i(\alpha+\beta)\phi_{B}}C_{0}(x,p_{B\perp},\phi_{B};\alpha,\beta) \ , \tag{51}$$

with $\alpha,\beta=\pm\frac{1}{2}$. The reason for this transformation will become clear presently. Now we expand C_{0} (we will drop the subscript zero in what follows), which will still be periodic in ϕ_{B}, as

$$C(x,p_{B\perp},\phi_{B};\alpha,\beta)=\sum_{m=-\infty}C_{m}(x,p_{B\perp};\alpha,\beta)e^{im\phi_{B}} \ . \tag{52}$$

Next we substitute for the spinors, using the explicit forms given in the Appendix, and group the resulting terms into a matrix form, using a vector representation for the amplitude:

$$\mathbf{C}_{m}=\begin{bmatrix} C_{m}(+\frac{1}{2},+\frac{1}{2}) \\ C_{m}(+\frac{1}{2},-\frac{1}{2}) \\ C_{m}(-\frac{1}{2},+\frac{1}{2}) \\ C_{m}(-\frac{1}{2},-\frac{1}{2}) \end{bmatrix} . \tag{53}$$

We find, for the spinor terms (omitting normalization factors),

$$-(\underline{\Gamma}^{(+)}e^{i(\phi_{B}-\phi_{q})}+\underline{\Gamma}^{(0)}+\underline{\Gamma}^{(-)}e^{-i(\phi_{B}-\phi_{q})}) \ , \tag{54}$$

with

$$\begin{aligned} \underline{\Gamma}^{(+)}&=\begin{bmatrix} A\bar{A} & A\bar{B} & B\bar{A} & B\bar{B} \\ A\bar{C} & 0 & B\bar{C} & 0 \\ C\bar{A} & C\bar{B} & 0 & 0 \\ C\bar{C} & 0 & 0 & 0 \end{bmatrix} , \\ \underline{\Gamma}^{(-)}&=\begin{bmatrix} 0 & 0 & 0 & C\bar{C} \\ 0 & 0 & C\bar{B} & -C\bar{A} \\ 0 & B\bar{C} & 0 & -A\bar{C} \\ B\bar{B} & -B\bar{A} & -A\bar{B} & A\bar{A} \end{bmatrix} , \\ \underline{\Gamma}^{(0)}&=\begin{bmatrix} 0 & A\bar{C} & C\bar{A} & B\bar{C}+C\bar{B} \\ A\bar{B} & -A\bar{A} & B\bar{B}+C\bar{C} & -B\bar{A} \\ B\bar{A} & B\bar{B}+C\bar{C} & -A\bar{A} & -A\bar{B} \\ B\bar{C}+C\bar{B} & -C\bar{A} & -A\bar{C} & 0 \end{bmatrix} , \end{aligned} \tag{55}$$

where

$$\begin{aligned} A&=\sqrt{2}m_{B}(p_{B}^{+}-p_{B}^{-}-q_{B}^{+}+q_{B}^{-}) \\ &\quad -2(p_{B}^{+}q_{B}^{-}-p_{B}^{-}q_{B}^{+}) \ , \\ B&=-q_{\perp}[2m_{B}+\sqrt{2}(p_{B}^{+}+p_{B}^{-})] \ , \\ C&=p_{B\perp}[2m_{B}+\sqrt{2}(q_{B}^{+}+q_{B}^{-})] \ . \end{aligned} \tag{56}$$

$\bar{A}$, $\bar{B}$, and $\bar{C}$ are related to A, B, and C by the replacement of all the B variables with their b counterparts, and the notation q_{B} is meant to show the mass dependence, and imply the z dependence [e.g., $q_{b}^{-}=(m_{b}^{2}+q_{\perp}^{2})/2q_{b}^{+}$; $q_{b}^{+}=(1-z)P^{+}$].

Since the only other angular dependence in the kernel comes from

$$\mathbf{p}_{B\perp}\cdot\mathbf{q}_{\perp}=p_{B\perp}q_{\perp}\cos(\phi_{B}-\phi_{q})$$

terms in the denominators, it can be seen that the transformation used on the amplitude [Eq. (51)] has had the effect of making the kernel depend on the azimuthal angles through $\phi_{B}-\phi_{q}$ only. Given this fact, and the periodicity of the integrand, we can easily see that the equation is diagonal in m, with the right-hand side taking the form

$$\begin{aligned} -\frac{2\bar{\lambda}^{2}}{(2\pi)^{2}}m_{B}m_{b}\frac{1}{2\pi}\int_{0}^{2\pi}d\phi_{B}\int dq_{\perp}q_{\perp}\int_{0}^{1}\frac{dz}{x-z}\frac{1}{[x(1-x)z(1-z)]^{1/2}} \\ \times\frac{1}{\Pi N}(\underline{\Gamma}^{(+)}e^{-i(m-1)\phi_{B}}+\underline{\Gamma}^{(0)}e^{-im\phi_{B}}+\underline{\Gamma}^{(-)}e^{-i(m+1)\phi_{B}})\left[\frac{\theta(z-x)}{\tilde{D}_{a}}-\frac{\theta(x-z)}{\tilde{D}_{b}}\right]\mathbf{C}_{m}(z,q_{\perp}) \end{aligned} \tag{57}$$

after shifting ϕ_B so as to remove the ϕ_q dependence, and performing the, now trivial, ϕ_q integration.

The fact that the equation has become diagonal in m can be explained by noticing that because of the transformation of the amplitude [Eq. (51)], m is now the eigenvalue of the third component of the total angular momentum J_3, rather than that of the orbital angular momentum L_3 (which it would have been if we had expanded the original amplitude). Thus we are seeing a consequence of the azimuthal symmetry which is still explicit in light-front variables.

In order to facilitate the asymptotic analysis some further transformations will be performed. First we absorb the $(x-z)$ factor in the denominator into $\tilde{D}_a$ and $\tilde{D}_b$. We then isolate the angular dependence in the denominator by defining A_a and A_b through

$$\tilde{D}_a = -2p_{B\perp}q_\perp[A_a-\cos(\phi_B)]\ ,$$
$$\tilde{D}_b = 2p_{B\perp}q_\perp[A_b-\cos(\phi_B)]\ . \tag{58}$$

Explicitly,

$$A_a=\frac{1}{2p_{B\perp}q_\perp}\left[\frac{z-x}{x}m_B^2+\frac{z-x}{1-z}m_b^2+m_a^2+\frac{z}{x}p_{B\perp}^2+\frac{1-x}{1-z}q_\perp^2-(z-x)M^2\right],$$
$$A_b=\frac{1}{2p_{B\perp}q_\perp}\left[\frac{x-z}{z}m_B^2+\frac{x-z}{1-x}m_b^2+m_a^2+\frac{x}{z}q_\perp^2+\frac{1-z}{1-x}p_{B\perp}^2-(x-z)M^2\right], \tag{59}$$

where $A_a, A_a>1$ for a bound state.

The angular integration can now be performed (by contour integration for example) to give, for the right-hand side of the equation,

$$-\frac{2\tilde{\lambda}^2}{(2\pi)^2}m_Bm_b\int dq_\perp q_\perp\int_0^1\frac{dz}{x-z}\frac{1}{[x(1-x)z(1-z)]^{1/2}}\frac{1}{\Pi N}\times\left[\frac{\theta(z-x)}{\sqrt{A_a^2-1}}(\underline{\Gamma}^{(+)}F_a^{|m-1|}+\underline{\Gamma}^{(0)}F_a^{|m|}+\underline{\Gamma}^{(-)}F_a^{|m+1|})+\frac{\theta(x-z)}{\sqrt{A_b^2-1}}(\underline{\Gamma}^{(+)}F_b^{|m-1|}+\underline{\Gamma}^{(0)}F_b^{|m|}+\underline{\Gamma}^{(-)}F_b^{|m+1|})\right]\frac{1}{2p_{B\perp}q_\perp}\mathbf{C}_m(z,q_\perp)\ , \tag{60}$$

where $F_a=1/(A_a+\sqrt{A_a^2-1})$, etc.

Next we remove the normalization and $[x(1-x)z(1-z)]^{1/2}$ factors, from the kernel, through the definition

$$\psi_m=\left[\left[\frac{1}{\sqrt{2}}(p_B^++p_B^-)+m_B\right]\left[\frac{1}{\sqrt{2}}(p_b^++p_b^-)+m_b\right]\times x(1-x)\right]^{-1/2}\mathbf{C}_m\ . \tag{61}$$

We also multiply both numerator and denominator of the kernel by $x(1-x)z(1-z)$ to explicitly remove any singular terms.

In order to establish the asymptotic behavior we are going to examine the contributions to a particular amplitude ψ_0 from regions where either $p_{B\perp}\gg q_\perp$ or $q_\perp\gg p_{B\perp}$. This being the case we will consider terms of the form $p_{B\perp}q_\perp$ small compared to $p_{B\perp}^2$ or $q_\perp^2$. From inspection of Eq. (55), it is clear that if either $p_{B\perp}$ or $q_\perp$ is large, no terms in the Γ's are larger than the diagonal $A\bar{A}$ terms. Further, for $m=0$ the F_a and F_b terms asymptotically suppress the contribution of $\Gamma^{(+)}$ and $\Gamma^{(-)}$ by a factor of $\max(p_{B\perp},q_\perp)$, leaving the kernel dominated by $\Gamma^{(0)}$.

Thus it seems reasonable, as a first approximation, to restrict our attention to the equation for a single element of ψ_0. If we make the additional assumption (for the sake of simplicity) that $P^+\gg M,m_B,m_b$ we arrive at an expression for the right-hand side of our equation:

$$-\frac{2\tilde{\lambda}^2}{4(2\pi)^2}\int_\epsilon^\infty dq_\perp q_\perp\int_0^1 dz[x(1-x)q_\perp^2+z(1-z)p_{B\perp}^2]\times\left[\frac{\theta(z-x)}{(1-x)z}+\frac{\theta(x-z)}{(1-z)x}\right]\psi_0(z,q_\perp)\ , \tag{62}$$

where we have set the lower limit on the transverse momentum integration to a constant ϵ of the same order of magnitude as the masses in the equation but small in comparison to $p_{B\perp}$, in order to explicitly remove any possible appearance of bad behavior at small transverse momenta due to the approximations we have made in the kernel.

At this point we need to consider the left-hand side of the equation, where we will use Δ to collectively denote the mass correction terms. We have

$$\left[\frac{1}{\sqrt{2}}(p_B^++p_B^-)+m_B\right]\left[\frac{1}{\sqrt{2}}(p_b^++p_b^-)+m_b\right]\times x(1-x)(\delta+\lambda^2\Delta)\psi_0\ . \tag{63}$$

The terms which are most difficult to deal with here are the mass correction terms. We have explicitly evaluated them and find that their transverse momentum dependence is $\sim p_{B\perp}^2$ for large $p_{B\perp}$. Given that we are free to choose x to be of moderate value (since we are only interested in the transverse momentum behavior) this is all we need to know about the mass correction terms here. This transverse momentum behavior is of the same order as that of δ.

It is clear therefore that the leading transverse momentum behavior of the left-hand side of the equation is $p_{B\perp}^6$, and we have, in total,

$$\psi_0(x,p_{B\perp})\sim\frac{\tilde{\lambda}^2}{p_{B\perp}^6}\int_\epsilon^\infty dq_\perp q_\perp\int_0^1 dz[x(1-x)q_\perp^2+z(1-z)p_{B\perp}^2]\left[\frac{\theta(z-x)}{(1-x)z}+\frac{\theta(x-z)}{(1-z)x}\right]\psi_0(z,q_\perp)\ . \tag{64}$$

In keeping with the approximations we have made, we now split the transverse momentum integration into two intervals about $q_\perp=p_{B\perp}$, and consider only the leading terms (neglecting the effects of the region where $q_\perp\approx p_{B\perp}$). This yields (denoting the θ function terms as $[\theta]$ for clarity)

$$\psi_0(x,p_{B\perp})\sim\frac{\tilde{\lambda}^2}{p_{B\perp}^4}\int_\epsilon^{p_{B\perp}} dq_\perp q_\perp\int_0^1 dz\, z(1-z)[\theta]\psi_0(z,q_\perp)+\frac{\tilde{\lambda}^2}{p_{B\perp}^6}\int_{p_{B\perp}}^\infty dq_\perp q_\perp^3\int_0^1 dz\, x(1-x)[\theta]\psi_0(z,q_\perp)\ . \tag{65}$$

In order for the second integral to converge, we need ψ_0 to have an asymptotic behavior $\sim 1/p_{B\perp}^n$, where $n>4$. We can see, however, that the lower integration range gives an $n=4$ behavior, which will dominate for large $p_{B\perp}$. Had we attempted to regulate the upper limit with a cutoff $q_{\max}$, we would have found that solutions are possible for any finite $q_{\max}$, but not for $q_{\max}=\infty$. Thus no solution is possible for the $m=0$ case.

For different values of m the situation is different. If we consider $m=1$, we find that $\Gamma^{(0)}$ and $\Gamma^{(-)}$ are suppressed, and the problematic term is the $A\bar{A}$ in $\Gamma^{(+)}$ (the reverse is true for $m=-1$). For $|m|\geq 2$ all the Γ's are suppressed, and there seems to be no obstacle to solving for the amplitudes. This association of the divergence problems of the equation with the behavior of the (unsuppressed) $A\bar{A}$ terms is made more interesting when we note the spin components that they correspond to.

For $m=0$ (or $J_3=0$) these terms couple the spin states $(+\frac{1}{2},-\frac{1}{2})$ and $(-\frac{1}{2},+\frac{1}{2})$, i.e., $S_3=0$ (since $\Gamma^{(0)}$ dominates). For $m=\pm1$ (or $J_3=\pm1$) they couple the states $(\pm\frac{1}{2},\pm\frac{1}{2})$, i.e., $S_3=\pm1$ (since $\Gamma^{(\pm)}$ dominates), respectively. Thus in all cases the problematic terms seem to be related to $L_3=0$.

This analysis leads us to state that the bound-state equation is well defined for states with $L_3\neq0$ and which are not coupled to states with $L_3=0$. A simple classical picture of states with $L_3=0$ would indicate that they have a zero impact parameter projection in the transverse plane, and consequently can become arbitrarily close. Thus it is for these states that we would expect any problem with high transverse momenta to manifest itself most clearly.

It should be noted that the decision to concentrate on the diagonal terms was made primarily for simplicity of presentation, since some of the off diagonal terms are of comparable magnitude in one or the other of the two momentum regimes we consider. However, the only off diagonal elements which are comparable in magnitude to the diagonal elements in both of the regimes are the cross terms between the $(\pm\frac{1}{2},\mp\frac{1}{2})$ states in $\Gamma^{(0)}$ and so the above conclusions remain unchanged.

V. GENERAL CONSIDERATIONS

We have just seen that, when taken at face value, the bound-state equation we have derived has no solutions for amplitudes with, or coupled to amplitudes with, $L_3=0$. This is somewhat unexpected given that the equation has the form one might expect for a bound-state equation in the "equal-time" formalism.

The factor that seems to be responsible for this deviation from the behavior we might expect, and more importantly for the bad behavior of the kernel, is the transverse momentum dependence of the light-front "energy" for on-shell momenta

$$p^-=\frac{m^2+p_\perp^2}{2p^+}\ . \tag{66}$$

As discussed earlier, it is this behavior which is responsible for the asymptotically linear dependence of the free particle spinors on the transverse momentum, which in turn is the cause of the high-momentum dependence of the diagonal terms in Eq. (55). It would seem to be of interest therefore to try and gain some insight into the origin of this behavior.

The difference between "equal-time" and "light-front" coordinates lies in the definition of the "time" coordinate (and one of the space coordinates). Hornbostel [9] uses a coordinate system which interpolates between these two sets, and which will prove useful here. We define

$$\begin{bmatrix}x^+\\x^-\end{bmatrix}=\begin{bmatrix}\sin\left[\frac{\theta}{2}\right] & \cos\left[\frac{\theta}{2}\right]\\ \cos\left[\frac{\theta}{2}\right] & -\sin\left[\frac{\theta}{2}\right]\end{bmatrix}\begin{bmatrix}x^0\\x^3\end{bmatrix}, \tag{67}$$

where we consider $\theta\in[\pi/2,\pi]$, the extremes corresponding to "light-front" and "equal-time" coordinates, respectively. The transverse coordinates $\mathbf{x}_\perp$ are unchanged.

Within this framework the mass-shell condition reads

$$\cos(\theta)(p_-^2-p_+^2)+2\sin(\theta)p_+p_--p_\perp^2=m^2\ . \tag{68}$$

For the "equal-time" case ($\theta=\pi$) we have

$$p_+^2-p_-^2=m^2+p_\perp^2 \tag{69}$$

or

$$p_+=\sqrt{m^2+p_\perp^2+p_-^2}\ , \tag{70}$$

which, given that $p_+=p^0$, gives us the usual asymptotic linear dependence on the transverse momenta.

For the general case we solve the mass-shell equation for $1/p_+$ [to avoid problems when $\cos(\theta)\to 0$]. Denoting $\cos(\theta)=C$ and $\sin(\theta)=S$ we find

$$\frac{1}{p_+}=\frac{Sp_-\pm\sqrt{(Sp_-)^2-(m^2+p_\perp^2+Cp_-^2)C}}{(m^2+p_\perp^2-Cp_-^2)} . \tag{71}$$

For $\theta\in(\pi/2,\pi)$ we find that asymptotically $p_+\sim p_{\perp 2}/\sqrt{p_\perp^2}=p_\perp$. It is only when $\theta=\pi/2$ that the $\sim\sqrt{p_\perp^2}$ in the numerator of Eq. (71) is lost, and we recover $p_+\equiv p^-$ as shown in Eq. (66) (the second root goes to zero, implying that p_+ goes to infinity). Thus we have established that the high transverse momentum dependence only occurs exactly on the light front.

One of the primary motivations for using light-front quantization is the triviality of the vacuum. This triviality is due to the fact that p^+ and p^- are either both positive or both negative. If we define p^0 to be positive, it then follows that p^+ and p^- are both positive also. At a general interpolating angle this is not the case. For $\theta\in(\pi/2,\pi]$ we know that $\sin(\theta/2)>\cos(\theta/2)$. It follows that p^+ and p^- can be of the same or different sign depending on the value of p^3, and therefore that it is possible to have disconnected vacuum diagrams which do not violate p^+ conservation. It is only when $\theta=\pi/2$ that we have the required behavior, and hence the triviality of the vacuum.

It is therefore the case that we have either a nontrivial vacuum and linear dependence of the "energy" on the transverse momenta, or a trivial vacuum and a quadratic dependence of "energy" on the transverse momenta. It thus appears as though it is the same feature of the light-front coordinates which renders the vacuum trivial, that causes divergences in the bound-state equation we have derived.

VI. CONCLUSION

We have examined in the lowest nontrivial light-front Tamm-Dancoff approximation the case of fermions and mesons coupled via a Yukawa interaction in $3+1$ dimensions. As a result of this examination we have obtained a bound-state equation [Eq. (46)] which, despite appearing to be of a reasonable form, has no solutions for amplitudes associated with $L_3=0$.

The primary motivation for this study was to establish whether or not the sector-dependent renormalization scheme proposed by Perry and Harindranath [7] for the $(1+1)$-dimensional case could also be used in $3+1$ dimensions. We find that most of the properties of this scheme are retained in $3+1$ dimensions. There are spurious divergences produced by the use of light-front variables. These are eliminated in $3+1$ dimensions much as they were in $1+1$ dimensions. This is undoubtedly due to the fact that in $3+1$ dimensions the "extra" transverse dimensions are unaffected by the change to light-front variables, and do not change in number or form these particular divergences. The mass renormalizations are also still successful, as is evidenced by the fact that in the only regime where the individual particle masses have any unambiguous meaning, namely, when they are on shell, the particles are seen to have their physical masses. Notwithstanding these positive features, we find that the transverse degrees of freedom demand that the coupling constant be renormalized (in order to render to mass correction terms finite), and also stop the kernel from decreasing fast enough to admit any solutions in certain cases.

It is not inconceivable that it is possible to discover a renormalization scheme that renders the bound-state equation physically sensible. Indeed given that we would expect well-defined results in the Tamm-Dancoff limit, it seems quite likely that such a renormalization is possible. The real issue is whether it is possible to discover a scheme which makes it straightforward to isolate the physical contributions at any particular order of the approximation. This is what the sector-dependent scheme proposed by Wilson, Perry, and Harindranath accomplished in $1+1$ dimensions. In the equal-time case it was the nontriviality of the vacuum which ultimately frustrated attempts to discover such a scheme. What we have seen here is perhaps an indication that while light-front quantization removes this problem, it replaces it with another. At this point we would only make the observation that the sector-dependent scheme seems not to generalize to $3+1$ dimensions as had been hoped.

We are currently investigating how we might modify the renormalization procedure in order to remove the problems we have encountered. We regard the method used for renormalizing the coupling constant as tentative, and are considering whether it is possible to absorb all the divergences in this manner. Also it is undoubtedly not insignificant that we only encounter difficulties for amplitudes related to $L_3=0$.

A slightly different perspective might be gained by noticing that it is the expansion of the bound state in terms of a free particle basis which necessitates the use of the problematic free particle spinors. It is clear from the outset that is not an ideal basis, since the constituents of a bound state cannot be on shell, and this suggests another approach which we are also investigating (such an approach would be loosely equivalent to modifying the functional dependence of p^- on the transverse momentum in some way).

It is of some interest that Brodsky and Pauli [10,11] have examined a bound-state Tamm-Dancoff equation on the light front for QED and have found similar divergences to those discussed here. This seems to confirm that the problems we have encountered are not restricted to the Yukawa-type theory we have considered. These authors proceed by imposing a cutoff on the transverse momenta and comment on the fact that their numerical solutions seem quite stable against changes in this cutoff, provided that the coupling constant is kept small. This behavior is not too surprising, since the divergences are quite weak (i.e., logarithmic).

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APPENDIX: LIGHT-FRONT VARIABLES AND THE HAMILTONIAN

In order to facilitate comparison, our definitions follow those of Perry and Harindranath [7] wherever possible. Assuming the same "normal" space-time metric as Bjorken and Drell [12] we define the light-front time and longitudinal space variables as

$$x^{\pm}=(1/\sqrt{2})(x^0\pm x^3)\ ,$$

with the transverse space components $\mathbf{x}_\perp\equiv(x^1,x^2)$ unchanged. With these definitions it follows that the scalar product becomes

$$x\cdot y=x^+y^-+x^-y^+-\mathbf{x}_\perp\cdot\mathbf{y}_\perp\ .$$

It should also be noted that we have $x^{\pm}=x_{\mp}$. As an immediate consequence of the mixing of plus and minus indices which occurs in the scalar product we are led to define p^- as the light-front energy (i.e., p^- is conjugate to x^+, the light-front time).

We similarly define

$$\gamma^{\pm}=(1/\sqrt{2})(\gamma^0\pm\gamma^3)$$

and thus find that, for example,

$$\gamma\cdot p=\gamma^+p^-+\gamma^-p^+-\gamma_\perp\cdot\mathbf{p}_\perp\ .$$

Where we need an explicit representation for the γ matrices, we have used the Weyl representation

$$\gamma^0=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \gamma^i=\begin{bmatrix}0&\sigma_i\\-\sigma_i&0\end{bmatrix},$$

$$\gamma^5=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

We use a Lagrangian density describing fermions and bosons interacting via a Yukawa interaction term

$$\mathcal{L}=(i/2)(\bar\psi\gamma^\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma^\mu\psi)-m_F\bar\psi\psi$$
$$+(1/2)\partial_\mu\phi\partial^\mu\phi-(1/2)m_B^2\phi^2-\lambda\bar\psi\psi\Gamma\phi\ ,$$

where we are using $\Gamma=i\gamma_5$, although for the most part we do not need to use this explicitly in deriving the Hamiltonian. Following Chang, Root, and Yan [13] we use Schwinger's action principle to derive the equal x^+ commutation relations (more details on this principle can be found in, for example, Ref. [14]). We find

$$[\phi(x),\phi(y)]_{x^+=y^+}=-(i/4)\delta^2(\mathbf{x}_\perp-\mathbf{y}_\perp)\epsilon(x^--y^-)\ ,$$

where $\epsilon(x)=\theta(x)-\theta(-x)$, and,

$$\{\psi_\alpha^{(+)\dagger}(x),\psi_\beta^{(+)}(y)\}_{x^+=y^+}$$
$$=(1/\sqrt{2})\Lambda_{\beta\alpha}^{(+)}\delta^2(\mathbf{x}_\perp-\mathbf{y}_\perp)\delta(x^--y^-)\ ,$$

where $\Lambda^{(+)}=(1/2)\gamma^-\gamma^+=(1/\sqrt{2})\gamma^0\gamma^+$, and $\psi^{(+)}=\Lambda^{(+)}\psi$.

To obtain the light-front Hamiltonian $\mathcal{P}^-$ we start by defining the energy-momentum tensor

$$T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\frac{\partial\phi_r}{\partial x_\nu}-\mathcal{L}g^{\mu\nu}\quad\text{where }\phi_r=\phi,\psi,\psi^\dagger\ ,$$

which enables us to find the light-front Hamiltonian density

$$\mathcal{P}^-=\frac{1}{2}m_B^2\phi^2+\bar\psi(m_F+\lambda\Gamma\phi)\psi-\frac{i}{2}\bar\psi\gamma^-\partial^+\psi$$
$$+\frac{i}{2}\partial^+\bar\psi\gamma^-\psi+\frac{i}{2}(\nabla_\perp\bar\psi\cdot\gamma_\perp\psi-\bar\psi\gamma_\perp\cdot\nabla_\perp\psi)$$
$$+\frac{1}{2}\nabla_\perp\phi\cdot\nabla_\perp\phi\ .$$

Upon integrating the Hamiltonian density to obtain the Hamiltonian, we can obtain a more convenient form by integrating by parts and using the field equations derived from the full Lagrangian $\mathcal{L}$:

$$P^-=\int d^2\mathbf{x}_\perp dx^-(\tfrac{1}{2}m_B^2\phi^2+\tfrac{1}{2}\nabla_\perp\phi\cdot\nabla_\perp\phi$$
$$+i\sqrt{2}\psi^{(+)\dagger}\partial^-\psi^{(+)})\ .$$

We should note that this expression for P^- is really only a formal one, in that it avoids appearing to contain an interaction term (i.e., containing Γ) by instead containing a "time" derivative, $\partial^-=\partial_+=\partial/\partial x^+$.

In order to obtain a usable form for the Hamiltonian we need to expand the fields ψ and ϕ in terms of a free field basis We first note that we are choosing the normalization [15]

$$u^{(+)\dagger}(p)u^{(+)}(p)=p^+/\sqrt{2}m\ ,$$

which is equivalent in usual equal-time formalism to the normalization:

$$u^\dagger(p)u(p)=\frac{p^0}{m}=\frac{E}{m}\ .$$

For the fermion field we write

$$\psi^{(+)}(x)=\sum_s\int\frac{d^2\mathbf{p}_\perp dp^+}{(2\pi)^{3/2}}\left[\frac{m}{p^+}\right]^{1/2}\theta(p^+)$$
$$\times[b(p,s)u^{(+)}(p,s)e^{-i(p^+x^--\mathbf{p}_\perp\cdot\mathbf{x}_\perp)}$$
$$+d^\dagger(p,s)v^{(+)}(p,s)$$
$$\times e^{+i(p^+x^--\mathbf{p}_\perp\cdot\mathbf{x}_\perp)}]\ ,$$

where

$$\{b(p,r),b^\dagger(q,s)\}=\{d(p,r),d^\dagger(q,s)\}$$
$$=\delta(p^--q^-)\delta^2(\mathbf{p}_\perp-\mathbf{q}_\perp)\delta_{r,s}\ ,$$

with all other anticommutators equal to zero. Similarly for the boson field we write

$$\phi(x)=\int\frac{d^2\mathbf{p}_\perp dp^+}{(2\pi)^{3/2}}\left[\frac{1}{2p^+}\right]^{1/2}\theta(p^+)\times[a(p)e^{-i(p^+x^--\mathbf{p}_\perp\cdot\mathbf{x}_\perp)}+a^\dagger(p)e^{+i(p^+x^--\mathbf{p}_\perp\cdot\mathbf{x}_\perp)}]\ ,$$

where

$$[a(p),a^\dagger(q)]=\delta(p^--q^-)\delta^2(\mathbf{p}_\perp-\mathbf{q}_\perp)\ ,$$

with all other commutators equal to zero. It should be noted that we are working in the Heisenberg picture, and as a consequence we perform our expansions at a fixed light-front time, $x^+=0$.

We will need, on occasion, an explicit representation for the spinors. We have chosen this such that the spinors have definite z components of spin in the rest frame. Explicitly,

$$u_\uparrow(p)=\frac{1}{N}\begin{bmatrix}m+\sqrt{2}p^-\\-p_\perp e^{i\phi_p}\\m+\sqrt{2}p^+\\p_\perp e^{i\phi_p}\end{bmatrix},\quad u_\downarrow(p)=\frac{1}{N}\begin{bmatrix}-p_\perp e^{-i\phi_p}\\m+\sqrt{2}p^+\\p_\perp e^{-i\phi_p}\\m+\sqrt{2}p^-\end{bmatrix},$$

where

$$N=2\{m[(1/\sqrt{2})(p^++p^-)+m]\}^{1/2}\ .$$

Following Perry and Harindranath [7] we split the "interaction" Hamiltonian P_I^- into three parts:

$$P_I^-=P_V^-+P_F^-+P_S^-\ .$$

The division is such that P_V^- changes the particle number by one, P_F^- changes the particle number by two and P_S^- leaves the particle number unchanged. We find

$$\begin{aligned}P_V^-=\frac{\lambda m_F}{\sqrt{2}(2\pi)^{3/2}}\sum_{s,t}\int dp\,dq\,dr\frac{1}{(p^++q^++r^+)^{1/2}}\theta(p^+)\theta(q^+)\theta(r^+)\\
\times[b^\dagger(p,s)b(q,t)a(r)\bar{u}(p,s)\Gamma u(q,t)\delta(p-q-r)\\
+b^\dagger(p,s)b(q,t)a^\dagger(r)\bar{u}(p,s)\Gamma u(q,t)\delta(p-q+r)\\
+b^\dagger(p,s)d^\dagger(q,t)a(r)\bar{u}(p,s)\Gamma v(q,t)\delta(p+q-r)\\
+d(p,t)b(q,s)a^\dagger(r)\bar{v}(p,s)\Gamma u(q,t)\delta(-p-q+r)\\
-d^\dagger(q,t)d(p,s)a(r)\bar{v}(p,r)\Gamma v(q,s)\delta(-p+q-r)\\
-d^\dagger(q,s)d(p,r)a^\dagger(r)\bar{v}(p,r)\Gamma v(q,s)\delta(-p+q+r)]\ ,\end{aligned}$$

where for compactness we have used the notation $dp\equiv d\mathbf{p}_\perp dp^+$ and $\delta(p)\equiv\delta(\mathbf{p}_\perp)\delta(p^+)$.

Next we find

$$\begin{aligned}P_F^-=-\frac{\lambda^2 m_F}{4(2\pi)^3}\sum_{s,t}\int dp\,dq\,dr\,dw\frac{1}{(p^++q^++r^++w^+)^{1/2}}\theta(p^+)\theta(q^+)\theta(r^+)\theta(w^+)\\
\times\Bigg[b^\dagger(p,s)b(q,t)a(r)a(w)\bar{u}(p,s)\gamma^+u(q,t)\frac{\delta(p-q-r-w)}{-q^+-w^+}\\
+b^\dagger(p,s)b(q,t)a^\dagger(r)a^\dagger(w)\bar{u}(p,s)\gamma^+u(q,t)\frac{\delta(p-q+r+w)}{-q^++w^+}\\
+b^\dagger(p,s)d^\dagger(q,t)a^\dagger(w)a(r)\bar{u}(p,s)\gamma^+v(q,t)\frac{\delta(p+q-r+w)}{q^++w^+}\\
+b^\dagger(p,s)d^\dagger(q,t)a^\dagger(r)a(w)\bar{u}(p,s)\gamma^+v(q,t)\frac{\delta(p+q+r-w)}{q^+-w^+}\\
+d^\dagger(p,s)b(q,t)a^\dagger(w)a(r)\bar{v}(p,s)\gamma^+u(q,t)\frac{\delta(-p-q-r+w)}{-q^++w^+}\\
+d(p,s)b(q,t)a^\dagger(r)a(w)\bar{v}(p,s)\gamma^+u(q,t)\frac{\delta(-p-q+r-w)}{-q^+-w^+}\\
-d^\dagger(q,t)d(p,s)a(r)a(w)\bar{v}(p,s)\gamma^+v(q,t)\frac{\delta(-p+q-r-w)}{q^+-w^+}\\
-d^\dagger(q,t)d(p,s)a^\dagger(r)a^\dagger(w)\bar{v}(p,s)\gamma^+v(q,t)\frac{\delta(-p+q+r+w)}{q^++w^+}\Bigg]\end{aligned}$$

and, for the final part of the interaction term,

$$P_S^- = -\frac{\lambda^2 m_F}{4(2\pi)^3} \sum_{s,t} \int dp\, dq\, dr\, dw \frac{1}{(p^+ q^+ r^+ w^+)^{1/2}} \theta(p^+)\theta(q^+)\theta(r^+)\theta(w^+)$$

$$\times \left[b^\dagger(p,s) b(q,t) a^\dagger(w) a(r) \bar{u}(p,s)\gamma^+ u(q,t) \frac{\delta(p-q-r+w)}{-q^+ + w^+} \right.$$

$$+ b^\dagger(p,s) b(q,t) a^\dagger(r) a(w) \bar{u}(p,s)\gamma^+ u(q,t) \frac{\delta(p-q+r-w)}{-q^+ - w^+}$$

$$+ b^\dagger(p,s) d^\dagger(q,t) a(r) a(w) \bar{u}(p,s)\gamma^+ v(q,t) \frac{\delta(p+q-r-w)}{q^+ - w^+}$$

$$+ d(p,s) b(q,t) a^\dagger(r) a^\dagger(w) \bar{v}(p,s)\gamma^+ u(q,t) \frac{\delta(-p-q+r+w)}{-q^+ + w^+}$$

$$- d^\dagger(q,t) d(p,s) a^\dagger(w) a(r) \bar{v}(p,s)\gamma^+ v(q,t) \frac{\delta(-p+q-r+w)}{q^+ + w^+}$$

$$\left. - d^\dagger(q,t) d(p,s) a^\dagger(r) a(w) \bar{v}(p,s)\gamma^+ v(q,t) \frac{\delta(-p+q+r-w)}{q^+ - w^+} \right] .$$

The terms in P_F^- and P_S^- are referred to as instantaneous. The reason for this lies in the fact that the Lagrangian contains no meson-meson interaction, which implies that a term of the form, for example, $b^\dagger(p,s)b(q,t)a(r)a(w)$ must correspond to fermion exchange. Since all fields in the Hamiltonian are defined to be at equal light-front time, this exchange is considered to be instantaneous.

For P_M^-, in which we include all terms diagonal in creation and annihilation operators, we find

$$P_M^- = \int d^2\mathbf{p}_\perp dp^+ \theta(p^+) \frac{1}{2p^+} a^\dagger(p) a(p) \left[m_B^2 + p_\perp^2 + \frac{\lambda^2}{2(2\pi)^3} \alpha(p^+) \right]$$

$$+ \int d^2\mathbf{p}_\perp dp^+ \theta(p^+) \frac{1}{2p^+} \sum_s b^\dagger(p,s) b(p,s) \left[m_F^2 + p_\perp^2 + \frac{\lambda^2}{2(2\pi)^3} \beta(p^+) \right]$$

$$+ \int d^2\mathbf{p}_\perp dp^+ \theta(p^+) \frac{1}{2p^+} \sum_s d^\dagger(p,s) d(p,s) \left[m_F^2 + p_\perp^2 + \frac{\lambda^2}{2(2\pi)^3} \gamma(p^+) \right] ,$$

where

$$\alpha(p^+) = \wp \int d^2\mathbf{r}_\perp dr^+ \theta(r^+) \left[\frac{1}{(p^+ - r^+)} - \frac{1}{(p^+ + r^+)} \right] ,$$

$$\beta(p^+) = \wp \int d^2\mathbf{r}_\perp dr^+ \theta(r^+) \frac{p^+}{r^+(p^+ - r^+)} ,$$

$$\gamma(p^+) = \int d^2\mathbf{r}_\perp dr^+ \theta(r^+) \frac{p^+}{r^+(p^+ + r^+)} ,$$

where $\wp$ denotes the principal part. It should be noted that α, β, and γ, the "self-induced inertias," are all divergent, and as such we must assume that the above expressions are regulated in some way, such that we can manipulate them.

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