Mishaps with Feynman parametrization at finite temperature

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By employing Feynman parametrization at finite temperature two groups have claimed that the real part of the one-loop self-energy for scalar bosons has a unique limit as $P^\mu \to 0$. The specific claim is that the limits $p_0 \to 0$ and $|\mathbf{p}| \to 0$ are interchangeable. One calculation uses real-time methods (Bedaque and Das); one uses imaginary time (Gribosky and Holstein). I show that the limits do not commute and trace the error to subtleties in the use of Feynman parametrization. The correct answer depends on the ratio p_0/p and is obtained in five different ways.

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I. INTRODUCTION

Finite-temperature Green's functions usually depend on both external energy p_0 and momentum $|\mathbf{p}|$ instead of on the four-vector P^μ. Moreover, there are often branch cuts in p_0. The best studied examples occur in massless theories where the function

$$F(p_0,p)=\frac{p_0}{2p}\ln\left[\frac{p_0+p}{p_0-p}\right]$$

occurs repeatedly in the high-temperature limit: $T \gg p, p_0$. In QCD at high temperature this logarithm occurs in the gluon self-energy [1] and in the quark self-energy [2]. Moreover, it occurs in all one-loop diagrams that have zero or two external quark lines and any number of external gluons because of the Abelian Ward identities for hard thermal loops discovered by Braaten and Pisarski [3] and by Frenkel and Taylor [4]. It automatically occurs in the effective Lagrangian for hard thermal loops [5]. The same logarithm occurs in the one-loop diagrams for the graviton self-energy at finite temperature [6]; and in higher-order graviton diagrams because of the Abelian Ward identities there [7].

The branch points at $p_0=\pm p$ mean that the limits $p_0 \to 0$ and $p \to 0$ do not commute:

$$F(p_0,0)=1\ ,$$

$$F(0,p)=0\ .$$

Consequently Taylor series expansions in the spacelike region $|p_0|<p$ are different from Taylor series expansion in the timelike region $|p_0|>p$ and there is no general expansion in the four-vector P^μ. It also means that in coordinate space the effects are inherently nonlocal [5].

In a more straightforward problem, Gribosky and Holstein [8] and recently Bedaque and Das [9] have calculated the one-loop self-energy of a scalar field with $\lambda\phi^3$ interactions and found a result in which the limits $p_0 \to 0$ and $p \to 0$ do commute. Their answer can be expanded in powers of P^μ and hence has a derivative expansion in coordinate space. If their result is valid, it casts doubt over all the previous results for QCD [1–5] and gravity [6,7]. The two new calculations [8,9] employ Feynman parametrization, which has not previously been used at finite temperature. I believe that is the source of the error.

The one-loop self-energy may be separated into the $T=0$ contribution and the temperature-dependent part π':

$$\pi=\pi|_{T=0}(P^2)+\pi'(p_0,p)\ .$$

Since there is no difficulty with the zero-temperature part, this paper will discuss only π'. The simplest method of calculation, i.e., without Feynman parametrization, gives a self-energy which takes on a different value in the two limits:

$$\lim_{p\to 0}\mathrm{Re}\,\pi'(0,p)=\frac{\lambda^2}{8\pi^2}\int_0^\infty \frac{dk}{\omega}n(\omega)\ , \tag{1}$$

$$\lim_{p_0\to 0}\mathrm{Re}\,\pi'(p_0,0)=\frac{\lambda^2}{8\pi^2}\int_0^\infty \frac{k^2dk}{\omega^3}n(\omega)\ , \tag{2}$$

where n is the Bose-Einstein function,

$$n(\omega)=\frac{1}{\exp(\omega/T)-1},\quad \omega\equiv(k^2+m^2)^{1/2}\ . \tag{3}$$

These answers are correct. In fact, the limiting value depends continuously on the manner in which the limit is taken. To display this set

$$p_0=\alpha p\quad (0\le\alpha\le\infty) \tag{4}$$

so that α is the slope in the p_0 vs p plane shown in Fig. 1. Then the general result is [10]

$$\lim_{p\to 0}\mathrm{Re}\,\pi'(\alpha p,p)=\frac{\lambda^2}{8\pi^2}\int_0^\infty \frac{dk}{\omega}n(\omega)\frac{k^2(1-\alpha^2)}{k^2-\alpha^2\omega^2}\ . \tag{5}$$

The values (1) and (2) are the extrema ($\alpha=0$ and ∞, respectively) of a continuum of limiting values. The dependence on p_0/p is known to occur in all one-loop diagrams for this theory [11].

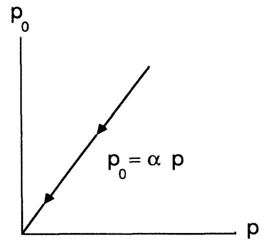

FIG. 1. The general path that is used to take the limit $p_0 \to 0$ and $p \to 0$ keeping a fixed value of the ratio $\alpha = p_0/p$. The standard cases are $\alpha = 0$ and $\alpha = \infty$.

A. Why the limit depends on p_0/p

The result (5) will later be obtained by four different methods. However, it is useful to first present a fifth derivation that will show the physical reason why the limits $p_0 \to 0$ and $p \to 0$ do not commute. The noncommuting limits can be traced to the cut structure of the self-energy. To one-loop order the self-energy satisfies the simple dispersion relation

$$\mathrm{Re}\pi'(p_0,p) = \frac{1}{\pi}\int_{-\infty}^{\infty} du \frac{\mathrm{Im}\pi'_R(u,p)}{u - p_0} . \qquad (6)$$

(Beyond one-loop order this dispersion relation is not correct.) Here π_R is the retarded self-energy, which is related to the advanced and Feynman self-energies, at one-loop order, by $\mathrm{Im}\pi_R = -\mathrm{Im}\pi_A = \tanh(p_0/2T)\mathrm{Im}\pi_F$. To one-loop order, the real parts of π_R, π_A, and π_F are all equal and need no distinguishing subscript. There are two contributions to the imaginary part in (6): $P^2 \le 0$ and $P^2 \ge 4m^2$. The spacelike contribution only exists at $T \ne 0$. It comes from the absorption of a virtual, spacelike field of momentum P^μ by a real, on-shell particle in the plasma. Distinguish the effects of the two cuts by

$$\mathrm{Re}\pi'(p_0,p) = A(p_0,p) + B(p_0,p) ,$$

where the contribution from the spacelike cut is

$$A(p_0,p) = \frac{1}{\pi}\int_{-p}^{p} du \frac{\mathrm{Im}\pi'_R(u,p)}{u - p_0} . \qquad (7)$$

Since $\mathrm{Im}\pi'(u,p)$ is an odd function of u, this can be written as an integral from 0 to p. Then change integration variables to $v = u/p$ to get

$$A(p_0,p) = \frac{2}{\pi}\int_0^1 v\, dv \frac{\mathrm{Im}\pi'_R(vp,p)}{v^2 - (p_0/p)^2} .$$

This clearly depends on the ratio $p_0/p \equiv \alpha$. It will depend on this ratio even as $p \to 0$. To check the explicit value (5), one needs the imaginary part, derived in Appendix A:

$$\lim_{p\to 0} \mathrm{Im}\pi'_R(vp,p) = \frac{\lambda^2}{16\pi} v n(\omega) , \qquad (8)$$

where $\omega = m/(1-v^2)^{1/2}$. Substituting above and changing integration variables to $k = mv/(1-v^2)^{1/2}$ gives

$$\lim_{p\to 0} A(\alpha p,p) = \frac{\lambda^2}{8\pi^2}\int_0^\infty \frac{k^2 dk}{\omega^3} n(\omega) \frac{m^2}{k^2 - \alpha^2\omega^2} . \qquad (9)$$

The right-hand side depends on the value of $\alpha = p_0/p$. The limit always depends on the value of α. There is no unique value as $P^\mu \to 0$.

By contrast, the usual cut contributes

$$B(p_0,p) = \frac{2}{\pi}\int_{u_0}^{\infty} u\, du \frac{\mathrm{Im}\pi'_R(u,p)}{u^2 - p_0^2} , \qquad (10)$$

where $u_0 = (4m^2 + p^2)^{1/2}$. For B it does not matter how p_0 and p go to zero:

$$B(0,0) = \frac{2}{\pi}\int_{2m}^{\infty} du \frac{\mathrm{Im}\pi'_R(u,0)}{u} .$$

Appendix A shows that

$$\mathrm{Im}\pi'_R(u,0) = \frac{\lambda^2}{16\pi}\left[1 - \frac{4m^2}{u^2}\right]^{1/2} n(u/2) . \qquad (11)$$

Substituting this and changing integration variables so that $u = 2(k^2+m^2)^{1/2}$ gives

$$B(0,0) = \frac{\lambda^2}{8\pi^2}\int_0^\infty \frac{k^2 dk}{\omega^3} n(\omega) . \qquad (12)$$

The sum of (9) and (12) is the same as (5), which will later be obtained by Feynman diagram methods. The above derivation shows that all of the α dependence comes from the new cut.

B. Feynman parametrization

What goes wrong with Feynman parametrization is rather interesting. At $T = 0$ each Feynman diagram represents a function that is analytic when the external energies p_0 are off the real axis. Consequently one usually calculates the full analytic function (real and imaginary parts) by a single integration using Feynman's famous formula [12]

$$\frac{1}{(1+i\epsilon)(b+i\epsilon)} = \int_0^1 \frac{dx}{[a(1-x)+bx+i\epsilon]^2} , \qquad (13)$$

where a and b are polynomials in the internal and external momenta.

At $T \ne 0$ the real-time Feynman amplitude is not the boundary value of a single analytic function. Consequently it is necessary to do one calculation for the real part of an amplitude and another for the imaginary part. In calculating the real part, integrands of the form $\delta(a)/b$ arise with $1/b$ defined by the principal value. Direct integration has been the standard method of evaluation and is repeated here in Sec. II. If instead one writes

$$\delta(a) = \frac{i}{2\pi}\left[\frac{1}{a+i\epsilon} - \frac{1}{a-i\epsilon}\right] , \qquad (14)$$

$$\mathcal{P}\frac{1}{b} = \frac{1}{2}\left[\frac{1}{b+i\epsilon} + \frac{1}{b-i\epsilon}\right] , \qquad (15)$$

then it seems natural to use Feynman parametrization to combine the denominators. Because the product $\delta(a)/b$ contains denominators whose imaginary parts have opposite signs it is natural to expect that

$$\frac{1}{(a+i\epsilon)(b-i\epsilon)} \overset{?}{=} \mathcal{P}\int_0^1 \frac{dx}{D^2} , \qquad (16)$$

$$D = a(1-x)+bx+i\epsilon(1-2x) , \qquad (17)$$

where the integral is defined as the principal value because the imaginary part of D vanishes as $x=\frac{1}{2}$. The problem is that (16) fails at $a+b=0$. The correct statement is

$$\frac{1}{(a+i\epsilon)(b-i\epsilon)} = \mathcal{P}\int_0^1 \frac{dx}{D^2} + \frac{4\pi i \delta(a+b)}{a-b+2i\epsilon} . \qquad (18)$$

This is proven in Appendix B. It will be shown in Sec. II that the second term produces the dependence on the manner in which p_0 and p are taken to zero.

Incidentally, it is easy to show that the naive formula (16) is wrong by a simple counterexample. Consider the function

$$f(r,s) \equiv \int_{-\infty}^{\infty} \frac{dk}{(k-r+i\epsilon)(k-s-i\epsilon)} .$$

Performing the integration gives $f(r,s)=2\pi i/(s-r+2i\epsilon)$. If instead one uses (16) and interchanges the order of integration, then

$$f(r,s) \overset{?}{=} \mathcal{P}\int_0^1 dx \int_{-\infty}^{\infty} dk \frac{1}{D'^2} ,$$

where $D'=k-r+(r-s)x+i\epsilon(1-2x)$. Because there is now a double pole in k, the integral over k vanishes: $f(r,s)\overset{?}{=}0$. Obviously this is wrong. In this example the extra term in (18) is the entire answer:

$$f(r,s) = \int_{-\infty}^{\infty} dk \frac{4\pi i}{s-r+2i\epsilon}\delta(2k-r-s) = \frac{2\pi i}{s-r+2i\epsilon} .$$

This paper is organized as follows. Section II performs the real-time calculation in two ways. Straightforward integration of $\delta(a)/b$ gives the correct answer (5). Then the Feynman parametrization formula (18) is used and the same answer results. Section III performs the imaginary-time calculation in two ways. Conventional integration and a trivial analytic continuation agree with (5). When Feynman parametrization is used the usual formula (13) applies since the imaginary-time denominators never vanish. However, the analytic extension from $p_{0l}=i2\pi lT$ to continuous, real values is quite difficult. Naive extension gives a function in which the limits $p_0\to 0$ and $p\to 0$ do commute. However, this extension is incorrect because it has an infinite number of unphysical branch points in the complex p_0 plane. The correct extension requires finding a function that vanishes at $p_{0l}=i2\pi lT$ but cancels all the unphysical branch points. For the correct extension the limits $p_0\to 0$ and $p\to 0$ do not commute and the correct limit again agrees with (5). In summary, Feynman parametrization makes an easy calculation very difficult both in real time and in imaginary time.

The effective potential is not dealt with at all. It is known that at $T\neq 0$ the effective potential does not have the same diagrammatic expansion as at $T=0$ and to calculate it in the real-time formalism requires an extra rule [13], which effectively replaces the time-ordered propagators by retarded or advanced propagators [14].

II. REAL-TIME CALCULATION

A. Conventional integration

It is useful to first compute the real part of the one-loop self-energy by simple integration and confirm that it has the limiting form (5). The real part is given by

$$\mathrm{Re}\pi'(p_0,p) = -\lambda^2 \int \frac{d^4K}{(2\pi)^3} n(|k^0|)\frac{\delta(a)}{b} , \qquad (19)$$

$$a \equiv K^2-m^2, \quad b\equiv (K+P)^2-m^2 , \qquad (20)$$

where n is the Bose-Einstein function. Integrating k^0 against the δ function gives

$$\mathrm{Re}\pi'(p_0,p) = -\lambda^2 \int \frac{d^3k}{(2\pi)^3} n(\omega)\frac{I}{2\omega} ,$$

$$I \equiv \frac{1}{p_0^2+2\omega p_0-(\mathbf{k}+\mathbf{p})^2} + (p_0 \leftrightarrow -p_0) .$$

One is left with two angular integrals of the form

$$\mathcal{P}\int_0^{\pi} \frac{\sin\theta d\theta}{u-2pk\cos\theta} = \frac{1}{2pk}\mathrm{Re}\left[\ln\left[\frac{u+2pk}{u-2pk}\right]\right] . \qquad (21)$$

(When $-2kp<u<2kp$ the real part of the logarithm discards the $\pm i\pi$.) This gives

$$\mathrm{Re}\pi'(p_0,p) = \frac{-\lambda^2}{16\pi^2}\int_0^{\infty} \frac{kdk}{\omega p} n(\omega)\mathrm{Re}[\ln(R)] , \qquad (22)$$

$$R = \frac{[p_0^2-p^2+2\omega p_0+2kp][p_0^2-p^2-2\omega p_0+2kp]}{[p_0^2-p^2+2\omega p_0-2kp][p_0^2-p^2-2\omega p_0-2kp]} . \qquad (23)$$

Zero-momentum limit. Now set $p_0=\alpha p$ and then let $p\to 0$. The result is

$$\lim_{p\to 0} \mathrm{Re}\pi'(\alpha p,p) = \frac{\lambda^2}{8\pi^2}\int_0^{\infty} \frac{dk}{\omega} n(\omega)\frac{k^2(1-\alpha^2)}{k^2-\alpha^2\omega^2} . \qquad (24)$$

This depends on the value of α and is the result quoted in (5).

B. Feynman parametrization

To employ Feynman parametrization one starts with (19) and expresses $\delta(a)/b$ as the product of (14) with (15). Combining denominators using the Feynman parametrization formulas (13) and (18) gives

$$\delta(a)\mathcal{P}\frac{1}{b} = -\int_0^1 \frac{dx}{2}\mathrm{Im}\left[\frac{1}{C^2}+\frac{1}{D^2}\right] + \frac{2\pi\delta(a+b)}{a-b} ,$$

$$C = a(1-x)+bx+i\epsilon , \qquad (25)$$

$$D = a(1-x)+bx+i\epsilon(1-2x) .$$

Both the x integration and the fraction $1/(a-b)$ are defined by their principal value. Denote the self-energy contribution that comes from the x integration by π_x and the part that comes from the Dirac δ function by π_δ. The complete self-energy is

$$\mathrm{Re}\pi'(p_0,p)=\pi_x(p_0,p)+\pi_\delta(p_0,p)\ , \tag{26}$$

$$\pi_\delta(p_0,p)=\lambda^2\int\frac{d^4K}{(2\pi)^3}n(|k_0|)\frac{2\pi\delta(a+b)}{a-b}\ . \tag{27}$$

The Feynman parametrized part was computed by Bedaque and Das with the result [Eq. (23) of [9]]

$$\pi_x=\frac{-\lambda^2}{4}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\int_0^{1/2}dx\,J\ ,$$

$$J\equiv\frac{2}{\phi}[n(\phi+xp_0)+(p_0\leftrightarrow -p_0)]\ ,$$

$$\phi\equiv[k^2+m^2-x(1-x)P^2]^{1/2}\ .$$

For π_x the limits $p_0\to 0$ and $p\to 0$ do indeed commute. Using $\partial J/\partial m^2=\partial J/\partial k^2$ one can integrate by parts with respect to k and get

$$\pi_x(p_0,p)=\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\int_0^{1/2}dx\,J\ .$$

Regardless of how $p_0\to 0$ and $p\to 0$, the value is the same:

$$\pi_x(0,0)=\frac{\lambda^2}{8\pi^2}\int_0^\infty\frac{dk}{\omega}n(\omega)\ . \tag{28}$$

This is the final result of Bedaque and Das [9].

Correction term. The correction term (27) must now be calculated. After shifting the integration variable $K^\mu\to K^\mu-P^\mu/2$, it becomes

$$\pi_\delta=-\lambda^2\int\frac{d^4K}{(2\pi)^4}n\left[\left|k^0-\frac{p^0}{2}\right|\right]\frac{\pi\delta(K^2-m_{\mathrm{eff}}^2)}{K\cdot P}\ ,$$

where $m_{\mathrm{eff}}^2=m^2-P^2/4$. This integral contains the principal value of $1/K\cdot P$. Performing the integrations over k_0 and over the angles gives

$$\pi_\delta=\frac{\lambda^2}{16\pi^2}\int_0^\infty k\,dk\frac{L}{p}\mathrm{Re}\left[\ln\left[\frac{\Omega p_0+kp}{\Omega p_0-kp}\right]\right]\ ,$$

$$L=\frac{1}{\Omega}\left[n\left[\left|\Omega+\frac{p_0}{2}\right|\right]-n\left[\left|\Omega-\frac{p_0}{2}\right|\right]\right]\ , \tag{29}$$

$$\Omega=\left[k^2+m^2-\frac{P^2}{4}\right]^{1/2}\ .$$

Note that if $x=\frac{1}{2}$ then $\phi=\Omega$; and if $P^2=0$ then $\Omega=\omega$.

To investigate the zero-momentum limit, set $p_0=\alpha p$. Then L becomes the derivative of the Bose-Einstein function so that

$$\lim_{p\to 0}\pi_\delta(\alpha p,p)=\frac{\lambda^2\alpha}{16\pi^2}\int_m^\infty d\omega\frac{dn}{d\omega}\mathrm{Re}\left[\ln\left[\frac{\alpha\omega+k}{\alpha\omega-k}\right]\right]\ .$$

Integrating by parts gives

$$\lim_{p\to 0}\pi_\delta(\alpha p,p)=\frac{\lambda^2}{8\pi^2}\int_0^\infty\frac{dk}{\omega}n(\omega)\frac{\alpha^2m^2}{k^2-\alpha^2\omega^2}\ . \tag{30}$$

Note that this is *not* the same as the contribution of the spacelike cut given in (9). However, adding this to (28) gives

$$\lim_{p\to 0}[\pi_x(\alpha p,p)+\pi_\delta(\alpha p,p)]=\frac{\lambda^2}{8\pi^2}\int_0^\infty\frac{dk}{\omega}n(\omega)\frac{k^2(1-\alpha^2)}{k^2-\alpha^2\omega^2}\ . \tag{31}$$

This is in perfect agreement for any value of α with the simpler approach (24).

III. IMAGINARY-TIME CALCULATION

In the imaginary-time formulation all energy variables are discrete and imaginary. The Green's functions must then be extended to continuous, complex values of the energy and ultimately to continuous, real values of the energy. The extension of the propagator that is analytic in the upper half plane of complex p_0 is called the retarded propagator D_R; the extension of the propagator that is analytic in the lower half plane is the advanced propagator D_A. The time-ordered or Feynman propagator D_F is given (for real and complex values of p_0) by

$$D_F=[1+n(p_0)]D_R-n(p_0)D_A\ ; \tag{32}$$

where n is the Bose-Einstein function. It then follows that to one-loop order the self-energies π_R, π_A, and π_F all have the same real part. To one-loop order the problem amounts to extending the function $\pi(p_{0l})$ defined only for $p_{0l}=i2\pi lT$, where l is an integer, to a function whose only singularities are on the real p_0 axis. There can be no singularities in either the upper or lower half plane. For the conventional calculation this extension is automatic. However, if Feynman parametrization is used the analytic extension is quite difficult, but knowing the results of Sec. II allows us to guess the right answer and obtain complete agreement with the real-time method.

A. Conventional integration

The one-loop self-energy is given by

$$\pi(p_{0l},p)=\frac{\lambda^2T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\frac{1}{D_1D_2}\ ,$$

$$D_1=[2\pi nT]^2+\mathbf{k}^2+m^2\ , \tag{33}$$

$$D_2=[2\pi(n+l)T]^2+(\mathbf{k}+\mathbf{p})^2+m^2\ ,$$

where $p_{0l}=i2\pi lT$. Performing the sum over n gives

$$\pi(p_{0l},p)=-\lambda^2\int\frac{d^3k}{(2\pi)^3}\coth(\omega/2T)\frac{I}{4\omega}\ ,$$

$$I\equiv\frac{1}{p_{0l}^2+2\omega p_{0l}-(\mathbf{k}+\mathbf{p})^2}+(p_{0l}\leftrightarrow -p_{0l})\ .$$

Since $\coth(\omega/2T)=1+2n(\omega)$, the temperature-dependent part is

$$\pi'(p_{0l},p)=-\lambda^2\int\frac{d^3k}{(2\pi)^3}n(\omega)\frac{I}{2\omega}\ .$$

The angular integration gives

$$\pi'(p_{0l},p)=\frac{-\lambda^2}{16\pi^2}\int_0^\infty\frac{kdk}{\omega p}n(\omega)\ln[R(p_{0l})]\ , \quad (34)$$

where R is the function given in (23). If p_{0l} is made complex and continuous, the only zeros or poles of R occur for p_0 on the real axis. It is perfectly appropriate to have singularities on the real axis. Thus the analytic extension of (34) is trivially obtained by letting p_{0l} be real. On the real axis the three self-energies are given by

$$\begin{aligned}\pi_R(p_0)&=\pi(p_0+i\epsilon)\ ,\\ \pi_A(p_0)&=\pi(p_0-i\epsilon)\ ,\\ \pi_F(p_0)&=\pi(p_0+i\epsilon p_0)\ .\end{aligned} \quad (35)$$

The real parts of all three are the same (to this order) and coincide with the real-time result (22). The limit $p_0\to 0$ and $p\to 0$ is therefore the same as (24).

B. Feynman parametrization

An alternative is to apply Feynman parametrization directly to (33). Since D_1 and D_2 are always positive and never vanish, the usual Feynman formula (13) is valid with no modifications:

$$\pi(p_{0l},p)=\frac{\lambda^2 T}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\int_0^1\frac{dx}{D^2}\ ,$$

$$D=(1-x)D_1+xD_2\ .$$

This is the procedure of Gribosky and Holstein [8]. They perform the sum on n and obtain, for the temperature-dependent part [Eq. (3.33) of [8]],

$$\pi'(p_{0l},p)=\frac{-\lambda^2}{8}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\int_0^1 dx\, J\ ,$$

$$J\equiv\frac{2}{\phi}[n(\phi+xp_{0l})+n(\phi-xp_{0l})]\ ,$$

$$\phi\equiv[k^2+m^2-x(1-x)P_l^2]^{1/2}\ ,$$

where $P_l^2\equiv p_{0l}^2-\mathbf{p}^2$. Because p_{0l} is imaginary and discrete, it is only necessary to integrate from $x=0$ to $x=\frac{1}{2}$:

$$\pi'(p_{0l},p)=\frac{-\lambda^2}{4}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\int_0^{1/2}dx\, J\ . \quad (36)$$

If p_{0l} is replaced by arbitrary complex p_0 one obtains

$$\pi_x(p_0,p)=\frac{-\lambda^2}{4}\frac{\partial}{\partial m^2}\int\frac{d^3k}{(2\pi)^3}\int_0^{1/2}dx\, J\ , \quad (37)$$

in agreement with Bedaque and Das for p_0 real [9]. However, (37) is not the correct analytic continuation of (36). The problem is that (37) has branch points in the complex p_0 plane at

$$\hat{p}_0=\mp(E+i2\pi sT)\ , \quad (38a)$$

$$E\equiv[(2\pi sT)^2+2m^2+\tfrac{1}{2}\mathbf{p}^2]^{1/2}\ , \quad (38b)$$

for every integer s. Therefore one must add to (37) a function $\pi_\delta(p_0,p)$ with two special properties: (a) it must vanish for $p_0=i2\pi lT$ and (b) it must have branch points at (38), and nowhere else, that cancel those of π_x. It is extremely difficult to use these two criteria to construct the unknown function π_δ. However, the previous real-time calculation provides us with the solution

$$\pi_\delta=\frac{\lambda^2}{16\pi^2}\int_0^\infty k\, dk\frac{L}{p}\mathrm{Re}\left[\ln\left[\frac{\Omega p_0+kp)}{\Omega p_0-kp}\right]\right]\ , \quad (39)$$

$$L\equiv\frac{1}{\Omega}[n(\Omega+p_0/2)-n(\Omega-p_0/2)]\ . \quad (40)$$

This automatically vanishes for p_0 on the imaginary axis because the logarithm has no real part there. The remainder of this section will show that the branch points of (37) are exactly canceled by the addition of (39). The real part of the self-energy is the sum of (37) and (39) evaluated for p_0 real. This is exactly the same as obtained in Sec. II B and therefore the behavior for small p_0 and p is exactly the same as (24).

Branch points of π_x. Using $\partial J/\partial m^2=\partial J\partial k^2$ one can integrate by parts with respect to k and get

$$\pi_x(p_0,p)=\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\int_0^{1/2}dx\, J\ .$$

We will now show that the poles in J lead to branch points in p_0. For most values of p_0 the poles in J cause no trouble because the integration contours for k and x can be distorted away from the poles of J. However, at certain values of p_0, the poles in J occur at the end points $x=\frac{1}{2}$, $k=0$ and the contours cannot be distorted. This produces branch points in π_x at those values of p_0. The phenomenon of such end-point singularities is fully discussed in Eden, Landshoff, Olive, and Polkinghorne [15]. To find the branch points, let $\hat{x}(p_0,k)$ be the location of the pole in J:

$$J\to\frac{N(p_0,k)}{x-\hat{x}(p_0,k)}\ .$$

Then $\hat{x}$ satisfies

$$\phi(p_0,k,\hat{x})\pm\hat{x}p_0=-i2\pi sT\ ,$$

where s is any integer. Integrating over x will only produce a singularity if the pole in x coincides with the end point of the integration, $x=\frac{1}{2}$. (At the $x=0$ end point there is no dependence on p_0.) Thus the singular part is

$$\pi_x\to\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\, N(p_0,k)\int^{1/2}\frac{dx}{x-\hat{x}(p_0,k)}$$

$$=\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\, N(p_0,k)\ln[\hat{x}(p_0,k)-\tfrac{1}{2}]\ ,$$

where $N(p_0,k)\equiv\pm 2T/[p_0\phi(p_0,k,\frac{1}{2})]$. This is only singular when the argument of the logarithm vanishes.

Let $\hat{k}(p_0)$ be the value of k that satisfies

$$\hat{x}(p_0,\hat{k})=\tfrac{1}{2}\ .$$

Near the branch point,

$$\hat{x}(p_0,k)-\tfrac{1}{2}\rightarrow[k-\hat{k}(p_0)]\left[\frac{\partial\hat{x}}{\partial\hat{k}}\right],$$

$$\pi_x\rightarrow\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\,N(p_0,k)\ln[k-\hat{k}(p_0)]\ .$$

Integrating over k will only produce a singularity at those values of p_0 for which $\hat{k}(p_0)$ coincides with the $k=0$ end point of the integration. The singular part is

$$\pi_x\rightarrow\frac{\lambda^2}{16\pi^2}N(p_0,0)\int_0 dk\,\ln[k-\hat{k}(p_0)]$$
$$=-\frac{\lambda^2}{4\pi^2|\hat{p}_0|^2}\hat{k}(p_0)\ln\hat{k}(p_0)\ . \qquad (41)$$

This has branch points at those values of p_0 that make $\hat{k}(p_0)$ vanish. Define $\hat{p}_0$ by $\hat{k}(\hat{p}_0)=0$. This is equivalent to solving

$$\phi(\hat{p}_0,0,\tfrac{1}{2})\pm\tfrac{1}{2}\hat{p}_0=-i2\pi sT\ .$$

The explicit form of $\hat{p}_0$ is given by (38) and, in terms of that,

$$\hat{k}(p_0)=[\mp E(p_0-\hat{p}_0)]^{1/2}\ . \qquad (42)$$

Branch points of π_δ. Now we will show that the only singularities of the function (39) are branch points that exactly cancel those in (41). The singularities will occur at the same location because $\phi(p_0,k,\tfrac{1}{2})=\Omega(p_0,k)$. The first step is to use

$$\frac{1}{\exp(u)-1}=\frac{d}{du}\ln[1-\exp(-u)]$$

to rewrite L and integrate (39) by parts to obtain

$$\pi_\delta(p_0,p)=\frac{\lambda^2}{32\pi^2}\frac{p_0}{p}\ln\left[\frac{p_0+p}{p_0-p}\right]+\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\,M(p_0,k)\ln(Q)\ ,$$
$$M(p_0,k)=\frac{Tp_0(P^2-4m^2)}{2\Omega(\Omega^2p_0^2-k^2p^2)}\ , \qquad (43)$$
$$Q=\frac{\sinh[(2\Omega+p_0)/4T]}{\sinh[(2\Omega-p_0)/4T]}\ .$$

The logarithm of Q has branch points at $\hat{k}(p_0)$ satisfying

$$\Omega(p_0,\hat{k})\pm\tfrac{1}{2}p_0=-i2\pi sT\ .$$

The singular part of (43) is therefore

$$\pi_\delta\rightarrow\frac{\lambda^2}{16\pi^2}\int_0^\infty dk\,M(p_0,k)\ln[k-\hat{k}(p_0)]\ .$$

The integration over k will only be singular at those values of p_0 for which $\hat{k}(p_0)$ coincides with the $k=0$ end point of the integration. The singular part is

$$\pi_\delta\rightarrow\frac{\lambda^2}{16\pi^2}M(p_0,0)\int_0 dk\,\ln[k-\hat{k}(p_0)]$$
$$=\frac{\lambda^2}{4\pi^2|\hat{p}_0|^2}\hat{k}(p_0)\ln\hat{k}(p_0)\ , \qquad (44)$$

with $\hat{k}(p_0)$ given by (42). The branch points of (44) precisely cancel those of (41) as claimed. Hence the real part of the self-energy is indeed the sum of (37) and (39); and the limit as p_0 and p approach zero is the same as (24).

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APPENDIX A: CALCULATION OF IMAGINARY PARTS

This appendix calculates the imaginary parts of the self-energy that are quoted in (8) and (11) for use in the dispersion relation.

$P^2<0$. As shown in [16], the imaginary part is

$$\mathrm{Im}\pi'_R(u,p)=\frac{\lambda^2}{4}\int\frac{d^3k}{(2\pi)^3}\frac{2\pi}{2\omega 2\tilde{\omega}}[\delta(u+\omega-\tilde{\omega})(n-\tilde{n})+\delta(u-\omega+\tilde{\omega})(\tilde{n}-n)]\ ,$$
$$\omega=(k^2+m^2)^{1/2}\ ,$$
$$\tilde{\omega}=[(\mathbf{k}+\mathbf{p})^2+m^2]^{1/2}\ ,$$

where n are the Bose-Einstein functions: $n=n(\omega)$, $\tilde{n}=n(\tilde{\omega})$. The first term is due to the absorption of energy u by the first particle with statistical weight $n(1+\tilde{n})$ minus the weight for the inverse process, $(1+n)\tilde{n}$, for a net statistical weight of $n-\tilde{n}$; the second is due to the absorption of energy u by the second particle and the inverse process with the net statistical weight $\tilde{n}(1+n)-(1+\tilde{n})n=\tilde{n}-n$. Changing integration variables from $\mathbf{k}$ to $-\mathbf{k}-\mathbf{p}$ makes the integral over the second δ function equal to the first. Next let θ be the angle between $\mathbf{k}$ and $\mathbf{p}$ and perform the angular integration by

$$\int_0^\pi \sin\theta\,d\theta\,\delta(u+\omega-\tilde{\omega})=\frac{\tilde{\omega}}{kp}\ .$$

For u and p small, the energy-conserving δ function requires $\cos\theta=\omega u/kp$. As in Sec. I, set $u=vp$ with $v\le 1$. Then $\cos\theta=v\omega/k$ has solutions for $1\ge\cos\theta\ge v$ corresponding to $k_{\min}\le k\le\infty$, where $k_{\min}=mv/(1-v^2)^{1/2}$. Thus

$$\mathrm{Im}\pi'_R(vp,p)=\frac{\lambda^2}{16\pi}\int_{k_{\min}}^\infty\frac{kdk}{\omega p}[n(\omega)-n(\omega+vp)]\ .$$

As $p\rightarrow 0$ the integrand becomes a total derivative with the result

$$\lim_{p\rightarrow 0}\mathrm{Im}\pi'_R(vp,p)=\frac{\lambda^2}{16\pi}vn(\omega_{\min})\ , \qquad (\mathrm{A}1)$$

where $\omega_{\min}=m/(1-v^2)^{1/2}$.

$P^2>4m^2$. The imaginary part is

$$\mathrm{Im}\pi_R(u,p)=\frac{\lambda^2}{4}\int\frac{d^3k}{(2\pi)^3}\frac{2\pi}{2\omega 2\tilde{\omega}}\delta(u-\omega-\tilde{\omega})(1+n+\tilde{n}) ,$$

where the statistical weight comes from $(1+n)(1+\tilde{n})-n\tilde{n}$. The $T=0$ contribution has statistical weight 1 and will be dropped; the $T\neq 0$ piece has weight $n+\tilde{n}$. The $T\neq 0$ part at $p=0$ is

$$\mathrm{Im}\pi'_R(u,0)=\frac{\lambda^2}{4}\int\frac{d^3k}{(2\pi)^3}\frac{\pi}{\omega^2}\delta(u-2\omega)n(\omega) .$$

Direct integration gives

$$\mathrm{Im}\pi'_R(u,0)=\frac{\lambda^2}{16\pi}\left[1-\frac{4m^2}{u^2}\right]^{1/2}n(u/2) . \tag{A2}$$

APPENDIX B: PROOF OF FEYNMAN PARAMETRIZATION

This appendix will prove (18), the relation for Feynman parametrizing the quantity

$$I=\frac{1}{(a+i\epsilon)(b-i\epsilon)} \tag{B1}$$

when a and b are real. Let $E=(a+i\epsilon)z_1+(-b+i\epsilon)z_2$. Then I can be written

$$I\equiv\int_0^\infty dz_1\int_0^\infty dz_2 \exp(iE) .$$

Change variables to $r=z_1-z_2$ and $s=(z_1+z_2)/2$. In terms of these variables,

$$E=(a-b+2i\epsilon)s+(a+b)r/2 , \tag{B2}$$

$$I=\int_0^\infty ds\int_{-2s}^{2s} dr \exp(iE) . \tag{B3}$$

Next define

$$D=a(1-x)+bx+i\epsilon(1-2x) ,$$

$$F_\eta\equiv\int_0^{1/2-\eta}\frac{dx}{D^2}+\int_{1/2+\eta}^{1}\frac{dx}{D^2} ,$$

where η is positive, real. In the limit of small η this becomes the principal value of the Feynman parameter integral, but for now the size of η is arbitrary. Rewrite F_η as

$$F_\eta=-\int_0^\infty r\,dr\int_0^{1/2-\eta}dx \exp(iDr) + \int_{-\infty}^0 r\,dr\int_{1/2+\eta}^{1}dx \exp(iDr) .$$

There is no problem with convergence at $r=\pm\infty$ because for the corresponding range of x, the imaginary part of D always has the correct sign. Now change variables from Feynman x to $s=r(-x+\frac{1}{2})$. This makes $Dr=E$ and

$$F_\eta=-\int_0^\infty dr\int_{\eta r}^{r/2}ds \exp(iE) - \int_{-\infty}^0 dr\int_{-\eta r}^{-r/2}ds \exp(iE) .$$

Interchange the order of integration to get

$$F_\eta\equiv-\int_0^\infty ds\int_{2s}^{s/\eta}dr \exp(iE) - \int_0^\infty ds\int_{-s/\eta}^{-2s}dr \exp(iE) . \tag{B4}$$

Clearly (B3) and (B4) are not equal. Let their difference be G_η:

$$I=F_\eta+G_\eta , \quad G_\eta\equiv\int_0^\infty ds\int_{-s/\eta}^{s/\eta}dr \exp(iE) . \tag{B5}$$

Up until now the value of η has been arbitrary. In the limit of small η

$$\lim_{\eta\to 0}F_\eta=\mathcal{P}\int_0^1\frac{dx}{D^2} , \tag{B6}$$

$$\lim_{\eta\to 0}G_\eta=\int_0^\infty ds\int_{-\infty}^{\infty}dr \exp(iE) = \frac{i}{a-b+2i\epsilon}2\pi\delta\left[\frac{a+b}{2}\right] . \tag{B7}$$

This proves the result stated in (18).

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