

## Gauge covariance and the fermion-photon vertex in three- and four-dimensional, massless quantum electrodynamics

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In the quenched approximation, the gauge covariance properties of three vertex *Ansätze* in the Schwinger-Dyson equation for the fermion self-energy are analyzed in three- and four-dimensional quantum electrodynamics. Based on the Cornwall-Jackiw-Tomboulis effective action, it is inferred that the spectral representation used for the vertex in the gauge technique cannot support dynamical chiral symmetry breaking. A criterion for establishing whether a given *Ansatz* can confer gauge covariance upon the Schwinger-Dyson equation is presented and the Curtis and Pennington *Ansatz* is shown to satisfy this constraint. We obtain an analytic solution of the Schwinger-Dyson equation for quenched, massless three-dimensional quantum electrodynamics for arbitrary values of the gauge parameter in the absence of dynamical chiral symmetry breaking.

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### I. INTRODUCTION

The Schwinger-Dyson equations (SDE's) provide a valuable nonperturbative tool for studying field theories. Phenomena such as confinement and dynamical chiral symmetry breaking, which cannot be explained by perturbative treatments, can be understood in terms of the behavior of particle propagators obtained by solving nonlinear integral equations. However, the full set of SDE's for any particular field theory contains an infinite tower of equations and is thus intractable. A common approach for dealing with gauge field theories is to approximate the fermion-gauge-boson vertex by a suitable *Ansatz* depending only on the dressed single particle propagators. The problem is then reduced to that of solving a finite set of coupled equations for the fermion and gauge-boson propagators.

Ideally, of course, one would solve the SDE for the vertex itself. However, this equation involves the kernel of the fermion-antifermion Bethe-Salpeter equation which cannot be expressed in a closed form; i.e., the skeleton expansion of this kernel involves infinitely many terms. Some approximation or truncation of the system must therefore be made at a very early stage. An effective way to do this is to make an *Ansatz* for the vertex satisfying certain criteria which the solution of the vertex equation must itself satisfy. At the present time this latter approach is the most efficacious manner in which to proceed since it allows for a study of the relative importance of particular vertex characteristics while avoiding the technical difficulties associated with solving the vertex equation directly. However, we expect that it will soon be necessary to study the vertex equation itself in order to make further progress.

A primary purpose of this paper is to compare the effectiveness of three commonly used vertex *Ansätze*, specifically with regard to their ability to ensure the gauge invariance of the theory. We begin with the requirement that any acceptable *Ansatz*  $\Gamma_\mu(p, q)$  must satisfy at least the following criteria: (a) it must satisfy the Ward-Takahashi (WT) identity; (b) it must be free of any kinematic singularities [i.e., expressing  $\Gamma_\mu(p, q)$  as a function of  $p$  and  $q$  and a functional of the fermion propagator  $S(p)$ , then  $\Gamma_\mu$  should have a unique limit as  $p^2 \rightarrow q^2$ ]; (c) it must reduce to the bare vertex in the free field limit (i.e., when dressed propagators are replaced by bare propagators); and (d) it must have the same transformation properties as the bare vertex  $\gamma_\mu$  under charge conjugation  $C$  and Lorentz transformations (such as  $P$  and  $T$ , for example).

Criterion (b) follows from Ref. [1] and criterion (c) is related to this since together they are necessary to ensure that the vertex *Ansatz* has the correct perturbative limit. The charge conjugation element of criterion (d) is essential since it constrains the properties of  $\Gamma_\mu(p, q)$  under  $p \leftrightarrow q$ .

One should also demand a further condition, namely, (e) that local gauge covariance should be respected.

In fact, a criticism of the SDE approach to solving gauge field theory has been the apparent violation of gauge symmetry directly at the level of the equation being addressed. Ensuring gauge covariance of the solutions of the SDE goes some way toward answering this criticism and allowing for a direct comparison of SDE results with those obtained from lattice gauge theory, for example.

Although condition (a) is a consequence of gauge invariance, it is only a statement about the longitudinal

part of the vertex, and says nothing about the transverse part. By itself it is insufficient to ensure condition (e) [2]. A well-defined set of transformation laws which describe the response of the propagators and vertex in quantum electrodynamics to an arbitrary gauge transformation are given in an early paper by Landau and Khalatnikov [3] (LK). These laws leave the SDE's and the WT identity form invariant and one can, in principle, ensure condition (e) by choosing an *Ansatz* for  $\Gamma$  which is covariant under the action of the LK transformations. Unfortunately, however, the transformation rule for the vertex is quite complicated, making this procedure difficult to implement. Here we will adopt a slightly different procedure. The LK transformation rule for the fermion propagator is relatively straightforward, and we are able to check *a posteriori* whether solutions for propagators obtained from a particular vertex *Ansatz* transform appropriately.

Herein we discuss three- and four-dimensional, Euclidean, quenched quantum electrodynamics (QED<sub>3</sub> and QED<sub>4</sub>, respectively) and when discussing both we choose to work with four-component spinors [4]. (In formulating the theory in Euclidean space we adopt the strategy of Ref. [5].) In describing the theory as “quenched” we mean that fermion loop contributions to the photon propagator are ignored; i.e., vacuum polarization corrections are neglected.

We remark that QED<sub>3</sub> has been much studied in recent years because of its similarities with quantum chromodynamics (*viz.* confinement and chiral symmetry breaking), because its dimensioned coupling provides a natural scale which makes it a useful tool for modeling theories relevant to unification and because it is not plagued by ultraviolet divergences. For our purposes, however, it is the fact that in both QED<sub>3</sub> and QED<sub>4</sub> the fermion SDE is solved by the combination of bare vertex and bare fermion propagator that makes these theories interesting. The LK transform of the bare fermion propagator from Landau to any other covariant gauge is readily found. Any *Ansatz* for the vertex which does not admit the transformed propagator as a solution for an arbitrary value of the gauge parameter can then be eliminated as a possible candidate and is unlikely to form a basis for a gauge covariant vertex in realistic models of non-Abelian theories.

We describe the vertex *Ansätze* we are considering in detail in Sec. II. The *Ansatz* of Ref. [6] is equivalent to that employed in recent studies of the SDE using the

gauge technique [7, 8]. (The “gauge technique” assumes that the elements of the SDE's, propagators, etc., have spectral representations in terms of which the SDE's are reformulated and then solved for directly.) We show in Sec. III, using the Cornwall-Jackiw-Tomboulis (CJT) effective action [9] (of which the fermion SDE can be interpreted as the Euler-Lagrange stationary point equation) that this vertex *Ansatz* cannot support dynamical chiral symmetry breaking simply because it leads to independent equations for the vector  $\sigma_V$  and scalar  $\sigma_S$  pieces of the fermion propagator,  $S(p) = -i\gamma \cdot p \sigma_V(p) + \sigma_S(p)$ , the equation for  $\sigma_S$  being homogeneous. This is true of any *Ansatz* that yields independent equations for  $\sigma_V$  and  $\sigma_S$  in the chiral limit. (This is exemplified in the QCD model of Refs. [10].)

In Sec. III we discuss the fermion SDE in QED<sub>3</sub> and QED<sub>4</sub> in some detail and give numerical solutions to the QED<sub>3</sub> fermion SDE for various vertex *Ansätze*. In these studies we concentrate mainly on the case of no dynamical mass generation (although the vacuum of massless QED<sub>3</sub> is generally believed to be chirally asymmetric [4, 11], as may be that of quenched QED<sub>4</sub> [12]) and demonstrate analytically that the vertex *Ansatz* proposed in Ref. [13] leads to a SDE which is solved by the LK transform of the bare vertex. The remaining two *Ansätze*, however, do not satisfy this test. The observation of LK covariance enables us to obtain an analytic solution to the quenched, massless SDE in QED<sub>3</sub> for arbitrary values of the gauge parameter in the absence of dynamical chiral symmetry breaking. We summarize our results and conclusions in Sec. IV. In an appendix we summarize the LK transformations for QED and give the LK transformed three-dimensional free massless fermion propagator for an arbitrary positive value of the covariant gauge parameter.

## II. FERMION-PHOTON VERTICES

The most general form for a fermion-photon vertex satisfying criteria (a) to (d) above has been given by Ball and Chiu [1] and, in Euclidean space, it can be written as

$$\Gamma_\mu(p, q) = \Gamma_\mu^{\text{BC}}(p, q) + \Gamma_\mu^T(p, q), \quad (1)$$

where

$$\Gamma_\mu^{\text{BC}}(p, q) = \frac{1}{2} [A(p) + A(q)] \gamma_\mu + \frac{(p+q)_\mu}{p^2 - q^2} \left\{ [A(p^2) - A(q^2)] \frac{\gamma \cdot p + \gamma \cdot q}{2} - i [B(p^2) - B(q^2)] \right\}, \quad (2)$$

and  $\Gamma_\mu^T$  is an otherwise unconstrained transverse piece satisfying

$$(p - q)_\mu \Gamma_\mu^T(p, q) = 0, \quad \Gamma_\mu^T(p, p) = 0. \quad (3)$$

$\Gamma_\mu^{\text{BC}}$  is given in terms of the dressed fermion propagator

$$S^{-1}(p) = i\gamma \cdot p A(p) + B(p), \quad (4)$$

where  $A$  and  $B$  are scalar functions of  $p^2 = p_\mu p_\mu$ . Our Euclidean space  $\gamma$  matrices satisfy  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ .

Chiral symmetry breaking in QED<sub>3</sub>, both in its quenched form [2] and in the presence of dynamical fermions [14], has been studied with some success by arbitrarily setting the transverse part (3) of the vertex equal to zero. The remaining part  $\Gamma_\mu^{\text{BC}}$  is the first of the three

*Ansätze* we will consider herein. When dynamical mass generation is also allowed for, it goes some way towards ensuring that the chiral condensate has only a weak dependence on the gauge parameter in QED<sub>3</sub> [2], and provides a value for the condensate in close agreement with that obtained from lattice simulations [11, 15]. However, our earlier numerical studies [16] have shown that the resultant fermion propagator does retain some dependence on the choice of gauge.

The second vertex we consider is that proposed by Haeri [6] which, in Euclidean space, can be written as

$$\Gamma_\mu^H(p, q) = i [\alpha_\mu S^{-1}(q) - S^{-1}(p) \alpha_\mu], \quad (5)$$

with  $\alpha_\mu = [\gamma \cdot p \gamma_\mu + \gamma_\mu \gamma \cdot q] / [p^2 - q^2]$ , or alternatively,

$$\Gamma_\mu^H(p, q) = \frac{p^2 A(p) - q^2 A(q)}{p^2 - q^2} \gamma_\mu + \frac{A(p) - A(q)}{p^2 - q^2} \gamma \cdot p \gamma_\mu \gamma \cdot q - i \frac{B(p) - B(q)}{p^2 - q^2} (\gamma \cdot p \gamma_\mu + \gamma_\mu \gamma \cdot q). \quad (6)$$

$\Gamma_\mu^H$  is easily seen to satisfy criteria (a) to (d) and must therefore be of the form Eq. (1).

It is interesting to note that the Haeri vertex is identical to the spectral representation of the vertex employed in the gauge technique [7, 8]:

$$S(p) \Gamma_\mu(p, q) S(q) = \int_{-\infty}^{\infty} d\omega \rho(\omega) \frac{1}{\not{p} - \omega} \gamma_\mu \frac{1}{\not{q} - \omega}, \quad (7)$$

where  $\rho(\omega)$  is the spectral density of the fermion propagator:

$$S(p) = \int_{-\infty}^{\infty} d\omega \frac{\rho(\omega)}{\not{p} - \omega}. \quad (8)$$

This result is true irrespective of whether the fermion acquires a mass and can be easily verified by direct substitution and comparison [after continuation of Eq. (5) to Minkowski space] [17].

Third we consider the *Ansatz* of Curtis and Pennington (CP). In order to ensure multiplicative renormalizability, they have proposed a vertex for which the transverse part  $\Gamma_\mu^{TCP}$  takes the form [13, 18]

$$\Gamma_\mu^{TCP}(p, q) = \frac{A(p) - A(q)}{2d(p, q)} \times [\gamma_\mu(p^2 - q^2) - (p + q)_\mu (\gamma \cdot p - \gamma \cdot q)], \quad (9)$$

with

$$d(p, q) = \frac{(p^2 - q^2)^2 + [M^2(p) + M^2(q)]^2}{p^2 + q^2}, \quad (10)$$

where  $M = B/A$ , yielding the *Ansatz*

$$\begin{aligned} \Gamma_\mu^{CP}(p, q) &= \Gamma_\mu^{BC} + \Gamma_\mu^{TCP} \\ &= \frac{p^2 A(p) - q^2 A(q)}{p^2 - q^2} \gamma_\mu \\ &\quad + \frac{(p + q)_\mu}{p^2 - q^2} \frac{A(p) - A(q)}{p^2 - q^2} (p^2 \gamma \cdot q - q^2 \gamma \cdot p) \\ &\quad + B\text{-dependent parts.} \end{aligned} \quad (11)$$

For QED<sub>4</sub>, the CP vertex gives a chirally symmetric fermion propagator which is exactly multiplicatively renormalizable at all momenta [19]. It has also been used in Landau gauge QED<sub>3</sub> in conjunction with a one-loop-corrected photon propagator [20], with the result that chiral symmetry is broken irrespective of the number of fermion flavors.

### III. THE QUENCHED SCHWINGER-DYSON EQUATION

We now turn our attention to a consideration of the quenched fermion SDE for QED<sub>3</sub> and QED<sub>4</sub>:

$$1 = (i\gamma \cdot p + m)S(p) + e^2 \int \frac{d^d q}{(2\pi)^d} D_{\mu\nu}(p - q) \gamma_\mu S(q) \Gamma_\nu(q, p) S(p). \quad (12)$$

By quenched we mean that virtual fermion loops are ignored in the gauge-boson propagator which corresponds to setting  $\Pi(k) = 0$  in Eq. (A5). Our aim is to study the gauge covariance properties of Eq. (12) with the *Ansätze* for the vertices described above. An *Ansatz* which leads to a fermion propagator which does not respond to a gauge transformation in the manner prescribed by the LK transformations, Eq. (A2), can reasonably be eliminated. As we will see, this provides an additional constraint on the transverse part of the vertex.

#### A. Haeri *Ansatz* and dynamical chiral symmetry breaking

We will first consider  $\Gamma_\mu^H$  of Eq. (6). An interesting observation is that, writing the propagator in the form

$$S(p) = -i\gamma \cdot p \sigma_V(p) + \sigma_S(p) \quad (13)$$

and defining the partially amputated vertex

$$\Lambda_\mu^H(p, q) = S(p) \Gamma_\mu^H(p, q) S(q), \quad (14)$$

Eq. (12) provides two decoupled equations, one for  $\sigma_V$  and one for  $\sigma_S$ , when  $m = 0$ , i.e., for massless fermions. This is obvious upon inspection since  $\Lambda_\mu^H$  involves  $\sigma_V$  multiplied only with odd numbers of  $\gamma$  matrices and  $\sigma_S$  multiplied only with even numbers. It is also worth noting that the equation for  $\sigma_S$  is always homogeneous and hence the solution is determined only up to an arbitrary multiplicative constant.

With Eq. (6) in Eq. (12) one always has the chiral-symmetry-preserving solution

$$S^W(p) = -i\gamma \cdot p \sigma_V^W(p) \quad (15)$$

for  $m = 0$  and, in addition, it is also probable that the equation admits a dynamical chiral-symmetry-breaking solution for  $m = 0$  which would have the form

$$S^{NG} = -i\gamma \cdot p \sigma_V^W(p) + \sigma_S^{NG}(p). \quad (16)$$

This was the case, for example, in the phenomenological QCD studies of Ref. [21]. We remark that in Eq. (15) and Eq. (16) the vector part of the propagator is necessarily the same. This is essential to the argument that follows

and is what sets this *Ansatz* apart from the others we consider.

The SDE is the stationary point equation for the CJT effective action [9] which, evaluated at this stationary point, is [22]

$$V[S] = \int \frac{d^d p}{(2\pi)^d} \{ \text{tr} \ln[1 - \Sigma(p)S(p)] + \frac{1}{2} \text{tr}[\Sigma(p)S(p)] \}. \quad (17)$$

One might measure the relative stability of these extremals by evaluating the difference  $V[S^{NG}] - V[S^W]$ . For an Abelian gauge theory with  $N_f$  flavors of fermion, one finds (for  $d = 3$  or  $4$  since we use 4 component spinors) that

$$V[S^{NG}] - V[S^W] = 2N_f \int \frac{d^d p}{(2\pi)^d} \ln \left[ 1 + \frac{1}{p^2} \frac{\sigma_S^2(p)}{\sigma_V^2(p)} \right] > 0, \quad (18)$$

since it is reasonable to assume that  $\sigma_S$  and  $\sigma_V$  are real for real Euclidean  $p^2$ . (Since the equation for  $\sigma_S$  is homogeneous, this difference can, in fact, be made arbitrarily large:  $\sigma_S \rightarrow \lambda \sigma_S$ .) Hence, based on the CJT effective action (which is the same as the auxiliary field effective action at the stationary point) one finds that  $\Gamma_\mu^H$  cannot support dynamical chiral symmetry breaking.

## B. Chirally symmetric solution and gauge covariance

For the remainder of this section we focus our attention on the chiral-symmetry-preserving solution of the massless SDE:  $S(p) = -i\gamma \cdot p \sigma_V(p)$ .

We first note that since

$$\int d\Omega_d \frac{1}{(p-q)^2} \left( (d-3)p \cdot q + 2 \frac{p \cdot (p-q)(p-q) \cdot q}{(p-q)^2} \right) \equiv 0 \quad (19)$$

then, in the Landau gauge, Eq. (12) admits the free propagator solution

$$S(p) = \frac{1}{i\gamma \cdot p} \quad (20)$$

for each of the vertices discussed herein because of criterion (c). We therefore immediately have the important result that if a given vertex *Ansatz* is to satisfy the gauge covariance criterion then, for arbitrary  $\xi$ , the associated SDE must have the LK transform of the free field propagator as its solution [Eq. (A2)].

In order to study this it is helpful to consider the massless SDE in configuration space:

$$\delta^d(x-y) = \gamma \cdot \partial^x S(x-y) + e^2 \int d^d z d^d x' d^d y' \gamma_\mu [D_{\mu\nu}^T(x-z) + \partial^z \partial^z \Delta(x-z)] S(x-x') \Gamma_\nu(z; x', y') S(y'-y), \quad (21)$$

where we have explicitly divided the gauge-boson propagator into a sum of a transverse, gauge-independent piece  $D_{\mu\nu}^T$  and longitudinal, gauge-dependent piece  $\Delta$ . Making use of the WT identity

$$\partial_\mu^z \Gamma_\mu(z; x', y') = S^{-1}(z-y') \delta^d(x'-z) - \delta^d(z-y') S^{-1}(x'-z) \quad (22)$$

and the identity  $\int_x \gamma \cdot \partial^x S(x, x') S^{-1}(x', z) = \gamma \cdot \partial^x \delta^d(x-z)$ , one obtains the massless SDE in the form

$$\delta^d(x-y) = \gamma \cdot \partial^x S(x-y) - e^2 \left( \int d^d z [\gamma \cdot \partial^z \Delta(x-z)] \delta^d(x-z) - [\gamma \cdot \partial^z \Delta(x-y)] \right) S(x-y) + e^2 \int d^d z d^d x' d^d y' \gamma_\mu D_{\mu\nu}^T(x-z) S(x-x') \Gamma_\nu(z; x', y') S(y'-y). \quad (23)$$

Now it is clear by inspection that if

$$\int d^d z d^d x' d^d y' \gamma_\mu D_{\mu\nu}^T(x-z) S(x-x') \Gamma_\nu(z; x', y') S(y'-y) = 0, \quad (24)$$

then Eq. (A2), with  $S(x; \xi = 0)$  given in Eq. (A8), is a solution of the massless SDE; i.e., it is a solution if the last term on the right-hand side of Eq. (12) is identically zero in Landau gauge.

Most studies of the SDE's are undertaken in momentum space and it is a simple matter to transcribe Eqs. (23) and (24). We see that the solution of the SDE is LK covariant if

$$\int \frac{d^d q}{(2\pi)^d} D_{\mu\nu}^T(p-q) \gamma_\mu S(q) \Gamma_\nu(q, p) = 0, \quad (25)$$

where  $D_{\mu\nu}^T(k) = (\delta_{\mu\nu} - k_\mu k_\nu / k^2) / k^2$  in the quenched theory, in which case the propagator satisfies

$$1 = i\gamma \cdot p S(p) + \xi e^2 \int \frac{d^d q}{(2\pi)^d} \frac{i\gamma \cdot (p-q)}{(p-q)^4} [S(p) - S(q)] \quad (26)$$

in the covariant gauge-fixing procedure.

It is now a simple matter to analyze the gauge covariance properties of our vertex *Ansätze*.

### 1. Ball-Chiu Ansatz

Using the BC vertex of Eq. (2) the QED<sub>3</sub> SDE takes the form

$$A(p) - 1 = \frac{-e^2}{4\pi^2 p^2} \int_0^\infty dq \frac{1}{A(q)} \left[ \xi \left( \frac{p^2 A(p) - q^2 A(q)}{p^2 - q^2} - \frac{p^2 A(p) + q^2 A(q)}{2pq} \ln \left| \frac{p+q}{p-q} \right| \right) - \left( 1 - \frac{p^2 + q^2}{2pq} \ln \left| \frac{p+q}{p-q} \right| \right) (p^2 + q^2) \frac{A(p) - A(q)}{p^2 - q^2} \right], \quad (27)$$

while in QED<sub>4</sub> it is

$$A(p) - 1 = \frac{e^2}{8\pi^2 p^2} \int_0^\infty dq \frac{q}{A(q)} \left[ \xi \left( A(p) \frac{p^2}{q^2} \theta(q-p) + A(q) \frac{q^2}{p^2} \theta(p-q) \right) - \frac{3}{4} \frac{A(p) - A(q)}{p^2 - q^2} (p^2 + q^2) \left( \frac{p^2}{q^2} \theta(q-p) + \frac{q^2}{p^2} \theta(p-q) \right) \right]. \quad (28)$$

It is clear that in neither of these equations is the right-hand side identically zero in the Landau gauge ( $\xi = 0$ ) and therefore this vertex cannot have the correct LK transformation properties. (This had already been established numerically in Ref. [2] for QED<sub>3</sub>.)

### 2. Haeri Ansatz

Using the Haeri *Ansatz* of Eq. (6) we find the following form of the SDE in QED<sub>3</sub>,

$$A(p) - 1 = \frac{-e^2}{4\pi^2 p^2} \int_0^\infty dq \frac{1}{A(q)} \left[ \xi \left( \frac{p^2 A(p) - q^2 A(q)}{p^2 - q^2} - \frac{p^2 A(p) + q^2 A(q)}{2pq} \ln \left| \frac{p+q}{p-q} \right| \right) + 2pq \ln \left| \frac{p+q}{p-q} \right| \frac{A(p) - A(q)}{p^2 - q^2} \right], \quad (29)$$

while in QED<sub>4</sub> it takes the form

$$A(p) - 1 = \frac{e^2}{8\pi^2 p^2} \int_0^\infty dq \frac{q}{A(q)} \left[ \xi \left( A(p) \frac{p^2}{q^2} \theta(q-p) + A(q) \frac{q^2}{p^2} \theta(p-q) \right) - 3 \frac{A(p) - A(q)}{p^2 - q^2} [p^2 \theta(q-p) + q^2 \theta(p-q)] \right]. \quad (30)$$

Again it is clear that the right-hand side of these equations is not zero in the Landau gauge and hence this vertex cannot have the LK transformation properties necessary to ensure gauge covariance.

### 3. Curtis-Pennington Ansatz

The CP vertex is a different matter. The QED<sub>3</sub> SDE is

$$A(p) - 1 = \frac{-e^2 \xi}{4\pi^2 p^2} \int_0^\infty dq \frac{1}{A(q)} \left( \frac{p^2 A(p) - q^2 A(q)}{p^2 - q^2} - \frac{p^2 A(p) + q^2 A(q)}{2pq} \ln \left| \frac{p+q}{p-q} \right| \right), \quad (31)$$

in which the right-hand side is clearly zero in the Landau gauge. Hence this vertex, or at least that part of it which contributes to the SDE, has the form necessary to ensure gauge covariance of the chirally symmetric fermion propagator.

It is possible to solve this equation analytically. The solution, for  $\xi > 0$ , is

$$\frac{1}{A(p)} = 1 - \frac{e^2 \xi}{8\pi p} \arctan \left( \frac{8\pi p}{e^2 \xi} \right), \quad (32)$$

as it should be since this corresponds to the LK transform of the massless free fermion propagator in QED<sub>3</sub>, as we show in the Appendix. [To obtain this result we first rewrote Eq. (31) in the form

$$1 - \frac{1}{A(p)} = \frac{-e^2 \xi}{8\pi^2 p} \left\{ \int_0^\infty dq \frac{q}{A(q)} \frac{d}{dq} \left( \frac{1}{q} \ln \left| \frac{p+q}{p-q} \right| \right) - \frac{1}{p^2 A(p)} \int_0^\infty dq q \frac{d}{dq} \left( q \ln \left| \frac{p+q}{p-q} \right| \right) \right\}. \quad (33)$$

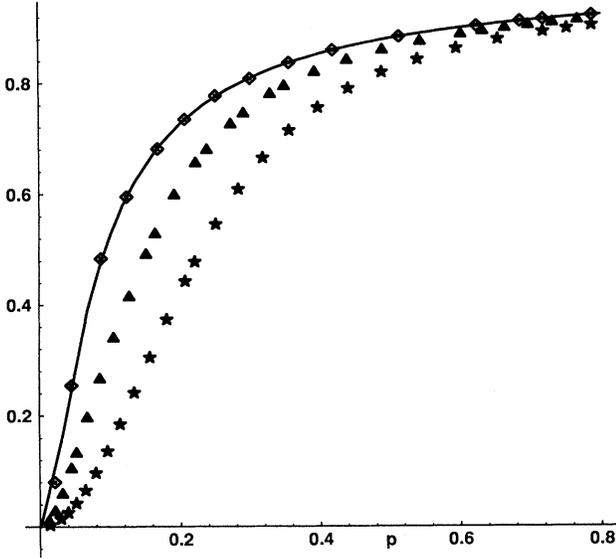


FIG. 1. This is a plot of  $1/A(p)$  as a function of  $p$  in QED<sub>3</sub> with  $\xi = 1$  (Feynman gauge) and  $e^2 = 1$ . The solid line is Eq. (32), the analytic solution expected from the LK transformation; the numerical results are  $\star$  = Haeri *Ansatz*,  $\triangle$  = Ball-Chiu *Ansatz*, and  $\diamond$  = Curtis-Pennington *Ansatz*. Clearly, the Curtis-Pennington *Ansatz* yields the correct solution.

Noting that the second integral in this equation is zero and using the identity

$$\frac{1}{\pi} \int_0^\infty dx \frac{x}{1+x^2} \ln \left| \frac{a+x}{a-x} \right| = \arctan a, \quad (34)$$

Eq. (32) follows.]

The SDE for QED<sub>4</sub> using the CP vertex is given in Ref. [19] and can be written formally as

$$A(p) - 1 = \frac{\xi \alpha_0}{4\pi p^2} \int_0^\infty dq^2 \left( \theta(p^2 - q^2) \frac{q^2}{p^2} + \theta(q^2 - p^2) \frac{p^2}{q^2} \frac{A(p)}{A(q)} \right) \quad (35)$$

with  $\alpha_0 = e^2/(4\pi)$ . In Ref. [19] this equation was solved by introducing an upper bound on the  $q^2$  integral. The actual form of the solution depends on the manner in which the divergent momentum integral is regularized. However, the fact that the right-hand side of Eq. (35) is proportional to  $\xi$  does not. This equation is, of course, Eq. (26) for  $d = 4$  and hence the CP vertex also satisfies criterion (e) in QED<sub>4</sub>.

To illustrate our discussion we present plots of numerical solutions for the function  $1/A(p)$  in QED<sub>3</sub> obtained from the BC and Haeri vertex equations, (27) and (29), at  $\xi = 1$ , Fig. 1, together with the CP vertex solution, Eq. (32), also at  $\xi = 1$ . It is clear that the BC and Haeri vertices do not give the correct LK transformed bare propagator as a solution and so fail to maintain the gauge covariance of the SDE.

To close this section we remark that Eq. (25) provides

us with a much needed additional constraint upon the vertex function which, while not a full implementation of criterion (e), nevertheless is a restriction on the form of the transverse part of the vertex: (e') In the absence of dynamical chiral symmetry breaking; i.e., for  $\sigma_S \equiv 0$ , the vertex must be such that Eq. (25) is satisfied, where  $D_{\mu\nu}^T(k)$  is the transverse part of the quenched photon propagator.

#### IV. SUMMARY

The Schwinger-Dyson equation (SDE) approach to the solution of a gauge field theory provides an intuitively attractive manner in which to address this problem and one which is less computationally intensive than lattice gauge theory, for example. A serious impediment to this application is the apparent lack of gauge covariance in all SDE studies to the present. In the fermion SDE this can be traced to inadequacies in the structure of the approximate and/or truncated fermion-gauge-boson vertex used in these studies. Addressing this violation of gauge symmetry in QCD is made difficult by the presence of ghost fields, however, progress can be made with Abelian theories. In addition to being interesting in their own right, the outcome of these studies can provide some understanding of necessary characteristics that should be incorporated in the construction of phenomenological, model SDE's for QCD. The results we have reported herein, which are summarized below, may be seen in this connection in addition to standing alone as a contribution to understanding gauge covariance in QED<sub>3</sub> and QED<sub>4</sub>.

We have considered three different *Ansätze* for the vertex in the quenched, massless QED<sub>3</sub> and QED<sub>4</sub> fermion SDE: (1) that due to Ball and Chiu [1]; (2) that due to Haeri [6]; and (3) that due to Curtis and Pennington [13].

In considering *Ansatz* (2) we observed that it is identical to that employed in the gauge technique and that, based on the CJT effective action, this *Ansatz* cannot support dynamical chiral symmetry breaking since a solution with no dynamically generated fermion mass; i.e., a solution of the form  $S(p) = -i\gamma \cdot p\sigma_V(p)$ , is always dynamically favored in this case. The feature of this vertex which entails this is the fact that it yields decoupled equations for  $\sigma_V$  and  $\sigma_S$  [Eq. (13)] when the fermion bare mass is zero. Whenever this is the case the CJT effective action will predict that the chirally symmetric solution is dynamically favored.

We obtained a necessary condition which must be satisfied by any vertex *Ansatz* if it is to confer gauge covariance on the quenched QED<sub>3</sub> and QED<sub>4</sub> SDEs. This condition is simple: the *Ansatz* must allow a free, massless propagator solution in Landau gauge; i.e.,  $S^{-1}(p) = i\gamma \cdot p$ , which provides a much needed constraint on the transverse piece of the vertex, Eq. (25). Only if this is the case can the solution of the SDE respond to a change in the gauge parameter as prescribed by the LK transformations, i.e., can the solution be gauge covariant. Only *Ansatz* (3) satisfies this constraint and it satisfies it both in QED<sub>3</sub> and QED<sub>4</sub>. In demonstrating this we obtained an analytic solution of the quenched, massless QED<sub>3</sub> SDE for arbitrary values of the gauge parameter in the absence of dynamical chiral symmetry breaking.

In going beyond the quenched approximation Eq. (19) is modified as follows:

$$\frac{1}{(p-q)^2} \rightarrow \frac{1}{(p-q)^2} \frac{1}{1 + \Pi(p-q)}, \quad (36)$$

where  $\Pi(p-q)$  is the photon polarization scalar. Subsequent to this modification it follows that the free, massless particle propagator is not a solution in Landau gauge when the bare vertex is used. In this case Eq. (23) combined with gauge covariance, Eq. (A2), does not require Eq. (24).

We may thus conclude that *Ansatz* (3) has another desirable feature, in addition to those discussed in Refs. [13, 18, 19]: that of ensuring gauge covariance of the quenched SDE, at least in the absence of dynamical mass generation. The other two *Ansätze* may be discarded since they manifestly cannot allow gauge covariance in QED<sub>3</sub> or QED<sub>4</sub>. Hence, one may make the inference that these two *Ansätze* are less likely to provide a good starting point in phenomenological SDE studies in QCD than *Ansatz* (3).

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#### APPENDIX: THE LANDAU AND KHALATNIKOV TRANSFORMATIONS

As pointed out in Sec. I, one would like to restrict the vertex further by imposing condition (e); namely, that the form of the vertex *Ansatz*, stated in terms of the dressed propagators, should be covariant under local gauge transformations. The gauge transformation laws relating the propagators and vertex of QED to their Landau gauge counterparts were first given by Landau and Khalatnikov [3]. These rules are most easily specified in coordinate space and we give below the corresponding Euclidean space transformation laws.

In an arbitrary gauge, the photon propagator is modified from its transverse, Landau gauge form  $D_{\mu\nu}(x; 0)$  by the addition of a longitudinal piece parametrized by an arbitrary function  $\Delta$ :

$$D_{\mu\nu}(x; \Delta) = D_{\mu\nu}(x; 0) + \partial_\mu \partial_\nu \Delta(x). \quad (A1)$$

The corresponding rule for the fermion propagator is

$$S(x; \Delta) = S(x; 0) e^{e^2[\Delta(0) - \Delta(x)]}, \quad (A2)$$

where  $e$  in the exponent is the gauge coupling constant. The rule for the fermion-photon vertex is

$$\begin{aligned} B_\mu(x, y, z; \Delta) &= B_\mu(x, y, z; 0) e^{e^2[\Delta(0) - \Delta(x-y)]} \\ &+ S(x-y; 0) e^{e^2[\Delta(0) - \Delta(x-y)]} \\ &\times \frac{\partial}{\partial z_\mu} [\Delta(x-z) - \Delta(z-y)], \quad (A3) \end{aligned}$$

where  $B_\mu$  is the nonamputated vertex defined in momentum space in terms of the amputated vertex  $\Gamma_\mu$  by

$$B_\mu(p, q) = S(p) \Gamma_\nu(p, q) S(q) D_{\mu\nu}(p-q). \quad (A4)$$

One can check directly that these transformations leave the WT identity and SDE form invariant [16].

In the usual covariant gauge-fixing procedure the photon propagator takes the form

$$D_{\mu\nu}(k; \xi) = \frac{1}{k^2[1 + \Pi(k^2)]} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \xi \frac{k_\mu k_\nu}{k^4}, \quad (A5)$$

which is obtained by taking  $\Delta$  in Eq. (A1) to be

$$\Delta(x) = -\xi \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik \cdot x}}{k^4}. \quad (A6)$$

Within this set of gauges, one finds that in QED<sub>3</sub> the transformation rule for the fermion propagator, Eq. (A2), becomes [23]

$$S(x; \xi) = S(x; 0) e^{-e^2 \xi |x|/8\pi}. \quad (A7)$$

The free massless propagator is  $S^{-1}(p; 0) = i\gamma \cdot p$  which corresponds to the following function in configuration space

$$S(x; 0) = \frac{\gamma \cdot x}{4\pi|x|^3}. \quad (A8)$$

Applying Eq. (A2) one obtains the LK transformed function in an arbitrary covariant gauge:

$$S(x; \xi) = \frac{\gamma \cdot x}{4\pi|x|^3} e^{-e^2 \xi |x|/8\pi}. \quad (A9)$$

For  $\xi > 0$  one may evaluate the Fourier amplitude directly to obtain

$$S(p; \xi) = \frac{-i\gamma \cdot p}{p^2} \left[ 1 - \frac{e^2 \xi}{8\pi p} \arctan \left( \frac{8\pi p}{e^2 \xi} \right) \right]. \quad (A10)$$

For completeness we give a formula for the LK transform of the bare vertex  $\gamma_\mu$  from Landau gauge to an arbitrary covariant gauge. Using Eqs. (A1), (A2), and (A3) one obtains the transformation rule for the partially amputated vertex

$$\Lambda_\mu(p, q) = S(p) \Gamma_\mu(p, q) S(q), \quad (A11)$$

which is simply

$$\Lambda_\mu(x, y, z; \Delta) = \Lambda_\mu(x, y, z; 0) e^{e^2[\Delta(0) - \Delta(x-y)]}. \quad (A12)$$

If the vertex is equal to the bare vertex in the Landau gauge,

$$\Lambda_\mu(p, q; 0) = -\frac{\gamma \cdot p \gamma_\mu \gamma \cdot q}{p^2 q^2}, \quad (A13)$$

one finds that, for arbitrary  $\xi$ ,

$$\Lambda_\mu(p, q; \xi) = \frac{-1}{16\pi^2} \gamma_\alpha \gamma_\mu \gamma_\beta \frac{\partial^2}{\partial p_\alpha \partial q_\beta} \int d^3 x d^3 y e^{i(p \cdot x - q \cdot y)} e^{-e^2 \xi |x-y|/8\pi} \frac{1}{|x|^3 |y|^3}. \quad (A14)$$

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