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Finite-temperature effective actions for gauge fields

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We consider the problem of computing the kinetic terms in the one-loop effective action. General results are presented for an expansion of the heat kernel of a second-order operator in powers of covariant derivatives. This is used to find the kinetic terms in the high temperature limit of the effective action up to the square of the gauge field strength.

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I. INTRODUCTION

It is currently conceived that the material in the early Universe underwent a series of phase transitions associated with the breaking of gauge symmetries [1, 2]. The physical properties of the various fields are described in local thermodynamic equilibrium by an effective action Γ, related to the free energy F by $\Gamma = \beta F$, where β is the inverse temperature [3, 4]. In some important circumstances, inhomogeneous fluctuations in the Higgs or gauge fields can play an important role, and a knowledge of the kinetic terms in the effective action is useful. One such situation which interests us at present is when the baryon number changes because of topological fluctuations in the gauge fields. Another situation is at the phase transition itself when the nucleation of phases can take place.

Various methods have been employed to calculate kinetic terms in the effective action. Since the coefficients of these terms are often not analytic in the coupling constants, the effect of these techniques is to resum infinite series of Feynman graphs. Some methods are based upon calculating the Green's functions [5–8], others on operator expansions [9]. The type of operator considered up until now has been rather restricted, as both of these methods become complicated in more general cases.

The method described here is based upon the heat kernel of an operator. There is a large literature on proper time expansions of the heat kernel for second-order operators of the form

$$\triangle = -\mathcal{D}^2 + X, \tag{1.1}$$

where $\mathcal{D}$ is a covariant derivative and X a matrix [10]. With a little ingenuity, it is possible in simple cases to regard this as an expansion in covariant derivatives and field strengths [11–13]. However, this runs into difficulties when the matrix X is not proportional to the unit matrix. We have therefore derived an iterative scheme which leads to the first terms of the heat kernel expansion of (1.1) in powers of covariant derivatives for the general operator.

The heat kernel expansions are used in this paper to calculate the kinetic terms in the effective action with one chiral fermion loop or one boson loop. Some of these terms have been calculated before in discussions of the effects of sphalerons on the baryon asymmetry of the Universe [14–16]. In particular, terms of the form $\mu N_{\rm CS}$, where μ is the baryon chemical potential and $N_{\rm CS}$ the Chern-Simons number of the gauge fields, have been obtained by evaluating Feynman graphs [17–21]. In our results, other one-loop terms in the effective action are included and we allow nonzero Higgs fields. We give the field strength squared terms which contribute to the nucleation rate of sphalerons. [This is proportional to $\exp(-\Delta\Gamma)$, where $\Delta\Gamma$ is the increase in effective action caused by a sphaleron.] We also calculate the leading terms involving the gradient of the Higgs fields.

Some of the terms in the one-loop effective action are infrared divergent and have to be modified by including higher-loop graphs. For a range of temperatures, the relevant graphs are ring diagrams. These diagrams introduce a temperature-dependent correction to the masses which we have included.

After summing ring diagrams, we find that the thermal corrections to the kinetic terms are generally smaller than the original terms. The most interesting terms are therefore those which are not represented in the original action. These may be topological, as with the Chern-Simons terms, nonanalytic or dependent on gradients of the thermodynamic variables. This last case can arise, for example, when the particle density is not homogeneous. Although we do not pursue this particular possibility further here, the relevant terms in the effective action have been calculated.

II. HEAT KERNEL EXPANSIONS

The heat kernel has been used traditionally in quantum field theory to study the effects of background fields. We will use the operator (1.1) on a manifold $\mathcal{M}$, with eigenvalues λ_n and eigenvectors u_n having the boundary conditions appropriate to the problem. The heat kernel is then defined by

$$K(x,x',t)=\sum_n u_n^\dagger(x)u_n(x')e^{-\lambda_n t}. \tag{2.1}$$

Its usefulness to us lies in the definition

$$\ln\det\triangle=-\int_{\mathcal{M}} d\mu(x)\,\zeta'(x,0), \tag{2.2}$$

the generalized ζ function being defined by

$$\zeta(x,s)=\frac{1}{\Gamma(s)}\mu_R^{2s}\int_0^\infty dt\,t^{s-1}\mathrm{tr}\,[K(x,x,t)] \tag{2.3}$$

for $s>d/2$, where d is the dimension of $\mathcal{M}$. This produces a regular function at $s=0$ after analytic continuation in s. It also leads to a dependence on the renormalization scale μ_R.

The covariant derivative can be split into space and gauge potential parts, $\mathcal{D}=\nabla-Y$. Under a gauge transformation U,

$${}^gY=U\,Y\,U^{-1}+\nabla U\,U^{-1},\qquad {}^gX=U\,X\,U^{-1}. \tag{2.4}$$

It follows directly from the definition of the heat kernel that the trace at coincident points is invariant under the gauge group. This trace has a famous asymptotic expansion in powers of t.

Here, we will define a slightly different expansion in powers of covariant derivatives:

$$\mathrm{tr}\,[K(x,x,t)]=(4\pi t)^{-d/2}\sum_i A_i(x,t) \tag{2.5}$$

where A_i contains i covariant derivatives of X.

The terms in the expansion can be derived from a momentum space expansion of the equation

$$\triangle K=-\dot K. \tag{2.6}$$

Define the transform

$$K(k,x',t)=\int d\mu(x)e^{-ik(x-x')}K(x,x',t) \tag{2.7}$$

and then

$$\int d\mu(x)\left\{k^2+Z(k,x)\right\}K(x,x',t)e^{-ik(x-x')}=-\dot K(k,x',t) \tag{2.8}$$

where

$$Z=X+2ik_\mu Y^\mu-Y^\mu Y_\mu-Y^\mu{}_{,\mu}. \tag{2.9}$$

The function $Z(k,x)$ is now replaced by an expansion in powers of $x-x'$ about $Z(k,x')$. After replacement of $x-x'$ with $\delta=i\partial/\partial k$, we get

$$\left\{k^2+\sum_{r=0}^\infty \frac{1}{r!}Z_{,\mu_1\cdots\mu_r}\delta^{\mu_1}\cdots\delta^{\mu_r}\right\}K=-\dot K. \tag{2.10}$$

The derivative expansions for K can be simplified by choosing a gauge in which $Y^\mu=0$ at $x=x'$ and $Y^\mu{}_{,\mu}=0$. Suppose that the expansions in ordinary derivatives are written

$$K=K_0\sum_n a_n,\qquad Z=\sum_n Z_n \tag{2.11}$$

with $a_0=1$, then, from the leading order,

$$K_0=e^{-(k^2+X_0)t} \tag{2.12}$$

and from order n,

$$\dot a_n=-K_0{}^{-1}\sum_{0<r+s\le n}\frac{1}{r!}Z_{s,\mu_1\ldots\mu_r}\delta^{\mu_1}\cdots\delta^{\mu_r}K_0a_{n-r-s}. \tag{2.13}$$

We can expand this expression further by introducing the eigenvalues of X_0, calling them m_i^2. Let T_i be the projection matrices onto the corresponding eigenspaces, then

$$K_0=\sum_i T_ie^{-(k^2+m_i^2)t}. \tag{2.14}$$

This allows us to write

$$\dot a_n=-\sum_{\substack{0<r+s\le n\\ i,j}}\frac{1}{r!}T_iZ_{s,\mu_1\cdots\mu_r}T_je^{-(m_j^2-m_i^2)t}\hat\delta^{\mu_1}\cdots\hat\delta^{\mu_r}a_{n-r-s} \tag{2.15}$$

where $\hat\delta=\delta-2ikt$.

Once the a_n have been obtained, we can recover expansion coefficients for the heat kernel, as in Eq. (2.5), by

$$A_n^{\mathrm{nc}}(x,t)=(4\pi t)^{d/2}\int d\mu(k)\mathrm{tr}[K_0(k,t)a_n(k,x,t)]. \tag{2.16}$$

The nc reminds us that we have chosen a gauge and have to use the gauge invariance to recover the expansion in covariant derivatives.

The first iteration of Eq. (2.15) gives

$$\dot a_1=\sum_{i,j}\left(T_iZ_{0,\mu}T_j2ik^\mu t-T_iZ_1T_j\right)e^{-(m_j^2-m_i^2)t}. \tag{2.17}$$

Integrating from $t=0$ leaves

$$a_1=\sum_{i,j}\left[T_iZ_{0,\mu}T_j2ik^\mu f_{ij}(t)-T_iZ_1T_jg_{ij}(t)\right], \tag{2.18}$$

where

$$f_{ij}(t)=m^{-2}(t-m^{-2}+m^{-2}e^{-m^2t})e^{m^2t}, \tag{2.19}$$

$$g_{ij}(t)=m^{-2}(1-e^{-m^2t})e^{m^2t}, \tag{2.20}$$

and $m^2 = m_i^2 - m_j^2$.

For the expansion coefficient A_1, we use

$$\mathrm{tr}(K_0 a_1) = \sum_i \left[\mathrm{tr}(T_i Z_{0,\mu} T_j) i k^\mu t^2 - \mathrm{tr}(T_i Z_1 T_i) t\right] e^{-(k^2+m_i^2)t}. \tag{2.21}$$

Only the last term survives the integral over k in (2.16):

$$A_1^{\mathrm{nc}} = -\sum_i \mathrm{tr}(T_i X_1 T_i)\, t e^{-m_i^2 t}. \tag{2.22}$$

Proceeding to the next order leads to

$$\begin{aligned}\mathrm{tr}(K_0 a_2) = &\sum_i \mathrm{tr}(T_i Z_{0,\mu\nu} T_i) K_i^{\mu\nu}(t) \\ &+ \sum_i \mathrm{tr}(T_i Z_2 T_i)(-t) e^{-(k^2+m_i^2)t} \\ &+ \sum_{i,j} \mathrm{tr}(T_i Z_{0,\mu} T_j Z_{0,\nu} T_i) F_{ij}^{\mu\nu}(t) \\ &+ \sum_{i,j} \mathrm{tr}(T_i Y_{0,\mu}^{\alpha} T_j Y_{0,\nu}^{\beta} T_i) k_\alpha k^\nu \delta_\beta^\mu G_{ij}(t) \\ &+ \sum_{i,j} \mathrm{tr}(T_i Z_1 T_j Z_1 T_i) H_{ij}(t)\end{aligned} \tag{2.23}$$

where

$$K_i^{\mu\nu}(t) = (-4k^\mu k^\nu t + 3\delta^{\mu\nu})\rho_i e^{-k^2 t}, \tag{2.24}$$

$$F_{ij}^{\mu\nu}(t) = [2t\kappa_{ij} k^\mu k^\nu + (\eta_{ij} - \kappa_{ij})\delta^{\mu\nu}] e^{-k^2 t}, \tag{2.25}$$

$$G_{ij}(t) = 4(\kappa_{ij} - \eta_{ij}) e^{-k^2 t}, \tag{2.26}$$

$$H_{ij}(t) = \chi_{ij}(t) e^{-k^2 t}. \tag{2.27}$$

We have defined

$$\rho_i(t) = -\tfrac{1}{6} t^2 e^{-m_i^2 t} \tag{2.28}$$

and

$$\begin{aligned}\kappa_{ij}(t) = &\, m^{-6}(1 + \tfrac{1}{3} m^4 t^2)(e^{-m_i^2 t} - e^{-m_j^2 t}) \\ &+ \tfrac{1}{2} m^{-4} t (e^{-m_i^2 t} + e^{-m_j^2 t}),\end{aligned} \tag{2.29}$$

$$\begin{aligned}\eta_{ij}(t) = &-m^{-6}(1 + \tfrac{1}{6} m^4 t^2)(e^{-m_i^2 t} - e^{-m_j^2 t}) \\ &- \tfrac{1}{2} m^{-4} t (e^{-m_i^2 t} + e^{-m_j^2 t}),\end{aligned} \tag{2.30}$$

$$\chi_{ij}(t) = -\tfrac{1}{2} m^{-2} t (e^{-m_i^2 t} - e^{-m_j^2 t}), \tag{2.31}$$

In the limiting cases where $i = j$,

$$\eta_{ii} = \tfrac{1}{12} t^3 e^{-m_i^2 t} \quad \text{and} \quad \chi_{ii} = \tfrac{1}{2} t^2 e^{-m_i^2 t}. \tag{2.32}$$

Integration over k now gives

$$\begin{aligned}A_2^{\mathrm{nc}} = &\sum_i \mathrm{tr}(T_i X_{0,\mu}{}^{\mu} T_i) \rho_i(t) \\ &+ \sum_i \mathrm{tr}(T_i Y_{0\mu,\nu} Y^{0\mu,\nu} T_i)(\tfrac{1}{3} t^2 e^{-m_i^2 t}) \\ &+ \sum_i \mathrm{tr}(T_i X_2 T_i)(-t e^{-m_i^2 t}) \\ &+ \sum_{i,j} \mathrm{tr}(T_i X_{0,\mu} T_j X_0{}^{\nu} T_i) \eta_{ij}(t) \\ &+ \sum_{i,j} \mathrm{tr}(T_i Y_0{}^{\mu,\nu} T_j Y_{0\alpha,\beta} T_i) \delta_\mu{}^{(\alpha} \delta_\nu{}^{\beta)} [-4\eta_{ij}(t)/t] \\ &+ \sum_{i,j} \mathrm{tr}(T_i Z_1 T_j Z_1 T_i) \chi_{ij}(t).\end{aligned} \tag{2.33}$$

In order to construct an expansion in covariant derivatives we introduce covariant tensors which reduce to the above terms in the chosen gauge. If we define the field strength by

$$\mathcal{F}_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu], \tag{2.34}$$

then $Y_{\mu,\nu} Y^{\mu,\nu}$ is replaced by $\frac{1}{2}\mathcal{D}^4$ and $Y_{\mu,\nu} Y^{\nu,\mu}$ by $\frac{1}{2}\mathcal{F}^2 + \frac{1}{2}\mathcal{D}^4$. The field strength counts as two powers of derivatives in the covariant expansion as given below.

The covariant expansion coefficients are given by

$$A_0 = \sum_i \mathrm{tr}(T_i) e^{-m_i^2 t}, \tag{2.35}$$

$$\begin{aligned}A_2 = &\sum_i \mathrm{tr}(T_i \mathcal{D}_\mu \mathcal{D}^\mu X_0 T_i) \rho_i(t) \\ &+ \sum_{i,j} \mathrm{tr}(T_i \mathcal{D}_\mu X_0 T_j \mathcal{D}^\mu X_0 T_i) \eta_{ij}(t) \\ &+ \sum_{i,j} \mathrm{tr}(T_i X_1 T_j X_1 T_i) \chi_{ij}(t) \\ &+ \sum_i \mathrm{tr}(T_i X_2 T_i)[6\rho_i(t)/t],\end{aligned} \tag{2.36}$$

$$A_4 = \sum_{i,j} \mathrm{tr}(T_i \mathcal{F}_{\mu\nu} T_j \mathcal{F}^{\mu\nu} T_i)[-\eta_{ij}(t)/t]\ . \tag{2.37}$$

Only the part of A_4 which remains when $\mathcal{D} X_0 = 0$ has been included. We will use the same simplification of this term in the remainder of this paper.

In the case that all of the eigenvalues of X_0 are identical, then the $i = j$ limits (2.32) can be used, and the results reduce to the usual ones [11, 10].

III. FINITE-TEMPERATURE RESULTS

Consider a system in local thermodynamic equilibrium where the ensemble averages of the fields vary over space. We define ensemble averages by path integrals on the Riemannian manifold $R^3 \times S^1$, and chemical potentials μ are replaced by potentials $A_0 = i\mu$ coupled to the corresponding charges [22–24]. The momentum space measure becomes

$$\int d\mu(k) = \sum_{n=-\infty}^{\infty} \frac{1}{\beta} \int \frac{d^3 k}{(2\pi)^3} \tag{3.1}$$

with $k^0 = 2\pi n/\beta$ for periodic fields (bosons) and $k^0 = 2\pi(n + \frac{1}{2})/\beta$ for antiperiodic fields (fermions).

We will give results for $\zeta_2(x, 0)'$ and $\zeta_4(x, 0)'$, the parts of ζ' which are second order in derivatives and second order in gauge field strengths, respectively. From the definition (2.3), we have

$$\zeta_n(x,s) = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}\int d\mu(k)\mathrm{tr}(K_0 a_n). \qquad (3.2)$$

Consider first the case when the chemical potential vanishes, denoting the ζ function for this case by $\zeta^{(0)}(x,s)$. The leading order term in its derivative expansion,

$$\zeta_0^{(0)}(x,s) = \sum_i \mathrm{tr}(T_i)\zeta_i(x,s), \qquad (3.3)$$

is the term from which the effective potential is derived [4, 3]. For the next term, Eqs. (3.2), (2.36), and (2.37) give

$$\begin{aligned}\zeta_2^{(0)}(x,s) &= \sum_i \mathrm{tr}(T_i\mathcal{D}_\mu\mathcal{D}^\mu X_0 T_i)\rho_i(s)\\ &\quad+\sum_{i,j}\mathrm{tr}(T_i\mathcal{D}_\mu X_0 T_j\mathcal{D}^\mu X_0 T_i)\eta_{ij}(s)\\ &\quad+\sum_{i,j}\mathrm{tr}(T_i X_1 T_j X_1 T_i)\chi_{ij}(s),\\ \zeta_4^{(0)}(x,s) &= \sum_{i,j}\mathrm{tr}(T_i\mathcal{F}_{\mu\nu}T_j\mathcal{F}^{\mu\nu}T_i)\xi_{ij}(s), \end{aligned} \qquad (3.4)$$

where

$$\eta_{ij}(s) = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}\int d\mu(k)\eta_{ij}e^{-k^2 t}, \qquad (3.5)$$

similarly for χ_{ij} and ξ_{ij} (defined as above, but with $\eta_{ij}\to -t^{-1}\eta_{ij}$). All of the Lorentz indices are spatial.

This result is valid for all temperatures, but if we are satisfied with a high temperature expansion the results of Appendix A can be used. These give

$$\rho_i(0)' \sim -\frac{T}{48\pi}m_i^{-1}, \qquad (3.6)$$

$$\chi_{ij}(0)' \sim \frac{T}{8\pi}(m_i+m_j)^{-1}, \qquad (3.7)$$

$$\eta_{ij}(0)' \sim \frac{T}{192\pi}\frac{4(m_i^2+m_j^2)}{m_i m_j(m_i+m_j)^3}, \qquad (3.8)$$

$$\xi_{ij}(0)' \sim -\frac{T}{96\pi}\frac{4(3m_i^2+4m_i m_j+3m_j^2)}{5(m_i+m_j)^3} \qquad (3.9)$$

for periodic fields. In the antiperiodic case,

$$\rho_i(0)' \sim -\frac{1}{48\pi^2}\ln(\mu_R/T), \qquad (3.10)$$

$$\chi_{ij}(0)' \sim \frac{1}{16\pi^2}\ln(\mu_R/T), \qquad (3.11)$$

$$\xi_{ij}(0)' \sim -\frac{1}{96\pi^2}\ln(\mu_R/T), \qquad (3.12)$$

$$\eta_{ij}(0)' \sim \frac{7}{768\pi^4 T^2}\zeta_R(3). \qquad (3.13)$$

If the chemical potential is nonzero, then the path integral has to be modified to include a timelike gauge-potential component $Y^0 = \mu Y$, for a fixed matrix Y. The previous results have to be modified by terms dependent on μ:

$$\zeta(x,s) = \sum_n \zeta^{(n)}(x,s)\mu^n. \qquad (3.14)$$

As before, a lower index is added to denote the number of spatial derivatives.

It is necessary to extend the calculations in Sec. II to include this background gauge field. This is achieved by replacing the matrix Z (2.9) with

$$Z = Z^{(0)} - 2ik^0\mu Y - \mu^2 Y^2. \qquad (3.15)$$

The projection matrices T_i are now defined by

$$Z T_i = \lambda_i T_i, \qquad (3.16)$$

where the leading order contribution to λ is $\lambda^{(0)} = m_i^2$. We will allow for the possibility that the Y matrices split the degeneracy of the eigenvalues of X_0, and therefore some of the masses can be equal, but $T_i Y T_j = 0$ if $m_i = m_j$ and $i \neq j$.

Standard perturbation theory can now be used on the (degenerate) eigenvalue problem (3.16). To first order we have

$$\lambda_i^{(1)}T_i^{(0)} = T_i^{(0)}Z^{(1)}T_i^{(0)}, \qquad (3.17)$$

and also

$$T_i^{(1)} = \sum_{\substack{m_i\neq m_j\\ j}}\frac{T_j^{(0)}Z^{(1)}T_i^{(0)}}{m_i^2-m_j^2}. \qquad (3.18)$$

The next order gives

$$\lambda_i^{(2)}T_i^{(0)} = T_i^{(0)}Z^{(2)}T_i^{(0)} + \sum_{\substack{m_i\neq m_j\\ j}}\frac{T_i^{(0)}Z^{(1)}T_j^{(0)}Z^{(1)}T_i^{(0)}}{m_i^2-m_j^2}, \qquad (3.19)$$

and so on.

These chemical potential expansions can be used in the derivative expansion of the heat kernel. To leading order in derivatives,

$$K_0 = \sum_i T_i e^{-(k^2+\lambda_i)t}. \qquad (3.20)$$

When expanded in powers of μ, the quadratic term in the ζ function becomes (supressing the 0 in $T_i^{(0)}$)

$$\zeta_0^{(2)}(x,s) = \sum_{i,j}\mathrm{tr}(T_i Y T_j Y T_i)\gamma_{ij}(s) \qquad (3.21)$$

where

$$\begin{aligned}\gamma_{ij}(s) &= \tfrac{1}{2}[\zeta_i(1,s)+\zeta_j(1,s)]\\ &\quad+\frac{\partial}{\partial\beta}\beta m^{-2}[\zeta_i(0,s)-\zeta_j(0,s)],\end{aligned} \qquad (3.22)$$

and $m^2 = m_i^2 - m_j^2$. $\zeta_i(p,s)$ is defined in Appendix A. For the degenerate case,

$$\gamma_{ii}(s) = -\beta\frac{\partial}{\partial\beta}\zeta_i(1,s). \qquad (3.23)$$

In the high temperature limit,

$$\gamma_{ij}(0)' \sim \begin{cases} \frac{1}{6}T^2 - \frac{1}{8\pi}(m_i+m_j)T & \text{bosons,} \\ -\frac{1}{12}T^2 & \text{fermions.} \end{cases} \tag{3.24}$$

The term $\zeta_0^{(2)}$ contributes to the effective potential, and has a bearing on phase transitions in systems with conserved particle numbers. It is possible to describe Bose-Einstein condensation in this way, and our results agree with the literature on electrically charged bosons [24] (in which case $Y = \sigma_2$, the Pauli matrix).

It is possible to proceed further with the derivative expansion. The next gauge invariant term involves two derivatives, but topological terms in three dimensions which are first order in the field strength may also be of interest. Terms such as this do not arise explicitly in the derivative expansion, but from derivative terms in X (as, for an example, with the chiral fermions in the next section). We let $Z^{(1)} = X^{(1)} - 2ik^0 Y$, and substitute into Eqs. (2.21) and (3.2). Thus, for the implicit terms,

$$\zeta_2^{(1)}(x,s) = -\sum_i \operatorname{tr}(T_i X_2^{(1)} T_i)\zeta_i(1,s) + \sum_{i,j} \operatorname{tr}(T_i X_2 T_j X_0^{(1)} T_i)\chi_{ij}(s) \tag{3.25}$$

[with the (0) superscripts on T_i suppressed].

At the next order in derivatives we make the same substitution into Eq. (2.23):

$$\zeta_3^{(1)}(x,s) = \sum_{i,j,k} \operatorname{tr}(T_i X_1 T_j X_2 T_k X_0^{(1)} T_i)\chi_{ijk}(s) \tag{3.26}$$

where

$$\chi_{ijk}(0)' \sim \begin{cases} \frac{1}{8\pi^2}(m_i^2 - m_k^2)^{-1}\log(\mu_R/T), & m_i \neq m_k, \\ -\frac{7}{128\pi^4}\zeta_R(3)T^{-2}, & m_i = m_k \end{cases}, \tag{3.27}$$

for antiperiodic fields in the high temperature limit.

Finally, we give results for two derivatives and two powers of the chemical potential, but with the fields constant over space. These terms are relevant for situations where the number density of particles is inhomogenous, but local thermodynamic equilibrium still holds. They are also needed to calculate the rate of quantum tuneling between phases of different density. From Eqs. (2.23) and (3.2),

$$\zeta_2^{(2)}(x,s)\mu^2 = \sum_{i,j} \operatorname{tr}(T_i Y T_j Y T_i)[\theta_{ij}\nabla^2\mu^2 + \upsilon_{ij}(\nabla\mu)^2], \tag{3.28}$$

where

$$\theta_{ij}(s) = \tfrac{1}{12}[\zeta_i(2,s) + \zeta_j(2,s)] - \tfrac{1}{6}\frac{\partial}{\partial\beta}\beta\chi_{ij}(s), \tag{3.29}$$

$$\upsilon_{ij}(s) = 2\frac{\partial}{\partial\beta}\beta[\tfrac{1}{6}\chi_{ij}(s) - \xi_{ij}(s)]. \tag{3.30}$$

In the high temperature limit,

$$\theta_{ij}(0)' \sim \frac{T}{96\pi}\frac{m_i+m_j}{m_i m_j}, \tag{3.31}$$

$$\upsilon_{ij}(0)' \sim -\frac{12}{T^4}(m_i^2 - m_j^2)^2 \tag{3.32}$$

for periodic fields.

IV. FERMION LOOPS

We shall consider the contributions to the effective action of a chiral SU(2) fermion, which can be built up into a quark or lepton in the full electroweak theory. The Lagrangian reads

$$\mathcal{L}_f = i\bar\psi_L\gamma\cdot D_L\psi_L + i\bar\psi_R\gamma\cdot D_R\psi_R - if\bar\psi_L\phi\psi_R - if\bar\psi_R\phi^\dagger\psi_L. \tag{4.1}$$

The gauge derivative $D = \nabla - igA$. There is a doublet Higgs field ϕ, a doublet fermion ψ_L, and a singlet fermion ψ_R. The chiral projector $P_L = \frac{1}{2}(1-\gamma_5)$, and the Riemannian γ matrices are taken to be Hermitian, with $\bar\psi = \psi^\dagger$.

The one-loop contribution to the effective action of the Higgs and gauge fields is obtained by expanding the Lagrangian about background fields ϕ and A:

$$\Gamma^{(1)} = \ln\det D_f \tag{4.2}$$

where

$$D_f = \begin{pmatrix} \gamma\cdot D & -f\phi \\ -f\phi^\dagger & \gamma\cdot D \end{pmatrix} P_c, \quad P_c = \begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix}. \tag{4.3}$$

The determinant of the fermion operator D_f is not yet defined, because the operator D_f maps left-chirality fermions into right-chirality ones. We can, however, define the determinant of $\triangle = D_f^\dagger D_f$, where $D_f^\dagger$ is the Hermitian conjugate of D_f. (When acting on the space of Dirac spinors, $D_f^\dagger = \gamma_5 D_f \gamma_5$ and it follows that $D^\dagger$ has the same nonzero spectrum as D_f.) We shall take

$$\Gamma^{(1)} = \tfrac{1}{2}\ln\det\triangle \tag{4.4}$$

where

$$\triangle = P_c \begin{pmatrix} -D^2 - g\sigma_{\mu\nu}F^{\mu\nu} + f^2\phi\phi^\dagger & f\gamma\cdot D\phi \\ f(\gamma\cdot D\phi)^\dagger & -D^2 + f^2\phi^\dagger\phi \end{pmatrix} P_c. \tag{4.5}$$

We have set $\sigma_{\mu\nu} = -\frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

The operator (4.5) is in a form in which we can use the general results of the previous section. We have

$$X = \begin{pmatrix} -g\sigma_{\mu\nu}F^{\mu\nu} + f^2\phi\phi^\dagger & f\gamma\cdot D\phi \\ f(\gamma\cdot D\phi)^\dagger & f^2\phi^\dagger\phi \end{pmatrix} \tag{4.6}$$

and

$$Y^\mu = \begin{pmatrix} igA^\mu & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{F}^{\mu\nu} = \begin{pmatrix} gF^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

If we set

$$X_0 = m_1^2 T_1 + m_2^2 T_2 + m_3^2 T_3, \tag{4.8}$$

then

$$T_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} P_\parallel & 0 \\ 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} P_\perp & 0 \\ 0 & 0 \end{pmatrix} \tag{4.9}$$

where $P_\parallel$ and $P_\perp$ are projections along and perpendicular to ϕ, and $m_1^2 = m_2^2 = f^2\phi^\dagger\phi$, $m_3^2 = 0$.
The traces read

$$\sum_i \mathrm{tr}(T_i\mathcal{D}_\mu\mathcal{D}^\mu X_0 T_i)\rho_i = 2D^2(m_1^2)\rho_1 + 2f^2\mathrm{tr}(P_\parallel D^2\phi\phi^\dagger)\rho_2 + 2f^2\mathrm{tr}(P_\perp D^2\phi\phi^\dagger)\rho_3, \tag{4.10}$$

$$\sum_{i,j} \mathrm{tr}(T_i\mathcal{D}_\mu X_0 T_j\mathcal{D}^\mu X_0 T_i)\eta_{ij} = 2D_\mu m_1^2 D^\mu m_1^2(\eta_{11}+\eta_{22}) + 4f^2m_1^2(D_\mu\phi)^\dagger P_\perp(D^\mu\phi)\eta_{23}, \tag{4.11}$$

$$\sum_{i,j} \mathrm{tr}(T_iX_1T_jX_1T_i)\chi_{ij} = 2f^2\mathrm{tr}(P_L\gamma^\mu\gamma^\nu)[(D_\mu\phi)^\dagger P_\parallel(D_\nu\phi)\chi_{12} + (D_\mu\phi)^\dagger P_\perp(D_\nu\phi)\chi_{13}], \tag{4.12}$$

$$\sum_{i,j} \mathrm{tr}(T_iX_2T_jX_2T_i)\chi_{ij} = g^2\mathrm{tr}(P_L\sigma_{\mu\nu}\sigma_{\alpha\beta})(F^{\mu\nu})_a(F^{\alpha\beta})_b(g_1^{ab}\chi_{22} + g_\perp^{ab}\chi_{33} + g_\parallel^{ab}\chi_{23}), \tag{4.13}$$

$$\sum_{i,j} \mathrm{tr}(T_i\mathcal{F}_{\mu\nu}T_j\mathcal{F}^{\mu\nu}T_i)\xi_{ij} = 2g^2(F_{\mu\nu})_a(F^{\mu\nu})_b(g_1^{ab}\xi_{22} + g_\parallel^{ab}\xi_{23} + g_\perp^{ab}\xi_{33}). \tag{4.14}$$

Group metrics g^{ab} are defined in Appendix B.

The high temperature approximation to the one-loop effective action can now be read off Eq. (3.13), using

$$\Gamma_n^{(1)} = \tfrac{1}{2}\int d\mu(x)\zeta_n'(x,0). \tag{4.15}$$

We will only give the result explicitly for the gauge fields (and we will leave out terms with two or more powers of $F^{0\nu}$):

$$\Gamma_4^{(1)} \sim \int d\mu(x)\{\, Z_1^{ab}(F_{\mu\nu})_a(F^{\mu\nu})_b + Z_2^{ab}\epsilon_{0\nu\alpha\beta}(F^{0\nu})_a(F^{\alpha\beta})_b\} \tag{4.16}$$

where

$$Z_1^{ab} = \frac{11}{96\pi^2}g^2\ln(\mu_R/T)g^{ab}, \qquad Z_2^{ab} = -\frac{1}{16\pi^2}g^2\ln(\mu_R/T)g^{ab}. \tag{4.17}$$

The metric g^{ab} is defined in Appendix B.

Results for the electroweak SU(2)×U(1) are obtained by adding the results (4.17) to similar results for the U(1), in which the SU(2) coupling g is replaced by $g'Y_L$ or $g'Y_R$, where Y_L and Y_R are the hypercharges of the left- and right-chirality fermions. (The $g'Y_R$ contribution to Z_2 also has a factor -1.)

The first term in the result is related simply to the charge renormalization and could have been obtained from a simple Feynman graph calculation. This is due to the fact that the mass has dropped out of the result, an effect which does not occur for bosonic loops as we shall see in the next section. The second term in the result is a total divergence of a Chern-Simons term. It becomes important in a slowly evolving system, for example in the expanding universe.

Terms in the effective action which arise when there is a nonvanishing chemical potential μ can also be found using results from the previous section. If μ is a (constant) Lagrange multiplier for the charge $\bar\psi_L\gamma_0Y_L\psi_L + \bar\psi_L\gamma_0Y_R\psi_R$, then the fermion derivative D_f is replaced by $D_f + \mu\gamma_0 Y$, where

$$Y = \begin{pmatrix} Y_L & 0 \\ 0 & Y_R \end{pmatrix}. \tag{4.18}$$

The fluctuation operator is modified by the addition of

$$X_0^{(1)} = \begin{pmatrix} 0 & -f\gamma_0(Y_L\phi + \phi Y_R) \\ -f\gamma_0(\phi^\dagger Y_L + Y_R\phi^\dagger) & 0 \end{pmatrix}. \tag{4.19}$$

We will take Y to be the hypercharge, and let it commute with the SU(2) generators. The relevant traces are then

$$\sum_{i,j} \mathrm{tr}(T_iX_2T_jX_0^{(1)}T_i)\chi_{ij} = 0 \tag{4.20}$$

and

$$\begin{aligned}\sum_{i,j,k} &\mathrm{tr}(T_iX_1T_jX_2T_kX_0^{(1)}T_i)\chi_{ijk} \\ &= -\,2igf^2(Y_L+Y_R)\mathrm{tr}(P_L\sigma_{0\rho}\sigma_{\alpha\beta}) \\ &\quad\times\left[-\phi^\dagger F^{\alpha\beta}(P_\parallel\chi_{221} + P_\perp\chi_{231})D^\rho\phi \right.\\ &\quad\left. +(D^\rho\phi)^\dagger(P_\parallel\chi_{122} + P_\perp\chi_{132})F^{\alpha\beta}\phi\right].\end{aligned} \tag{4.21}$$

From Eq. (3.26), we have

$$\Gamma_3^{(1)} \sim \int d\mu(x)\epsilon_{0\rho\alpha\beta}\left\{(D^\rho\phi)^\dagger Z_3F^{\alpha\beta}\phi - \phi^\dagger F^{\alpha\beta}Z_3D^\rho\phi\right\} \tag{4.22}$$

where

$$Z_3 \sim -\frac{7i}{32\pi^4T^2}\zeta_R(3)\mu gf^2(Y_L+Y_R). \tag{4.23}$$

This term seems to be analogous to the Chern-Simons term that has been discussed before for chiral fermions at finite temperatures [17]. Here, and with the result for the one-loop action at $\mu = 0$, the derivative expansion fails when one of the masses vanishes. This problem can be removed by including corrections from ring diagrams. To leading order these affect the masses m_i, but the above result is independent of these masses and is therefore unchanged. However, the inclusion of ring diagrams prevents a rigorous comparison between this result and those for massless fields cited in the literature.

V. BOSON LOOPS

Now we turn to contributions to the effective action from a vector or Higgs scalar loop. The calculations are presented for vector representations of O(N), but the generalizations to other groups are also indicated.

The Higgs field Φ has a Lagrangian density

$$\mathcal{L}^{\phi} = -\tfrac{1}{2}(D_\mu \Phi)^T (D^\mu \Phi) - V(\Phi) \tag{5.1}$$

with a potential

$$V(\Phi) = -\tfrac{1}{2}\mu^2 \Phi^T \Phi + \tfrac{1}{4}\lambda(\Phi^T\Phi)^2. \tag{5.2}$$

The gauge field $A = A_a T^a$, where the T^a are Hermitian generators of the Lie algebra. Indices will be raised with the group metric $g^{ab} = \mathrm{tr}(T^a T^b)$. The Lagrangian density for the gauge fields is taken to be

$$\mathcal{L}^A = -\tfrac{1}{4} g^{ab} (F_{\mu\nu})_a (F^{\mu\nu})_b. \tag{5.3}$$

We use the 't Hooft gauge-fixing term, with background fields A and ϕ and perturbations $\tilde{A}$, $\tilde{\phi}$:

$$\mathcal{L}^{\mathrm{GF}} = -\tfrac{1}{2\alpha}\mathcal{F}_a \mathcal{F}^a \tag{5.4}$$

where $D = \nabla - igA$, and

$$\mathcal{F} = D_\mu A^\mu + i\alpha g \phi^T T_a \tilde{\phi} T^a. \tag{5.5}$$

The ghost field c Lagrangian density reads

$$\mathcal{L}_2^c = c^a[-D\cdot(D - ig\tilde{A})\delta_a^b + \alpha g^2 \phi^T T_a T^b \Phi] c_b. \tag{5.6}$$

If we write $\eta = (\tilde{A}, \tilde{\phi})$, then the total Lagrangian density becomes

$$\mathcal{L}^{\mathrm{tot}} = -\tfrac{1}{2}\eta^T \triangle \eta - \tfrac{1}{2} c^T \triangle_{\mathrm{gh}} c + \mathcal{L}^{\mathrm{int}} \tag{5.7}$$

where $\triangle$ and $\triangle_{\mathrm{gh}}$ are the fluctuation operators. They take the form of Eq. (5.1) in Feynman gauge $\alpha = 1$, which we use throughout.

For the vector and Higgs fields,

$$X = \begin{pmatrix} X^A + ig\,\mathrm{tr}(T_a F_\mu{}^\nu T^b) & -2igD_\mu \phi^T T_a \\ 2igT^b D^\nu \phi & X^\phi \end{pmatrix} \tag{5.8}$$

where

$$X^A = g^2 \phi^T T_a T^b \phi \delta_\mu{}^\nu, \tag{5.9}$$

$$X^\phi = g^2 T_a \phi \phi^T T^a - \mu^2 + \lambda \phi^T \phi + 2\lambda \phi\phi^T, \tag{5.10}$$

and

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} -g\,\mathrm{tr}(F_{\mu\nu}[T_a, T^b]) & 0 \\ 0 & gF_{\mu\nu} \end{pmatrix}. \tag{5.11}$$

We can write

$$X_0 = m_4^2 T_4 + m_5^2 T_5 + m_6^2 T_6 + m_7^2 T_7 \tag{5.12}$$

where

$$T_4 = \begin{pmatrix} 0 & 0 \\ 0 & P_\perp \end{pmatrix}, \quad T_5 = \begin{pmatrix} P_{\parallel a}{}^b & 0 \\ 0 & 0 \end{pmatrix},$$
$$T_6 = \begin{pmatrix} 0 & 0 \\ 0 & P_\parallel \end{pmatrix}, \quad T_7 = \begin{pmatrix} P_{\perp a}{}^b & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.13}$$

The projection matrices onto the broken and unbroken subgroups of the gauge group are defined in Appendix B. The mass m_5 is the vector mass, m_6 is the Higgs boson mass, and m_4 is the mass of the longitudinal vector component:

$$m_5^2 = \tfrac{1}{2}g^2\phi^T\phi, \quad m_6^2 = -\mu^2 + 3\lambda\phi^T\phi,$$
$$m_4^2 = \tfrac{1}{2}g^2\phi^T\phi + \lambda\phi^T\phi - \mu^2, \quad m_7^2 = 0. \tag{5.14}$$

In the vacuum state, $m_4 = m_5$. Apart from Eqs. (5.14), the form of the operator is also valid for other representations of simple groups, and the results given below will be widely applicable.

For the ghosts,

$$X^{\mathrm{gh}} = g^2 \phi^T T_a T^b \phi \tag{5.15}$$

and

$$\mathcal{F}_{\mu\nu} = -g\,\mathrm{tr}(F_{\mu\nu}[T_a, T^b]). \tag{5.16}$$

We have $X_0 = m_5^2 T_5 + m_7^2 T_7$ (no vector indices), with the masses defined above.

The kinetic terms in the one-loop effective action are given by

$$\Gamma_n^{(1)} = \int d\mu(x)\left\{-\tfrac{1}{2}\zeta_n'(x,0) + \zeta_n'(x,0)^{\mathrm{gh}}\right\} \tag{5.17}$$

with the high temperature limits $\zeta_n'(x,0)$ calculated using Eq. (3.13). The contribution which depends on the square of the field strength is

$$\Gamma_4^{(1)} = \int d\mu(x)\left\{Z^{ab}(F_{\mu\nu})_a(F^{\mu\nu})_b\right\} \tag{5.18}$$

where

$$\begin{aligned} Z^{ab} = {} & g^2 \frac{T}{192\pi}\left(\frac{1}{m_4} - \frac{C}{2m_5} + \frac{C_\perp}{2m_5} - \frac{C_\perp}{2m_7}\right) g_\perp^{ab} \\ & - g^2\left(\tfrac{1}{2}\xi_{46} + C\xi_{57} + \frac{T}{64\pi}\frac{C}{m_5 + m_7}\right) g_\parallel^{ab} \\ & + g^2 \frac{T}{192\pi}\frac{1}{m_6} g_1^{ab}. \end{aligned} \tag{5.19}$$

The metrics $g_\perp^{ab}$, $g_\parallel^{ab}$ and the Casimir constants C, $C_\perp$ are defined in Appendix B, ξ in Eq. (3.9).

This result, and also the one-loop correction to the Higgs field gradient terms [8], has an infrared divergence due to the vanishing of one of the masses (m_7). This divergence can be removed by including ring-diagram corrections, which have the effect of replacing the masses m_i by temperature-dependent masses $m_i(T)$. The terms that are of order $g^2 T/m_i$ become, for $m_i = 0$ or for very high temperatures, of order g. This means that, except for specially chosen models, they are subdominant but larger than the fermion loop terms.

VI. RING DIAGRAMS

It is a well established fact that the one-loop approximation fails to give all of the leading order terms at high temperatures. Some indication of the problems with the one-loop calculation are seen in the infrared divergences of the previous section. The leading corrections come from summing ring diagrams, which are obtained by replacing the one-loop result with

$$\Gamma^{(\text{ring})} = \tfrac{1}{2}\text{tr}\ \ln(\triangle + \pi) \tag{6.1}$$

where π is the vacuum polarization tensor [3, 4]. (At the phase transition temperature, even this approximation is likely to break down. It can be argued that the approximation works if we avoid coming too close to the transition temperature [25].) The ring diagram corrections are calculated in this section, for convenience rather than originality.

The leading terms in the Feynman diagram expansion for π are shown in Figs. 1 and 2. The vertices are constructed from the interaction Lagrangian density

$$\mathcal{L}^{\text{int}} = \mathcal{L}^{\text{int}}_g + \mathcal{L}^{\text{int}}_\lambda + \mathcal{L}^{\text{int}}_{\text{gh}}, \tag{6.2}$$

where the relevant terms are

$$\mathcal{L}^{\text{int}}_g = \tfrac{1}{2}g^2\tilde{\phi}^T\tilde{A}_\mu\tilde{A}^\mu\tilde{\phi} + \tfrac{1}{4}g^2\text{tr}([\tilde{A},\tilde{A}]^2) + ig(\nabla_\mu\tilde{\phi}^T)\tilde{A}^\mu\tilde{\phi} - ig\,\text{tr}(\nabla_\mu\tilde{A}_\nu[\tilde{A}^\mu,\tilde{A}^\nu]), \tag{6.3}$$

$$\mathcal{L}^{\text{int}}_{\text{gh}} = g\,\text{tr}(c[\tilde{A}^\mu,\nabla_\mu c]), \tag{6.4}$$

$$\mathcal{L}^{\text{int}}_\lambda = -\tfrac{1}{4}\lambda(\tilde{\phi}^T\tilde{\phi})^2. \tag{6.5}$$

The propagator lines represent the background field propagator $G(x,x')$. This has an expansion in powers of derivatives, from which we use the leading term $G_0(x,x')$. The coincidence limit $G_0(x,x) = \zeta_0(x,1)$ (taken as a matrix and untraced); hence,

$$G_0(x,x) = \sum_i T_i\zeta_i(0,1) \sim \tfrac{1}{12}T^2 - \tfrac{1}{2\pi}\sum_i T_i m_i T. \tag{6.6}$$

The asymptotic expansion is from Appendix B.

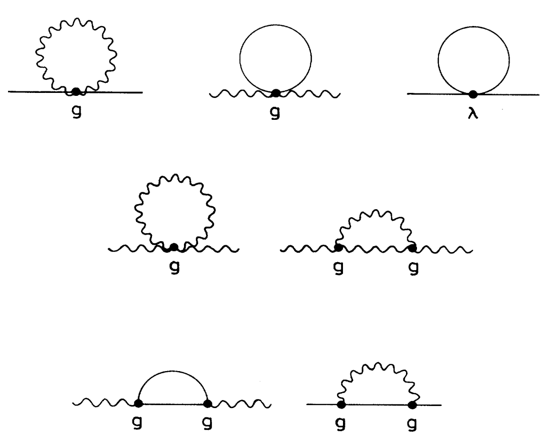

FIG. 1. Leading order Feynman graph contributions to the boson vacuum polarization tensor.

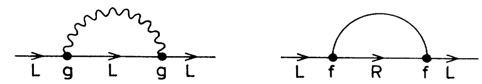

FIG. 2. Leading order Feynman graph contributions to the fermion vacuum polarization tensor.

We will denote the gauge-field interaction contribution to the vacuum polarization by π_g and the self-interaction contribution by π_λ. From Fig. 1,

$$\pi_g = g^2\{\,(1-\tfrac{1}{4})\Omega^n G_0(x,x)\Omega_n - (3-\tfrac{5}{2})\Omega^c G_0(x,x)\Omega_c\}\delta(x-x'), \tag{6.7}$$

$$\pi_\lambda = \lambda\ \{2\Lambda G_0(x,x)\Lambda + \Lambda\,\text{tr}[\Lambda G_0(x,x)]\}\,\delta(x-x'). \tag{6.8}$$

New matrices have been introduced merely to keep the indices tidy:

$$\Omega_n = \begin{pmatrix} 0 & (T_a)_{nj} \\ (T^b)_{in} & 0 \end{pmatrix},\quad \Omega_c = \begin{pmatrix} f_{ca}{}^b & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.9}$$

and

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix}. \tag{6.10}$$

Some terms which are not proportional to $\delta(x,x')$ are also present, but these are relatively small in the high temperature limit. Inserting the coincident limits of the propagators gives

$$\pi_g \sim \tfrac{1}{48}g^2T^2(3\Omega_n^2 - 2\Omega_c^2)\delta(x-x'), \tag{6.11}$$

$$\pi_\lambda \sim \tfrac{1}{6}\lambda T^2\Lambda(1+\tfrac{1}{2}\text{tr}\Lambda)\delta(x-x'). \tag{6.12}$$

When substituted into Eq. (6.1), these terms replace the original masses with temperature-dependent masses:

$$\begin{aligned} m_5^2(T) &= m_5^2 + \tfrac{1}{48}(3+2C)g^2T^2, \\ m_7^2(T) &= m_7^2 + \tfrac{1}{48}(3+2C)g^2T^2, \\ m_4^2(T) &= m_4^2 + \tfrac{1}{4}cg^2T^2 + \tfrac{1}{6}(1+\tfrac{1}{2}\text{tr}\Lambda)\lambda T^2, \\ m_6^2(T) &= m_6^2 + \tfrac{1}{4}cg^2T^2 + \tfrac{1}{6}(1+\tfrac{1}{2}\text{tr}\Lambda)\lambda T^2. \end{aligned} \tag{6.13}$$

Here, c is defined by $T_aT^a = c1$ and C is the quadratic Casimir constant defined in Appendix B. The temperature-dependent Higgs boson masses agree with the masses in the finite-temperature effective potential [4].

The ring corrections to the fermion loop have a slightly different form,

$$\Gamma^{(\text{ring})} = \tfrac{1}{2}\text{tr}\ \ln(\triangle + D_f^\dagger\pi), \tag{6.14}$$

because the fermion propagator $S = \triangle^{-1}D_f^\dagger$. We label the gauge interaction term by π_g and the Yukawa interaction term by π_f:

$$\pi_g(x,x') = \tfrac{1}{2}g^2\text{tr}[\Upsilon_n^T S_0(x,x')\Upsilon_m G_0(x,x')], \tag{6.15}$$

$$\pi_f(x,x') = -f^2\text{tr}[\Theta_n^T S_0(x,x')\Theta_m G_0(x,x')]. \tag{6.16}$$

where

$$\Upsilon_n = \begin{pmatrix} \gamma_\mu (T_a)_{in} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Theta_n = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{nj} \end{pmatrix}. \tag{6.17}$$

The leading high temperature term comes only from the parts of this expression which are quadratically divergent, and, for these,

$$D_f^\dagger \pi_g = -g^2 \mathrm{tr}[\Upsilon_n^T G_0(x,x) \Upsilon_m] \delta(x-x'), \tag{6.18}$$

$$D_f^\dagger \pi_f = -f^2 \mathrm{tr}[\Theta_n^T G_0(x,x) \Theta_m] \delta(x-x'). \tag{6.19}$$

The fermion masses become

$$\begin{aligned} m_1^2(T) &= m_1^2 + \tfrac{1}{6} f^2 T^2, \\ m_2^2(T) &= m_2^2 + \tfrac{1}{8} g^2 T^2 + \tfrac{1}{12} f^2 T^2, \\ m_3^2(T) &= m_3^2 + \tfrac{1}{8} g^2 T^2 + \tfrac{1}{12} f^2 T^2. \end{aligned} \tag{6.20}$$

These masses replace the original masses in the results for the effective action, Eqs. (4.17) and (5.19), removing the infrared problems.

Further improvements can be made by solving for the masses self-consistently, which is equivalent to summing "superdaisy" contributions to the effective action and gives results valid when $g^2 T^2$ is large. Another possibility is to sum ring diagram contributions to the effective potential, using higher terms in the derivative expansion of the propagators to obtain $O(g^4 F^2)$ corrections.

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APPENDIX A: INTEGRALS

The finite-temperature measure was given in Eq. (3.1). We define

$$K_i(t) = \int d\mu(k) e^{-(k^2+m_i^2)t}. \tag{A1}$$

Integration by parts can be used to show that

$$\int d\mu(k) k^\mu k^\nu e^{-(k^2+m_i^2)t} = \frac{\delta^{\mu\nu}}{2t} K_i(t) \tag{A2}$$

and also

$$\int d\mu(k) k^\mu k^\nu k^\alpha k^\beta e^{-(k^2+m_i^2)t} = \frac{4!\,\delta^{(\mu\nu}\delta^{\alpha\beta)}}{4t^2} K_i(t) \tag{A3}$$

for spatially directed k. In the timelike case,

$$\int d\mu(k) k^0 k^0 e^{-(k^2+m_i^2)t} = \frac{1}{2t}\left(1 + \beta \frac{\partial}{\partial \beta}\right) K_i(t). \tag{A4}$$

The effective action can be expressed in terms of ζ functions,

$$\zeta_i(p,s) = \frac{1}{\Gamma(s)} \mu_R^{2s} \int_0^\infty dt\, t^{p+s-1} K_i(t). \tag{A5}$$

The value at $s=0$ is defined by analytic continuation. We can use recurrence relations

$$\zeta_i(p+1,s) = -\frac{\partial}{\partial m_i^2} \zeta_i(p,s) \tag{A6}$$

to relate $\zeta_i(p,s)$ to $\zeta_i(0,s)$.

The high temperature expansion of $\zeta_i'(0,0)$, which enters the effective action, is well known from the free energy of a free field [4]. For bosons,

$$\begin{aligned} \zeta_i'(0,0) \sim{}& \tfrac{\pi^2}{45} T^4 - \tfrac{1}{12} T^2 m_i^2 + \tfrac{1}{6\pi} T m_i^3 \\ &+ \tfrac{m^4}{16\pi^2} \log(\mu_R/T) - \tfrac{1}{384\pi^4} \zeta_R(3) m_i^6/T^2, \end{aligned} \tag{A7}$$

where ζ_R is the Riemann ζ function. The renormalization scale μ_R enters in the logarithmic term and the coefficient of this term is related to a renormalization β function. For fermions,

$$\begin{aligned} \zeta_i'(0,0) \sim{}& -\tfrac{7}{8}\tfrac{\pi^2}{45} T^4 + \tfrac{1}{24} T^2 m_i^2 \\ &+ \tfrac{m^4}{16\pi^2} \log(\mu_R/T) - \tfrac{7}{384\pi^4} \zeta_R(3) m_i^6/T^2. \end{aligned} \tag{A8}$$

Tha absence of a term proportional to T is significant in the derivative expansion of the effective action.

Another quantity of interest is $\zeta_i(p,1)$ (regulated by removing poles) which determines the size of the vacuum fluctuations. The high temperature expansion gives

$$\zeta_i(0,1) \sim \begin{cases} \tfrac{1}{12} T^2 - \tfrac{1}{2\pi} m_i T & \text{bosons,} \\ -\tfrac{1}{24} T^2 & \text{fermions.} \end{cases} \tag{A9}$$

APPENDIX B: GROUP THEORY

The generators of the Lie algebra of the gauge group are represented by matrices T^a. These define a metric $g^{ab} = \mathrm{tr}(T^a T^b)$, which we have taken to be nondegenerate, with inverse g_{ab}. The structure constants $f^{ab}{}_c$ are defined by $[T^a, T^b] = i f^{ab}{}_c T^c$, and we also define a quadratic Casimir constant C by $f^{ac}{}_d f^{bd}{}_c = -C g^{ab}$.

The background Higgs fields ϕ are used to define a projection $P_\|$, by $\phi^T \phi P_\| = \phi \phi^T$, and an orthogonal projection $P_\perp = 1 - P_\|$. The matrices $P_\perp T^a P_\perp$ generate a Lie algebra, and the matrices $P_\| T^a P_\|$ generate another Abelian Lie algebra. We define

$$g_\perp^{ab} = \mathrm{tr}(P_\perp T^a P_\perp T^b P_\perp), \tag{B1}$$

$$g_1^{ab} = \mathrm{tr}(P_\| T^a P_\| T^b P_\|), \tag{B2}$$

$$g_\|^{ab} = 2\,\mathrm{tr}(P_\| T^a P_\perp T^b P_\|). \tag{B3}$$

These are not independent, because $g^{ab} = g_\perp^{ab} + g_1^{ab} + g_\|^{ab}$. We also define operators

$$P_{\perp a}{}^b = g_{ac}(g_\perp^{cb} - g_1^{cb}), \quad P_{\| a}{}^b = g_{ac}(g_\|^{cb} + 2 g_1^{cb}). \tag{B4}$$

The vector mass $g^2 \phi^T T_a T^b \phi = \frac{1}{2} g^2 \phi^T \phi P_{\| a}{}^b$. Furthermore,

$$f^{ac}{}_d f^{be}{}_h P_{\perp e}{}^d P_{\perp c}{}^h = -C_\perp g_\perp^{ab} \tag{B5}$$

defines a quadratic Casimir constant $C_\perp$ on the reduced symmetry group.

For the fundamental representation of the gauge group O(N), $g_1^{ab} = 0$ and ${P_{\perp a}}^b$ is a projection onto the Lie algebra of the reduced symmetry subgroup consisting of rotations about the background Higgs field. We have Casimir constants $C = N - 2$ and $C_\perp = N - 3$.

For the fundamental representation of SU(2), with $\phi^T = (0, \phi_0)$,

$$g_\perp = g_1 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad g_\| = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 0 \end{pmatrix}. \tag{B6}$$

The vector masses are $2g^2\phi^\dagger T_a T^b \phi$ for a complex representation. The quadratic Casimir constants are $C = 1$ and $C_\perp = 0$.

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