Perturbation theory in two-dimensional open string field theory

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We develop the covariant string field theory approach to open two-dimensional strings. Upon constructing the vertices, we apply the formalism to calculate the lowest-order contributions to the four- and five-point tachyon-tachyon tree amplitudes. Our results are shown to match the "bulk" amplitude calculations of Bershadsky and Kutasov. In the present approach, the pole structure of the amplitudes becomes manifest, and their origin from the higher string modes becomes transparent.

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I. INTRODUCTION

In the last couple of years there has been a flurry of activity in the $(d \leq 2)$-dimensional string theory. To a large extent, the excitement was prompted by the significant success of the *matrix model* approach [1,2]. The matrix model in $d=2$ has been shown to be equivalent to the simple scalar (*collective*) field theory with the cubic interaction only [3]. The amplitudes and the S matrix have been calculated exactly for the closed strings [4–10].

Part of the motivation for studying the matrix models is the hope that they can provide us with some insight about *string field theory* (SFT) in higher and critical dimensions. Originally, SFT was formulated for critical strings [11–17]. The covariant formulation of Witten uses the Becchi-Rouet-Stora-Tyutin (BRST) approach. It is, therefore, very important to establish the relation between the matrix models and BRST approaches. It has partially been accomplished for the first-quantized BRST formulation ("Liouville theory"). The S-matrix elements have been calculated, with discrete states and their symmetry W_∞ revealed [18–23]. Progress has been made for open strings as well. In their paper [24], Bershadsky and Kutasov calculated the bulk tree level tachyon amplitudes. They found the pole structure much more intriguing than in the closed string case. It is important to note that, at present, there is no satisfactory matrix model for open two-dimensional (2D) strings.

The first-quantized BRST approach corresponds to the free field theory. It is very interesting to study the interaction theory as well. Subcritical SFT has been discussed [25]. Some work has been done in the $d=2$ case [26–28]. One would like to (a) solidify our knowledge of open SFT in 2D, (b) check, by explicit calculation, that such a theory indeed reproduces the results of Ref. [24], (c) establish the precise connection between the covariant formulation and the simple scalar formulation in 2D, and (d) after gaining some insight dealing with the relatively simple open SFT, try to implement it to closed SFT, which has recently been put on firm ground [29], although admittedly a more complicated one. In this paper, we deal with (a) and (b) leaving the rest of the program to future investigation.

The paper is organized as follows. After the summary of our notation and conventions (Sec. II), we discuss some general properties of Witten's formulation and establish the vertices for the SFT in $d=2$ (Sec. III). Section IV is devoted to the component analysis of the classical and quantum open SFT. In Sec. V we calculate the four- and five-point correlation functions using perturbative SFT, and compare the results with [24]. Finally, we outline some open problems and possible directions of future work (Sec. VI).

II. NOTATIONS AND CONVENTIONS

In this section we summarize the notation and conventions. The matter field $X(z)$ and the Liouville field $\varphi(z)$ are denoted as a 2D vector $\phi^\mu(z)$, where $\mu=1,2$ corresponds to the matter and the Liouville sector, respectively. Also,

$$\langle \phi^\mu(z)\phi^\nu(w)\rangle \sim -\delta^{\mu\nu}\ln(z-w) \ . \tag{2.1}$$

The stress-energy tensor for the matter-Liouville system is given by

$$T^\phi(z) = -\frac{(\partial_z\phi^\mu)^2}{2} - iQ^\mu\frac{\partial_z^2\phi^\mu}{2}(z) \ , \tag{2.2}$$

where $Q^\mu=(0,-i2\sqrt{2})=(0,-iQ)$. The reparametrization ghosts are as in the critical string case:

$$\begin{aligned} &\langle c(z)b(w)\rangle = \langle b(z)c(w)\rangle \sim 1/(z-w) \ , \\ &T^{b,c}(z) = -2b\,\partial_z c + c\,\partial_z b \ . \end{aligned} \tag{2.3}$$

In constructing the field theory vertices it is useful to bosonize ghosts. One introduces a scalar field $\sigma(z)$ whose two-point function and stress-energy tensor read

$$\begin{aligned} &\langle \sigma(z)\sigma(w)\rangle \sim \ln(z-w) \ , \\ &T^\sigma(z) = \frac{(\partial_z\sigma)^2}{2} + 3\frac{\partial_z^2\sigma}{2} \ . \end{aligned} \tag{2.4}$$

Then, one can identify $c(z) \leftrightarrow e^{\sigma(z)}$, $b(z) \leftrightarrow e^{-\sigma(z)}$. We will often state both the *bosonized* and the *nonbosonized* formulas.

A conformal field of weight h, $O_h(z)$, has the mode expansion

$$O_h(z)\equiv\sum_n O_n z^{-n-h}\ , \tag{2.5}$$

with the coefficients O_n:

$$O_n=\oint O(z)z^{n+h-1}\frac{dz}{2\pi i}\ . \tag{2.6}$$

From (2.5) one can read off the mode expansions for the conformal fields $\partial_z\phi^\mu$, $\partial_z\sigma$, $c(z)$, and $b(z)$, making use of the fact that their conformal weights are $h_\phi=h_\sigma=1$, $h_c=-1$, and $h_b=+2$. Their expansion coefficients are denoted, respectively, as $-i\alpha_n^\mu$, σ_n, c_n, and b_n. They satisfy the (anti)commutation relations

$$\begin{aligned}&[\alpha_n^\mu,\alpha_m^\nu]=n\delta^{\mu\nu}\delta_{n+m,0}\ ,\\&[\sigma_n,\sigma_m]=n\delta_{n+m,0}\ ,\\&\{c_n,b_m\}=\{b_n,c_m\}=\delta_{n+m,0}\ .\end{aligned} \tag{2.7}$$

The coefficients in the expansion of the stress-energy tensor $T(z)$ are the *Virasoro* operators L_n and they satisfy the Virasoro algebra

$$|L_n,L_m|=(n-m)L_{n+m}+(c/12)n(n^2-1)\delta_{n+m,0}\ . \tag{2.8}$$

The coefficient c in (2.8) is the *central charge* and its value for the conformal fields we are interested in is

$$c_x=1,\quad c_\varphi=1+3Q^2,\quad c_{\rm gh}=-26\ . \tag{2.9}$$

From (2.9) it follows that if one defines the *total* Virasoro operator,

$$L_n^{\rm tot}\equiv L_n\equiv L_n^\phi+L_n^{\rm gh}\ , \tag{2.10}$$

the total central charge is $c^{\rm tot}\equiv c=0$ precisely when $Q=2\sqrt{2}$. This specific value of Q will appear over and over again as a consistency condition for the theory.

We define a Fock-space vacuum to be the SL(2,R)-invariant vacuum. This means that for conformal field O_h with weight h, the modes O_n satisfy

$$\begin{aligned}&O_n|0\rangle=0\quad\text{when } n\ge 1-h\ ,\\&\langle 0|O_{-n}=0\quad\text{when } n\ge 1-h\ .\end{aligned} \tag{2.11}$$

We introduce the Hermitian conjugation as follows (here and below in this section we closely follow the Ref. [18]):

$$\begin{aligned}&(\alpha_n^\mu)^\dagger=\alpha_{-n}^\mu,\quad n\neq 0\ ,\\&(\sigma_n)^\dagger=-\sigma_{-n},\quad n\neq 0\ ,\\&(c_n)^\dagger=c_{-n}\ ,\\&(b_n)^\dagger=b_{-n}\ ,\\&(\alpha_0^\mu)^\dagger=\alpha_0^\mu+Q^\mu\\&(\sigma_0)^\dagger=-\sigma_0+3\ .\end{aligned} \tag{2.12}$$

From (2.11) it follows that the SL(2,R) generators $\{L_{-1},L_0,L_1\}$ annihilate both $|0\rangle$ and $\langle 0|$ (thus the name for the vacuum). On the other hand, neither of the vacua is annihilated by c_{-1}, c_0, or c_1 so we define

$$\langle 0|c_{-1}c_0c_1|0\rangle=1\ . \tag{2.13}$$

This equation is the remainder of the fact that the ghost system has the background charge -3 and that one therefore needs three zero modes to saturate it. It is useful to introduce yet another vacuum which we refer to as *physical* vacuum $|\Omega\rangle$, where $|\Omega\rangle\equiv c_1|0\rangle$. We postulate that $|0\rangle$ has the *ghost number* g zero, so that $|\Omega\rangle$ has $g=1$. Physical vacuum is annihilated by *all* positive moded oscillators. Thus it is the state of the lowest L_0 ("energy") value. Expressed in the bosonized language:

$$|\Omega\rangle=c_1|0\rangle=e^{\sigma(0)}|0\rangle\ .$$

Since $[\sigma_0,\sigma(z)]=+\sigma(z)$, one infers that σ_0 is nothing but the bosonized version of the ghost number operator g. More generally, one can define a state:

$$|p,\lambda\rangle=:e^{ip^\mu\phi^\mu(0)}e^{\lambda\sigma(0)}:|0\rangle \tag{2.14}$$

[: : denotes the normal ordering with respect to the SL(2,R)-invariant vacuum]. For such a state,

$$\begin{aligned}&\alpha_0^\mu|p,\lambda\rangle=p^\mu|p,\lambda\rangle\ ,\\&\sigma_0|p,\lambda\rangle=\lambda|p,\lambda\rangle\ ,\\&\langle p,\lambda|\alpha_0^\mu=\langle p,\lambda|p^\mu\ ,\\&\langle p,\lambda|\sigma_0=\langle p,\lambda|\lambda\ ,\end{aligned} \tag{2.15}$$

and the basic scalar product is defined as (note that $\langle p,\lambda|$ is *not* the Hermitian conjugate of $|p,\lambda\rangle$)

$$\langle p_1,\lambda_1||p_2,\lambda_2\rangle=\delta(p_1-p_2)\delta(\lambda_1-\lambda_2)\ . \tag{2.16}$$

Here, $p^\mu\equiv(p,-iq)$, and $p,q\in R$. In the special case when $\lambda=1$, we get tachyon. To simplify the notation we use $|p,1\rangle\equiv|p\rangle$, unless otherwise stated. In Sec. IV and below, p refers to the general off-shell state, while k is reserved for the on-shell states.

An important $g=1$ operator is the BRST charge Q:

$$Q=\oint:c(z)\left[T^\phi+\frac{T^{b,c}}{2}\right]:\frac{dz}{2\pi i}\ . \tag{2.17}$$

Making use of (2.12) one easily verifies that Q is Hermitian, i.e., that $Q^\dagger=Q$. Also, the BRST charge is nilpotent provided that the Liouville background charge is $Q=2\sqrt{2}$. If one denotes by F the one-string Fock space built on the physical vacuum $|\Omega\rangle$, then Q and g endow F with the structure of the differential complex. Namely, g provides for the grading on the complex: $F=\oplus F^{(n)}$, where n is the ghost number of $F^{(n)}$, and Q is the differential. Corresponding cohomologies of Q, $H^{(n)}$, are the physical states of 2D theory [18–23]. These two operators play the crucial role in the construction of the gauge-invariant string field theory, as well.

III. WITTEN'S OPEN STRING FIELD THEORY IN $d=2$

In the Sec. II we mentioned that the physical states of the first-quantized strings are given by the BRST cohomology classes of F. Transition from the first-quantized to the gauge-invariant field theory formulation of the critical SFT is well known. Namely, physical states in the first-quantized approach are the classical solutions of the free field theory (see, e.g., Ref. [16]). In Ref. [11] a covariant formulation of the interacting SFT in $d=26$ is proposed. The present section applies the construction to the $d=2$ case.

A generic string field $|A\rangle$ is, by definition, an arbitrary linear combination:

$$|A\rangle=\sum_s |s\rangle a_s \ , \tag{3.1}$$

where $|s\rangle\in F$ and a_s are the coefficient functions, which can be either even or odd with respect to Grassmann parity (basically, we follow the conventions in Ref. [14] with the shift in g by $+\frac{3}{2}$). The coefficient functions depend on the center-of-mass coordinates only. Grading in the string field space is induced by the $F=\oplus F^{(n)}$ decomposition, so a generic string field $|A\rangle$ can be decomposed into g eigenstates:

$$\begin{aligned} &|A\rangle=\sum_n |A\rangle_{(n)} \ , \\ &g|A\rangle_{(n)}=n|A\rangle_{(n)}\equiv n|A\rangle_n \ . \end{aligned} \tag{3.2}$$

In particular, it is convenient to declare that Grassmann odd coefficient functions a_s *anticommute* with a ket $|l\rangle$ if g_l:$g|l\rangle=g_l|l\rangle$ is *odd* and that they *commute* otherwise. The *total parity* $(-)^{g_s}(-)^{a_s}$ of the string field $|A\rangle$ is denoted as $(-)^A$.

Next one considers multistring states which are simply the elements of the

$$F\otimes F\otimes\cdots\otimes F\equiv F^n \ .$$

A space dual to F^n we denote as F_{*n}, where F_{*n}:$F^n\mapsto C$. *More notation:* a mode O_n corresponding to the rth string we denote as O_n^r; vacua are labeled as $|\Omega\rangle^r$; an element from F_{*n} we generically denote as ${}_{12\cdots n}\langle V|\equiv{}_n\langle V|$. As an example, $|A\rangle_n^r$ is the rth string of ghost number n.

Multistring states describe the processes of splitting and joining of strings, that is, the string interactions. The simplest nontrivial operation is *integral* $\int$ which maps $F\mapsto R$ and carries $g=-3$. Another operation is the *string multiplication* $*$. The star operation carries $g=0$ and maps $F^2\mapsto F$ so that given the two string fields A and B, $A*B$ is again a string field. The operations $\int$ and $*$, together with BRST charge Q should satisfy Witten's axioms [11]:

$$\begin{aligned} &(A*B)*C=A*(B*C) \ , \\ &Q(A*B)=(QA)*B+(-)^A A*(QB) \ , \\ &\int A*B=(-)^{AB}\int B*A \ , \\ &\int QA=0 \ . \end{aligned} \tag{3.3}$$

Using the axioms (3.3) it is easy to construct a gauge-invariant theory of string fields, the classical action being

$$W_{\rm cl}=\frac{1}{2}\int\left[A_1*QA_1+\frac{2}{3}A_1*A_1*A_1\right] \tag{3.4}$$

and the gauge invariance ($A_0\equiv\Lambda$)

$$\Delta A_1=Q\Lambda+A_1*\Lambda-\Lambda*A_1 \ . \tag{3.5}$$

Witten's proposal can be realized by rephrasing the axioms in terms of the Fock-space oscillators [12,13]. Namely, the operations $\int$ and the string functional multiplication $*$ may be represented through the multipoint vertex operators ${}_n\langle V|\in F_{*n}$:

$$\begin{aligned} &\int A\equiv{}_1\langle V\|A\rangle^1 \ , \\ &|A*B\rangle_1\equiv{}_{1^\dagger 23}\langle V\|A\rangle^2|B\rangle^3 \ , \end{aligned} \tag{3.6}$$

and the derivative operator Q is the first-quantized BRST operator (2.17). Then, in the bosonic string case, the classical action (3.4) reads

$$W_{\rm cl}=\tfrac{1}{2}{}_{12}\langle V\|A\rangle_1^2 Q|A\rangle_1^1+\tfrac{1}{3}{}_{123}\langle V\|A\rangle_1^3|A\rangle_1^2|A\rangle_1^1 \ , \tag{3.7}$$

where ${}_{12}\langle V|={}_{123}\langle V\|V\rangle^3$ is the two-string vertex which is nothing but the inner product on the string field space.

To find ${}_{12\cdots n}\langle V|$ is the main problem in the construction of the theory. This can be done by solving the *overlap* equations or, equivalently, using the Neumann function method [12]. The overlap equations for a conformal field $O_h(z)$ with the conformal dimension h are given by

$${}_{12\cdots n}\langle V|[z^h O_h^r(z)-z^{-h}O_h^{r-1}(-z^{-1})]=0 \ , \tag{3.8}$$

where $r=1,\ldots,n$, and if $r-1=0$ we identify $r-1=n$. This leads, in particular, to the following identities for the modes of the conformal fields in question (see Sec. II):

$$\begin{aligned} &{}_{12\cdots n}\langle V|(\alpha_m^{\mu r}+(-)^m\alpha_{-m}^{\mu r-1})=0 \ , \\ &{}_{12\cdots n}\langle V|(\sigma_m^r+(-)^m\sigma_{-m}^{r-1})=0 \ , \\ &{}_{12\cdots n}\langle V|(c_m^r+(-)^m c_{-m}^{r-1})=0 \ , \\ &{}_{12\cdots n}\langle V|(b_m^r-(-)^m b_{-m}^{r-1})=0 \ , \end{aligned} \tag{3.9}$$

where $m\neq 0$ in all four equations (3.9). Zero modes are treated separately. We require that

$$\begin{aligned} &{}_{12\cdots n}\langle V|\left[\sum_{r=1}^n \alpha_0^{\mu r}+Q^\mu\right] \\ &\qquad={}_{12\cdots n}\langle V|\delta^{(2)}\left[\sum_{r=1}^n p^{\mu r}+Q^\mu\right]=0 \ , \\ &{}_{12\cdots n}\langle V|\left[\sum_{r=1}^n \sigma_0^r-3\right]={}_{12\cdots n}\langle V|\delta\left[\sum_{r=1}^n \lambda^r-3\right]=0 \ . \end{aligned} \tag{3.10}$$

The information above is sufficient, using the results and methods of the critical SFT, to figure out, almost without further ado, 2D SFT vertices. They are of the generic form

$$ {}_{12\cdots n}\langle V| = {}_{12\cdots n}\langle V|^{\phi}\,{}_{12\cdots n}\langle V|^{\sigma}\ , \tag{3.11}$$

where ${}_{12\cdots n}\langle V|^{\phi}$ is the matter and ${}_{12\cdots n}\langle V|^{\sigma}$ is the ghost part of the vertex. Then, for the one-string vertex one gets (here $\langle p|$ does not contain any ghost dependence and should not be confused with $\langle p,1|$)

$$\begin{aligned} {}_1\langle V|^{\phi} &= \int d^2p\, \delta^{(2)}(p^\mu+Q^\mu)\langle p|\exp\left[-\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\alpha_n^\mu\alpha_n^\mu\right]\exp\left[-Q^\mu\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\alpha_{2n}^\mu\right],\\ {}_1\langle V|^{\sigma} &= \int d\lambda\, \delta(\lambda-3)\langle\lambda|\exp\left[-\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\sigma_n\sigma_n\right]\exp\left[3\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\sigma_{2n}\right]. \end{aligned}\tag{3.12}$$

For the two-string vertex we get

$$\begin{aligned} {}_{21}\langle V|^{\phi} &= \int d^2p_1 d^2p_2\, \delta^{(2)}(p_1^\mu+p_2^\mu+Q^\mu)\,{}_2\langle p_2|\,{}_1\langle p_1|\exp\left[-\sum_{n=1}^{\infty}\frac{(-)^n}{n}\alpha_n^{\mu 1}\alpha_n^{\mu 2}\right],\\ {}_{21}\langle V|^{\sigma} &= \int d\lambda_1 d\lambda_2\, \delta(\lambda_1+\lambda_2-3)\,{}_2\langle\lambda_2|\,{}_1\langle\lambda_1|\exp\left[-\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sigma_n^1\sigma_n^2\right]. \end{aligned}\tag{3.13}$$

It is useful to express the ghost part of the vertex in the fermionic form as well:

$$ {}_{21}\langle V|^{b,c} = {}_2\langle\Omega|\,{}_1\langle\Omega|(c_0^1+c_0^2)\exp\left[\sum_{n=1}^{\infty}(-)^n(b_n^1c_n^2+b_n^2c_n^1)\right]. \tag{3.14}$$

The three-string vertex is

$$\begin{aligned} {}_{321}\langle V|^{\phi} = \int & d^2p_1 d^2p_2 d^2p_3\, \delta^{(2)}(p_1^\mu+p_2^\mu+p_3^\mu+Q^\mu)\,{}_3\langle p_3|\,{}_2\langle p_2|\,{}_1\langle p_1|\exp\left[\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty}\frac{1}{2}N_{nm}^{rs}\alpha_n^{\mu r}\alpha_m^{\mu s}\right]\\ &\times\exp\left[Q^\mu/3\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\alpha_{2n}^{\mu r}\right]\exp\left[Q^\mu/2\sum_{r,s=1}^{3}N_{00}^{rs}\alpha_0^{\mu r}\right]\exp\left[Q^\mu/3\sum_{r,s=1}^{3}\sum_{n=1}^{\infty}N_{n0}^{rs}\alpha_n^{\mu r}\right]e^{-3N_{00}}\ , \end{aligned}\tag{3.15}$$

$$\begin{aligned} {}_{321}\langle V|^{\sigma} = \int & d\lambda_1 d\lambda_2 d\lambda_3\, \delta(\lambda_1+\lambda_2+\lambda_3-3)\,{}_3\langle\lambda_3|\,{}_2\langle\lambda_2|\,{}_1\langle\lambda_1|\exp\left[\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty}\frac{1}{2}N_{nm}^{rs}\sigma_n^r\sigma_m^s\right]\\ &\times\exp\left[-\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{(-)^n}{2n}\sigma_{2n}^r\right]\exp\left[-\sum_{r,s=1}^{3}N_{00}^{rs}\sigma_0^r\right]\exp\left[-\sum_{r,s=1}^{3}\sum_{n=1}^{\infty}N_{n0}^{rs}\sigma_n^r\right]e^{3/2N_{00}}\ . \end{aligned}\tag{3.16}$$

The explicit expressions for the Neumann coefficients N_{nm}^{rs} which appear in (3.15) and (3.16) can be found in Ref. [17].

Note that the structure of the matter and bosonized ghosts vertices is the same, only the values for the insertions are different. More importantly, insertions are given, in accordance to the geometric considerations in Ref. [11], by the background charge of the conformal fields in question. For the matter field ϕ^μ it is $Q^\mu=(0,-i2\sqrt{2})$, and for the ghosts, it is -3. These are the values for which the matter field c-number anomalies cancel their ghost counterparts (this can be shown in a straightforward fashion, just as it was done in Ref. [12] for the critical strings), and for which the first-quantized BRST charge is nilpotent.

Now that we have completed the construction of the vertices, a comment is in order. The Liouville (non)conservation law is explicitly enforced on the vertices by its very construction. So, the correlation functions, calculated by means of the Feynman rules (see Sec. V), are necessarily "bulk." In other words, the cosmological constant in the theory is taken to vanish. It is intriguing to wonder whether this condition can be relaxed and the space-time gauge invariance preserved. An attempt in that direction was made in Ref. [26] in which the effect of the Liouville wall was simulated by the "semifree" field boundary conditions. The correlation functions calculated in that paper are, however, bulk. Much better understanding of the cosmological constant role in the theory is needed and will be the subject of further investigations.

IV. THE COMPONENT ANALYSIS IN 2D SFT

The purpose of this section is to set the groundwork for the explicit calculations of the amplitudes in Sec. V.

It is also instructive to see how component fields enter the classical action (Sec. IV A) in order to better understand what happens to them upon quantization (Sec. IV B).

Before moving on, let us introduce another piece of notation. Let us define, for a string field $|A\rangle$, the reflected field $|A\rangle^r$ [17,29]:

$$|A\rangle_1^r \equiv {}_{12}\langle V\|A\rangle^2 \ . \tag{4.1}$$

A string field $|A\rangle$ is *real* if $|A\rangle^r = |A\rangle^\dagger$, where the dagger denotes the Hermitian conjugation [see Eq. (2.12)]. From now on we restrict ourselves to the real string fields (this is necessary for the proper counting of the string degrees of freedom).

A. Classical field theory in $D=2$

A string field can be represented, in general, as

$$|A\rangle = |A\rangle^{(N=0)} + |A\rangle^{(N=1)} + |A\rangle^{(N=2)} + \cdots \ ,$$

where the ellipsis denotes the contributions of the levels $N \geq 3$. In this subsection we are interested in the classical SFT, so the field $|A\rangle$ has $g=1$. For such a field,

$$\begin{aligned} |A\rangle^{(N=0)} &= \Phi(p)|p\rangle \ , \\ |A\rangle^{(N=1)} &= [A_\mu(p)\alpha_{-1}^\mu|p\rangle - i\Psi(p)b_{-1}c_0|p\rangle] \ , \\ |A\rangle^{(N=2)} &= \left[-\frac{1}{2}H_{\mu\nu}(p)\alpha_{-1}^\mu\alpha_{-1}^\nu|p\rangle - iG_\mu(p)\alpha_{-2}^\mu|p\rangle + S(p)b_{-1}c_{-1}|p\rangle + B(p)b_{-2}c_0|p\rangle \right. \\ &\quad \left. - iB_\mu(p)\alpha_{-1}^\mu b_{-1}c_0|p\rangle \right] \ . \end{aligned} \tag{4.2}$$

It is straightforward, although tedious, to calculate the free field action $\frac{1}{2}\int A * QA$, which can be rewritten as

$$\frac{1}{2}\int A * QA = \frac{1}{2}\langle A|Q|A\rangle \ . \tag{4.3}$$

On the $N=0$ level there is only one field, the tachyon:

$$W_f^{(N=0)} = \frac{1}{2}\int d^2p_1 d^2p_2 \delta^{(2)}(p_1+p_2+Q)\phi(p_1) \times[(p_2/2)\cdot(p_2+Q)-1]\phi(p_2) \ . \tag{4.4}$$

Since

$$\frac{p}{2}\cdot(p+Q)-1 = \frac{1}{2}\left[p+\frac{Q}{2}\right]^2 ,$$

one sees that tachyon in 2D is indeed *massless*. Note that if one defines

$$\int_l \equiv \int d^2p_1 d^2p_2 \delta^{(2)}(p_1+p_2+Q) \ , \tag{4.5}$$

inside the integral,

$$p_1^\mu = -(p_2+Q)^\mu \ ,$$

then $\frac{1}{2}(p+Q/2)^2$ is invariant under the partial integration. On the next level, $N=1$, there are two fields A_μ and Ψ, the second one being an auxiliary nondynamical field:

$$W_f^{(N=1)} = \frac{1}{2}\int_l A_\mu(p_1)\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+1\right]A_\mu(p_2) - i\int_l A_\mu(p_1)p_2^\mu\Psi(p_2) + \int_l \Psi(p_1)\Psi(p_2) \ . \tag{4.6}$$

Finally, let us state the $N=2$ action:

$$\begin{aligned} W_f^{(N=2)} = {} & \frac{1}{4}\int_l H_{\mu\nu}(p_1)\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]H_{\mu\nu}(p_2) - \frac{1}{2}\int_l S(p_1)\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]S(p_2) \\ & + \int_l G_\mu(p_1)\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]G_\mu(p_2) + 2\int_l G_\mu(p_1)B_\mu(p_2) + \int_l B_\mu(p_1)B_\mu(p_2) \\ & + i\int_l B_\mu(p_1)p_2^\mu S(p_2) + i\int_l B_\mu(p_2)p_2^\nu H_{\mu\nu}(p_1) + 2\int_l B(p_1)B(p_2) - \frac{1}{2}\int_l H_\mu^\mu(p_1)B(p_2) \\ & + 2i\int_l G_\mu(p_1)\left[p_2-\frac{Q}{2}\right]_\mu B(p_2) + 3\int_l B(p_1)S(p_2) \ . \end{aligned} \tag{4.7}$$

For all levels $N \geq 1$ one can check that the auxiliary fields are annihilated (and only them) by the c_0 operator. They play the role of the Lagrange multipliers. The rest of the fields are dynamical. Among them are the Stueckelberg fields which appear as the coefficient functions corresponding to the "dynamical" ghost excitations, i.e., the ghost excitations annihilated by the b_0 operator. Such fields are essential for the construction of the *local* gauge-invariant theory. For the first time in our construction they appear on the second level, namely, the field $S(p)$. They are present on all of the higher levels.

As a final comment, note that the physical spectrum of the 2D string theory (the tachyon and the discrete states) can be naturally reproduced in the second-quantized language [27]. This boils down to solving the classical equation of motion (EOM) for the free theory $QA=0$ modulo gauge invariance $Q\Lambda$. The tachyon survives intact, since there are no gauge transformations associated with that field. All higher-order fields are subject to gauge fixing which kills all but the discrete degrees of freedom (naive counting of the 2D photon degrees of freedom, e.g., gives $2-2=0$). That the states obtained that way are precisely the same as the ones obtained in the first-quantized approach is not surprising, of course, since we are solving the same set of equations. What is less obvious, however, is the question of what happens to the discrete states upon including the interaction. This will be discussed elsewhere. From that end, let us just note that in the field theoretical formalism, such a question is a perfectly well-defined one.

B. Gauge fixing and the Feynman rules

In Sec. IV A we were discussing the classical SFT. Now, the time has come to quantize it. That is to say, we want to calculate the path integral

$$\int (da_s) e^{-W_{\rm cl}(a_s)} ,$$

where $W_{\rm cl}$ is discussed in detail in the previous section. Such an integral is an ill-defined object [due to the gauge invariance (3.5)], so it cannot serve as a starting point for the perturbation theory. The standard way out in the gauge theories is to fix the gauge. We choose Siegel's gauge $b_0|A\rangle=0$. After performing the Faddeev-Popov trick once we are left, in general, with some gauge condition(s) imposed on the *quantum* fields and with the Jacobian (Faddeev-Popov determinant) which can be represented as a path integral over ghosts. In an irreducible theory, such as the Yang-Mills theory, that proves to be enough, and from such an effective action one can read off the propagators and the vertices. In the case of the string theory, which is an infinitely reducible theory, such a gauge-fixed action has additional gauge invariances. So we have to gauge fix again. By doing so, we introduce ghosts of ghosts. This procedure continues *ad infinitum* [14]. It is convenient, following Thorn, to take the input (classical) field and all ghosts to be odd. Then, the proper gauge-fixed action is

$$W_{\rm GF}=\frac{1}{2}\int \left[A*QA+\frac{2}{3}A*A*A-2(b_0\beta)*A \right] , \tag{4.8}$$

where the field A is now the sum of the input field $A_{\rm inp}$ *and* all of the ghost fields. It contains fields of the all possible ghost numbers. The fact that $|A\rangle$ is odd means that if g_s is even, the corresponding coefficient function a_s should be Grassman odd and vice versa. The field β is a Lagrange multiplier (it also contains all possible ghost numbers) which enforces the gauge condition.

In order to get the Feynman rules, let us integrate over the β fields. We get $b_0|A\rangle=0$, or, using (3.1), $b_0|s\rangle=0$. The kinetic term becomes

$$\begin{aligned}\frac{1}{2}\int A*QA &= \frac{1}{2}\int A*Qb_0c_0A \\ &= \frac{1}{2}\int A*(-b_0Qc_0A) \\ &\quad +\frac{1}{2}\int A*c_0(L_0-1)A \\ &= \frac{1}{2}\int b_0A*Qc_0A+\frac{1}{2}\int A*c_0(L_0-1)A \\ &= \frac{1}{2}\int A*c_0(L_0-1)A . \end{aligned} \tag{4.9}$$

One can rewrite (4.9) in terms of the components fields:

$$\frac{1}{2}\int A*c_0(L_0-1)A=\frac{1}{2}\sum_{s,l}K_{sl}a_la_s , \tag{4.10}$$

where

$$K_{sl}\equiv {}_{21}\langle V||s\rangle_1 c_0(L_0-1)|l\rangle_2 .$$

To add more "meat" to this rather abstract looking expression let us analyze its content on the first couple of levels. One naive guess would be to just exclude from the input action all of the auxiliary fields (or, in other words, to set Ψ, B, B_μ, etc., equal to zero). Although this is not the complete answer, as one may guess, such "short" gauge-fixed action contains all the information necessary for the tree amplitude calculations. Nevertheless, it is instructive to see the whole structure. In fact, $|A\rangle=|A\rangle_{\rm inp}+|A\rangle_{\rm gh}$, where, to the second level,

$$\begin{aligned}|A\rangle_{\rm gh}=&-ib_{-1}|p\rangle\beta(p)+b_{-2}|p\rangle\delta(p)-i\alpha^{\mu}_{-1}b_{-1}|p\rangle\beta_\mu \\ &-ic_{-2}|p\rangle\rho+\alpha^{\mu}_{-1}c_{-1}|p\rangle\gamma_\mu .\end{aligned}$$

Here, in line with our conventions, the coefficient functions corresponding to the ghost numbers $g=0$ and 2 are Grassmann odd. One sees that the $g=0$ fields (ghosts) are of the form $\Lambda\theta$ where θ is a Grassmann number and Λ is a gauge parameter field. Ghost number 2 fields correspond to the antighosts. The kinetic part of the action becomes

$$\frac{1}{2}\int A*c_0(L_0-1)A=\frac{1}{2}\int_l\Phi\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2\right]\Phi+\frac{1}{2}\int_l A_\mu\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+1\right]A_\mu-\int_l S\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]S$$
$$+\frac{1}{4}\int_l H_{\mu\nu}\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]H_{\mu\nu}+\int_l G_\mu\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]G_\mu-i\int_l\beta\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+1\right]\gamma$$
$$-i\int_l\beta_\mu\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]\gamma_\mu-i\int_l\rho\left[\frac{1}{2}\left[p+\frac{Q}{2}\right]^2+2\right]\delta\equiv W_{11}+W_{20}\ . \tag{4.11}$$

In the last line we denoted the matter and the ghost contributions, respectively, as W_{11} and W_{20}.

Let us proceed to the interactions. Taking into account the signs convention, one has

$$\frac{1}{3}\int(A*A*A)=\frac{1}{3}\sum_{s,l,m}V_{slm}a_m a_l a_s(-)^{a_l}\ , \tag{4.12}$$

where the vertex functions are given by

$$V_{slm}={}_{321}\langle V\|s\rangle_1|l\rangle_2|m\rangle_3\ . \tag{4.13}$$

Inserting the coupling constant g in front of the interaction term, we get the final expression that we have been looking for:

$$W_{\rm GF}=\frac{1}{2}\sum_{s,l}K_{sl}a_l a_s+\frac{g}{3}\sum_{s,l,m}V_{slm}a_m a_l a_s(-)^{a_l}\ . \tag{4.14}$$

It is important to note that $g=n\neq1$ fields always couple to $g=1$ in pairs with $g=2-n$. This means that they can enter the amplitude only in the form of the closed ghost loops. Hence, they do not contribute to the tree amplitudes. This proves that our naive guess of the gauge-fixed action is indeed correct at the tree level. In the next section, we implement these results to get the leading contributions to the four- and five-point tachyon scattering amplitudes.

V. THE CALCULATION OF THE OPEN STRING AMPLITUDES IN $d=2$

Now we can proceed to the calculation of the lowest order contributions to the tachyon four- and five-point on-shell bulk amplitudes. An advantage of the component perturbation theory is its straightforwardness. The calculations of the correlation functions are very similar to those of the ordinary cubic scalar field theory. Let us consider four-point calculations in more detail. In that case, apart from an overall multiplicative numerical factor (which we will be often sloppy about), one has, to leading order,

$$g^2\langle\Phi(k_1)\cdots\Phi(k_4)V_{slk}a_k a_l a_s V_{uvn}a_n a_v a_u(-)^{a_l+a_v}\rangle\ . \tag{5.1}$$

Using the Wick's theorem, chopping the external tachyonic legs, and concentrating on one particular contribution corresponding to the s channel (other contributions are related to this one by the label permutations) one gets

$$A_s^{(4)}\propto V_{\Phi I\Phi}(k_2,p,k_1)D_{II}(p,p')V_{\Phi I\Phi}(k_4,p',k_3)\ . \tag{5.2}$$

Here, $D_{II}(p,p')$ is a propagator of an intermediate (off-shell) state I and the vertices $V_{\Phi I\Phi}(k_2,p,k_1)$ are the couplings of the two on-shell tachyons coupled to an intermediate state I. There are infinitely many intermediate states which contribute to the amplitude (5.2), so the *total* amplitude is the sum over all of them. Instead of calculating the whole sum (after all, this can be more efficiently done by the conformal mapping method [26]), we are interested in the pole structure itself. The method is *designed* precisely to give that structure. As a check, we have to compare the residues read off from (5.2) with the ones calculated from the total amplitude [24]. We confine ourselves to the first three poles. In that case the propagators are

$$\begin{aligned}
D_{\Phi\Phi}(p_1,p_2)&=\frac{1}{\frac{1}{2}(p+Q/2)^2}\delta(p_1+p_2+Q)\ ,\\
D_{AA}(p_1,p_2)&=\frac{\delta^{\mu\nu}}{\frac{1}{2}(p+Q/2)^2+1}\delta(p_1+p_2+Q)\ ,\\
D_{GG}(p_1,p_2)&=-\frac{1}{2}\frac{\delta^{\mu\nu}}{\frac{1}{2}(p+Q/2)^2+2}\delta(p_1+p_2+Q)\ ,\\
D_{SS}(p_1,p_2)&=-\frac{\delta^{\mu\nu}}{\frac{1}{2}(p+Q/2)^2+2}\delta(p_1+p_2+Q)\ ,\\
D_{HH}(p_1,p_2)&=\frac{1}{4}\frac{\delta^{\mu\kappa}\delta^{\lambda\nu}+\delta^{\mu\lambda}\delta^{\kappa\nu}}{\frac{1}{2}(p+Q/2)^2+2}\delta(p_1+p_2+Q)\ .
\end{aligned} \tag{5.3}$$

Omitting the common factor of

$$(4/3\sqrt{3})^{(p+Q/2)^2/2}\delta(k_1+k_2+p+Q)\ ,$$

the vertices $V_{\Phi I\Phi}$ are

$$\begin{aligned}
&V_{\Phi\Phi\Phi}(k_1,p,k_2)=1\\
&V_{\Phi A\Phi}(k_1,p,k_2)=N_{10}^{12}(k_2^\mu-k_1^\mu)\ ,\\
&V_{\Phi G\Phi}(k_1,p,k_2)=2N_{20}^{22}(p^\mu+\tfrac{1}{3}Q^\mu)-\tfrac{1}{3}Q^\mu\ ,\\
&V_{\Phi H\Phi}(k_1,p,k_2)=N_{11}^{22}\delta^{\mu\nu}+(N_{10}^{12})^2(k_2-k_1)^\mu(k_2-k_1)^\nu\ ,\\
&V_{\Phi S\Phi}(k_1,p,k_2)=\tfrac{1}{2}+\tfrac{1}{2}N_{11}^{22}\ .
\end{aligned} \tag{5.4}$$

Plugging (5.3) and (5.4) into (5.2) and integrating out the δ functions, one gets

$$A_s^{(4)} \propto g^2 \left[\frac{16}{27}\right]^{(k_1+k_2+Q/2)^2/2} \left[\frac{1}{\frac{1}{2}(k_1+k_2+Q/2)^2} + \frac{\frac{4}{27}(k_1-k_2)(k_3-k_4)}{\frac{1}{2}(k_1+k_2+Q/2)^2+1} + \frac{\frac{2}{81}(k_1+k_2)\cdot(k_1+k_2+Q)+\frac{8}{729}[(k_1-k_2)(k_3-k_4)]^2}{\frac{1}{2}(k_1+k_2+Q/2)^2+2} + \frac{\frac{124}{729}-\frac{10}{729}[(k_1-k_2)^2+(k_3-k_4)^2]}{\frac{1}{2}(k_1+k_2+Q/2)^2+2} + \cdots\right] \delta\left[\sum_{i=1}^{4} k_i + Q\right] . \tag{5.5}$$

Written in the form (5.5) the amplitude seems to have multiparticle poles (poles in more than one variable). This is not, however, the case. The reason for this is the δ function in (5.5). In fact, upon introducing the "masslike" variables $m_i = -\frac{1}{2}k_i^2$, and, for definiteness, choosing the kinematic region to be $(+++-)$, (5.3) becomes

$$A_{+++-} \propto g^2 (\tfrac{16}{27})^{m_3} \left[\frac{1}{m_3} + \frac{\frac{8}{27}(m_2-m_1)}{m_3+1} + \frac{\frac{120}{729}+\frac{76}{729}m_3+\frac{32}{729}(m_1-m_2)^2}{m_3+2} + \cdots\right] . \tag{5.6}$$

We see that, indeed, four-point amplitudes exhibit poles for the discrete values of the individual momenta only. Calculating the residues [here we use the kinematic relations (5.14) for $n=3, m=1$] one finally obtains

$$A_{+++-} \propto g^2 \left[\frac{1}{m_3} + \frac{1}{2}\frac{m_2-m_1}{m_3+1} + \frac{1-\frac{1}{2}m_1 m_2}{m_3+2} + \cdots\right] . \tag{5.7}$$

Five particle amplitudes can be calculated in the similar fashion. They are proportional to

$$A \propto g^3 V_{\Phi I \Phi}(k_2,p,k_1) D_{II}(p_1,p_1') V_{I\Phi J}(p_1',k_3,p_2) \times D_{JJ}(p_2,p_2') V_{\Phi J \Phi}(k_5,p_2',k_4) .$$

Thus, to calculate five-point amplitudes, one also needs to know the couplings of an on-shell tachyon to two intermediate off-shell particles, $V_{I\Phi J}$ [we again omit the common factors of $(4/3\sqrt{3})^{\sum_{i=1}^{2}[(p_i+Q/2)^2/2]}\delta(\sum_{i=1}^{2}p_i+k+Q)$]:

$$V_{\Phi\Phi\Phi}(p_1,k,p_2)=1 ,$$
$$V_{\Phi\Phi A}(p_1,k,p_2)=N_{10}^{12}(p_1-k)^{\mu} , \tag{5.8}$$
$$V_{A\Phi A}(p_1,k,p_2)=N_{11}^{12}\delta^{\mu\nu}+(N_{10}^{12})^2(k-p_2)^{\mu}(p_1-k)^{\nu} .$$

In the kinematic region (4.1) the amplitude reads

$$A_{++++-} \propto g^3 \left[\frac{16}{27}\right]^{1-(m_1+m_2)} \left[\frac{16}{27}\right]^{2m_4} \times \left[\frac{1}{1-(m_1+m_2)}\frac{1}{2m_4} + (N_{10}^{12})^2\left[\frac{3(m_3-m_1-m_2)+1-m_4}{[1-(m_1+m_2)](2m_4+1)} + \frac{2(m_2-m_1)}{2m_4[2-(m_1+m_2)]}\right] + 3[N_{11}^{12}-4(N_{10}^{12})^2]\frac{m_2-m_1}{(2m_4+1)[2-(m_1+m_2)]} + \cdots\right] . \tag{5.9}$$

Note that the kinematic constraints (5.14) in this case are not powerful enough to eliminate the two-particle poles, as was the case for the four-point amplitude. It is easy to see that it is a generic trait, for all $N=n+m\geq 5$. Another interesting property of the amplitude (5.9) is the presence of the *fake* poles in it. The point is that apart from the poles which we expect, there are also, on the first sight, poles for the half integer values of m_4. As one can easily check, however, their residues *vanish*. This follows from the kinematics as well as from the properties of the Neumann functions. As an example let us state that $N_{11}^{12}-4(N_{10}^{12})^2=0$ so that the fourth-term in (5.9) vanishes even before taking the residue. The second pole, however, vanishes only after the residue is taken. One can check that generically, both types of cancellations appear for the higher intermediate states. After taking the residues, one gets

$$A_{++++-} = g^3 \left[\frac{1}{1-(m_1+m_2)}\frac{1}{m_4} + \frac{1}{2}\frac{m_2-m_1}{m_4[2-(m_1+m_2)]} + \cdots\right] . \tag{5.10}$$

Let us now briefly summarize the relevant results of Bershadsky and Kutasov [24]. One denotes a tachyon on-shell vertex operator as $T_k^{(\pm)}, T_k^{(\pm)}=\int_\partial d\xi e^{ik_\pm X}$, where

$$k_\pm^\mu=(k,-Q/2\pm k) \ . \tag{5.11}$$

The plus and minus signs in (5.11) correspond to the different chiralities. Correlation functions for the finite boundary and bulk cosmological constant are not known. The bulk amplitudes, which satisfy the condition $\sum_{i=1}^N k_i^\mu+Q^\mu=0$, can be calculated explicitly, however. Namely, one finds that, in that case,

$$A_{\text{open}}(k_1,\dots,k_N)=\propto\int_1^\infty d\xi_{N-1}\int_1^{\xi_{N-1}}\cdots\int_1^{\xi_4}d\xi_3\langle T_{k_1}(0)T_{k_2}(1)T_{k_1}(\xi_3)\cdots T_{k_{N-1}}(\xi_{N-1})T_{k_N}(\infty)\rangle \ . \tag{5.12}$$

Upon performing the integrations (5.12) one sees that the result depends on the kinematics of the tachyons. The only, in principle, nonvanishing amplitudes are of the type

$$\begin{aligned}A^{(n,m)}&=\langle T_{k_1}^{(+)}\cdots T_{k_n}^{(+)}T_{k_{n+1}}^{(-)}\cdots T_{k_{n+m}}^{(-)}\rangle\\&=\prod_{i=1}^{n+m}\frac{1}{\Gamma(1-m_i)}F_n(k_1,\dots,k_n)\\&\qquad\times F_m(k_{n+1},\dots,k_{n+m}) \ ,\end{aligned}\tag{5.13}$$

where m's satisfy the kinematic constraints

$$\sum_{i=1}^n m_i=2-m, \quad \sum_{i=n+1}^{n+m} m_i=2-n \ , \tag{5.14}$$

and the form factors F_n are

$$F_n=\prod_{l=1}^{n-1}\left[\sin\pi\sum_{i=1}^l m_i\right]^{-1} \ . \tag{5.15}$$

To be able to compare (5.13) with (5.7), let us state the explicit form of $A^{(3,1)}$:

$$\begin{aligned}A^{(3,1)}&\propto\prod_{i=1}^4\frac{1}{\Gamma(1-m_i)}\frac{1}{\sin\pi m_1}\frac{1}{\sin\pi(m_1+m_2)}\\&=\frac{\Gamma(m_1)\Gamma(m_3)}{\Gamma(1-m_2)} \ .\end{aligned}\tag{5.16}$$

The poles in m_3 originate from the $\Gamma(m_3)$. Residue in $m_3=0$ is equal to 1. The residue in $m_3=-1$ is

$$(1-m_1)=(m_2-m_1)/2$$

(here we have used $m_1+m_2=2$). The next pole has the residue $\frac{1}{2}(1-m_2)(2-m_2)$, which can be rewritten as $(1-m_1m_2/2)$. One sees that the residues exactly match the ones in (5.7). It is indeed amusing to see how the messy coefficients in (5.5) all end up giving the simple (and correct) final answer (5.7). Of course, such a good agreement is not a coincidence, since the full answer, for the four-point amplitude, was obtained in the second-quantized framework using a different method [26]. Nevertheless, it is also instructive to check the agreement for the five-point functions.

The five-point amplitude $A^{(4,1)}$ is

$$A^{(4,1)}\propto\prod_{i=1}^5\frac{1}{\Gamma(1-m_i)}\frac{1}{\sin\pi m_1}\frac{1}{\sin\pi(m_1+m_2)}\frac{1}{\sin\pi m_4} \ . \tag{5.17}$$

Once again, one can see that the five-point amplitudes exhibit multiparticle, along with the single particle (discrete), poles. The inspection shows that (5.17) has the poles in two independent sets of variables, e.g., m_4 and m_1+m_2, for the negative (positive) integer values, respectively. Taking the corresponding residues in (5.17) it is easy to confirm that they give exactly the same pole structure as (5.10). This, together with the cancellation of the fake poles in (5.10) clearly indicates that the results obtained from the perturbation field theory agree with the first-quantized ones.

To conclude, let us outline the relationship between the method used in the calculations of the correlation functions in this paper and the one used in Ref. [26]. Basically, the conformal mapping method developed in Ref. [15], and used for 2D strings in Ref. [26] calculates the amplitude by transforming the field theoretic answer for the four-point on-shell amplitude into an integral which exactly matches the first-quantized expression. In that sense, it is by construction clear that it reproduces the first-quantized results in a, therefore, somewhat trivial fashion. On the other hand, the field theory written in components does not give, as we have seen, closed expressions for the amplitudes. To get them, one would have to sum over the perturbation series. What the method does provide for, however, is the manifest pole structure of the amplitudes. Origin of poles as coming from the higher string modes becomes transparent. It is a nontrivial check of the formalism that the residues exactly match ones obtained from Ref. [24].

VI. CONCLUDING REMARKS

There are several open questions and directions of possible future investigations. Let us mention three of them here. First, the natural question arises about the destiny of the discrete states upon including the interactions (on the classical and the quantum levels). Second is the important question of constructing the effective tachyonic action starting from Witten's SFT for the open strings and, possibly, connecting it to the collective field theory. Finally, there is the important problem of how (if at all) the cosmological constant can "arise" from the SFT. These are just some of many interesting questions one can ask. We hope, and this was the principal aim of this work, that we have established here the firm ground for these, and other, future investigations.

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