Entropy and action of dilaton black holes

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We present a detailed calculation of the entropy and action of $U(1)^2$ dilaton black holes and show that both quantities coincide with one quarter of the area of the event horizon. Our methods of calculation make it possible to find an explanation of the rule $S = A/4$ for all static, spherically symmetric four-dimensional black holes studied so far. We show that the only contribution to the entropy comes from the extrinsic curvature term at the horizon, which gives $S = A/4$ independently of the charge(s) of the black hole, presence of scalar fields, etc. Previously, this result did not have a general explanation, but was established on a case-by-case basis. The on-shell Lagrangian for maximally supersymmetric extreme dilaton black holes is also calculated and shown to vanish, in agreement with the result obtained by taking the limit of the expression obtained for black holes with regular horizon. The physical meaning of the entropy is discussed in relation to the issue of splitting of extreme black holes.

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I. INTRODUCTION

It has been known for some time that the area of the event horizon of a black hole behaves like the entropy of a thermodynamic system [1, 2]. After Hawking found that the temperature T of the black-hole thermal radiation was related to the surface gravity κ as $T = \frac{\kappa}{2\pi}$, it became clear that the analogy could be made more precise by identifying the entropy with one quarter of the area of the horizon. However, the physical origin of this identification was obscure.

Gibbons and Hawking gave more support to this identification, by performing a direct calculation of the partition function in the saddle-point approximation, obtaining the same result for the black holes known at that time (the Kerr-Newmann family) [3]. A more recent calculation for rotating black holes may be found in [4]. The result is to be interpreted as the intrinsic entropy of the gravitational field, even in the absence of thermal gravitons. The thermodynamics of two-dimensional black holes was studied recently in [5].

However, a general proof of the relation $S = A/4$ was absent. Each time when a new class of black holes was discussed, it was necessary to perform rather complicated calculations anew, and, surprisingly enough, all results obtained so far always supported the simple rule $S = A/4$. In particular, in one of our previous papers we presented the results of our calculation of the action and entropy for the recently discovered family of stringy dilaton black holes [6]. We again found that the entropy is given by one quarter of the area of the event horizon [7]. The purpose of this paper is twofold. First of all, we present all the explicit calculations for the stringy dilaton black holes, making a careful distinction between the entropy and the action. On the other hand, we are giving a general explanation of the rule $S = A/4$, which is applicable to all static, spherically symmetric black holes studied so far. We will use the conventions given in Ref. [7].

In Sec. II we calculate the entropy of a general static spherically symmetric black holes. We generalize Hawking's treatment of Schwarzschild black hole [8] and see that there is no need to calculate the action to get the entropy. In fact, one has to calculate only the contribution of the surface term (the integral of the extrinsic curvature K) on the horizon. All the gauge terms are shown to drop out. We then calculate this term for general static spherically symmetric black holes with regular horizon, and find that it always gives us one quarter of the area of the horizon.

In Sec. III we calculate the Euclidean action for $U(1)^2$ dilaton black holes. First we show that the on-shell bosonic action of dimensionally reduced string theory is a total derivative. Even though the gauge terms do contribute and the extrinsic curvature surface term is calculated at infinity, we get the same result: action equals entropy.

In Sec. IV we calculate the Lagrangian for the $N = 2$ supersymmetric black holes directly as the integrand of the volume integral, keeping all total derivative terms. This calculation explains, from the point of view of restored $O(2)$ symmetry between the two central charges, why the entropy of the maximally supersymmetric black holes vanishes.

The last section contains some discussion of the puzzle

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surrounding the physical meaning of the entropy of extreme dilaton black holes and a possible relation to the issue of splitting of them.

II. ENTROPY

The starting point in the thermodynamic study of a statistical system is the calculation of a thermodynamic function or potential. In the presence of a set of conserved charges C_i and their related potentials μ_i it is convenient to work in the grand canonical ensemble, where the fundamental object is the grand partition function

$$\mathcal{Z} = Tr e^{-\beta(H-\mu_i C_i)} \tag{1}$$

and the thermodynamic potential

$$W = E - TS - \mu_i C_i \ , \tag{2}$$

is related to the grand partition function by

$$e^{-\beta W} = \mathcal{Z} \ . \tag{3}$$

Once $\mathcal{Z}$ is known, all thermodynamic properties of the system can be obtained; for example, the entropy is given by

$$S = \beta(E - \mu_i C_i) + \ln \mathcal{Z} \ . \tag{4}$$

Gibbons and Hawking [3] discovered that the Euclidean partition function of quantum gravity, when evaluated in the saddle-point approximation expanded about one of the black-hole metrics known at that time, can be interpreted as an approximation to the *thermal* grand partition function of a system of temperature equal to the black-hole temperature. In this semiclassical approximation, and from the path integral representation of the partition function, we have

$$\mathcal{Z} = e^{-I_E} \ , \tag{5}$$

where I_E is the Euclidean on-shell action. An important observation made in Ref. [3] is that the Euclidean sections of the complexified metrics studied have only one boundary, spatial infinity $r \to \infty$. The reason is that the region inside the horizon is not present and that the manifold is regular on the horizon, provided that the Euclidean time τ has period $\beta = T^{-1}$. The extrinsic curvature surface term present in I_E has to be calculated only at infinity. We stress this fact by using the notation I_E^∞,

$$\ln \mathcal{Z} = -I_E^\infty \ . \tag{6}$$

Moreover, it was found that the action of the black-hole metric was equal to the entropy calculated using Eq. (4), and both were equal to one quarter of the area of the event horizon, the value suggested by the first law of black-hole thermodynamics [2].

The fact that the action coincides with the entropy was explained by Hawking using scaling arguments in Ref. [8]. In the same reference he also gave a prescription to calculate independently the term βE in the same approximation for the Schwarzschild case in which there are no charges involved. To calculate the (mean value of the) energy one considers the following equation for the amplitude for imaginary time evolution between two surfaces of Euclidean time τ_1 and τ_2 with given boundary conditions:

$$\langle \tau_1 | \tau_2 \rangle = e^{-(\tau_2-\tau_1)E} \ . \tag{7}$$

To justify this equation one should remember the standard quantum-mechanical relation $|t_1\rangle = e^{-i(t_1-t_2)H}|t_2\rangle$, which yields

$$\langle t_1 | t_2 \rangle = \langle t_1 | e^{-i(t_2-t_1)H} | t_1 \rangle \ . \tag{8}$$

Let us assume that the state $|t_1\rangle$ describes a thermodynamic system with the mean energy $\langle E\rangle = \langle t_1|H|t_1\rangle$. If the system is large and the fluctuations of its energy are relatively small, so that $\frac{\langle E\rangle^2 - \langle E^2\rangle}{\langle E\rangle^2} \ll 1$, then one may expand Eq. (8) neglecting these fluctuations. This gives

$$\langle t_1 | t_2 \rangle = e^{-i(t_2-t_1)\langle E\rangle} \ . \tag{9}$$

Equation (7) is the Euclidean version of (9) with $E \equiv \langle E\rangle$.

The dominant contribution to the path integral corresponding to a black hole comes from the stationary phase metric, which in this case is the Schwarzschild metric. Thus to find the energy E we have to calculate the action while taking into account the contribution to the surface term from the horizon as well as that from infinity. The reason for needing both surfaces is that, in the Schwarzschild geometry, the time translation Killing vector $\frac{\partial}{\partial\tau}$ is zero on the horizon, and so the surfaces of constant τ have two boundaries: one at the horizon and the other at infinity. So we have, for $\tau_2 - \tau_1 = \beta$,

$$\beta E = I_{E,h}^\infty \ . \tag{10}$$

Again, the notation $I_{E,h}^\infty$ stresses the fact that both the horizon and the surface at infinity contribute to the surface term. For the Schwarzschild case $E = M$, where M is the mass of the black hole. The condition $\frac{\langle E\rangle^2 - \langle E^2\rangle}{\langle E\rangle^2} \ll 1$ is satisfied for large Schwarzschild black holes with mass $M \gg M_p$, since in this case $\langle E\rangle^2 = M^2$, and $\langle E\rangle^2 - \langle E^2\rangle = \frac{M_p^2}{8\pi}$.

In the presence of conserved charges, one can consider constrained imaginary time evolution so that only metrics with the prescribed charges are considered in the path integral. This can be implemented by using Lagrange multipliers μ_i. The generalization of Eqs. (7) and (10) in our case is

$$\langle \tau_1 | \tau_2 \rangle = e^{-(\tau_2-\tau_1)(E-\mu_i C_i)} \tag{11}$$

and

$$\beta(E - \mu_i C_i) = I_{E,h}^\infty \ . \tag{12}$$

Now, if we substitute Eqs. (6) and (12) into (4) we get

$$S = I_{E,h}^\infty - I_E^\infty \ . \tag{13}$$

These two terms are explicitly given by

$$I^{\infty}_{E,h} = \frac{1}{16\pi}\int_M(-R+\mathcal{L}_{\rm matter}) + \frac{1}{8\pi}\int_h^{\infty}[K] \ ,$$
$$I^{\infty}_{E} = \frac{1}{16\pi}\int_M(-R+\mathcal{L}_{\rm matter}) + \frac{1}{8\pi}\int^{\infty}[K] \ , \tag{14}$$

and finally, substituting Eqs. (14) into (13), we have for the entropy of a general black hole

$$S = \frac{1}{8\pi}\int_h [K] \tag{15}$$

i.e., simply the *extrinsic curvature surface term at the horizon*. This is a remarkable fact that emphasizes the intrinsic gravitational nature of the entropy so calculated.

The next step is to calculate $[K]$ for a sufficiently general case. For us it will be the general case of a static, spherically symmetric asymptotically flat black hole. Extreme purely electric and magnetic black holes have no regular horizon and we will treat them in the last section. Let us start with the definition of K, the trace of the extrinsic curvature of the hypersurface with (spacelike) unit normal vector n^{μ}

$$K = h^{\mu\lambda}\nabla_{\mu}n_{\lambda} \ , \tag{16}$$

where

$$h_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_{\nu} \tag{17}$$

is the induced metric on the hypersurface. The term $[K]$ in Eq. (15) is the difference $K - K_0$, where K_0 is obtained by substituting into K the flat space metric. An infinite contribution is subtracted in this way, so that we obtain finite results for the action. However, were the results obtained without the subtraction of K_0 finite, it would not be needed.

For a spherically symmetric metric

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 - r^2 d\Omega^2 \ , \tag{18}$$

we find that

$$K = \frac{1}{\sqrt{-g_{rr}}}\left[\frac{1}{2}\frac{\partial_r g_{tt}}{g_{tt}} + \frac{2}{r}\right] \quad \text{and} \quad K_0 = \frac{2}{r} \ , \tag{19}$$

for surfaces of constant r. Recalling that the Euclidean time τ is compactified on $[0,\beta]$, we obtain for the surface integral calculated with outward-pointing n^{μ}

$$\frac{1}{8\pi}\int_r [K] = -\frac{1}{\kappa}\left(\frac{A(r)}{4}\right)\left[\frac{1}{2}\frac{\partial_r g_{tt}}{\sqrt{-g_{rr}g_{tt}}} + \frac{2}{r}\sqrt{\frac{g_{tt}}{-g_{rr}}} - \frac{2}{r}\sqrt{g_{tt}}\right] \ , \tag{20}$$

where the surface gravity $\kappa = \frac{2\pi}{\beta}$ is given by

$$\kappa = \frac{1}{2}\frac{\partial_r g_{tt}}{\sqrt{-g_{rr}g_{tt}}}\bigg|_{r=r_h} \ . \tag{21}$$

If the horizon occurs at a finite value of r, which we denote by r_h, and $g_{tt} = 0 = \frac{g_{tt}}{g_{rr}}$, we see that the term $\frac{1}{\kappa}$ coming from the τ integration is canceled by the first term in the square brackets by using expression (21) for the surface gravity, yielding

$$S = -\frac{1}{8\pi}\int_h [K] = \frac{A(r_h)}{4} \ . \tag{22}$$

Thus we have found that the entropy is again one quarter of the area of the event horizon. This result has been established for the general class of static spherically symmetric black holes (in particular, this includes charged axion-dilaton black holes) with metric given in Eq. (18).

III. ACTION

A. The on-shell action for axion-dilaton black holes is topological

The total (bosonic) action for stringy $d = 4$ dilaton-axion black holes is given by the volume integral

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,(\mathcal{L}_{\rm grav} + \mathcal{L}_{\rm dil} + \mathcal{L}_{\rm axion} + \mathcal{L}_{\rm gauge}) \ . \tag{23}$$

We will follow the presentation of the gravitational part of the action as given in [9], but using the notation of [7]:

$$\sqrt{-g}\,\mathcal{L}_{\rm grav} = \sqrt{-g}(-R) + \partial_{\mu}\sqrt{-g}\,\omega^{\mu} \ , \tag{24}$$

where the vector ω^{μ} in the total derivative term in the gravitational Lagrangian is

$$\omega^{\mu} = g^{\lambda\rho}\Gamma^{\mu}_{\lambda\rho} - g^{\lambda\mu}\Gamma^{\nu}_{\lambda\nu} \ . \tag{25}$$

In general, an infinite contribution (as the K_0 term of the previous section) will have to be subtracted from Eq. (23) in order to obtain finite results.

In the SO(4) version of the action of $N = 4, d = 4$ supergravity, or dimensionally reduced string theory,

$$\mathcal{L}_{\rm dil} = 2\partial^{\mu}\phi\partial_{\mu}\phi \ , \tag{26}$$
$$\mathcal{L}_{\rm axion} = \tfrac{1}{2}e^{4\phi}\partial^{\mu}a\partial_{\mu}a \ , \tag{27}$$
$$\mathcal{L}_{\rm gauge} = -\left(e^{-2\phi}F^{n}_{\mu\nu}F^{n\,\mu\nu} + e^{2\phi}\tilde{G}^{n}_{\mu\nu}\tilde{G}^{n\,\mu\nu}\right) + i\,a\left(F^{n}_{\mu\nu} * F^{n\,\mu\nu} + \tilde{G}^{n}_{\mu\nu} * \tilde{G}^{n\,\mu\nu}\right) \ , \tag{28}$$

where $g_{\mu\nu}, \phi, a, A^{n}_{\mu}, \tilde{B}^{n}_{\mu}, n = 1,2,3$, are the metric, dilaton, axion, and six vector fields. In order to calculate the on-shell action we will need the following equations of motion:

$$-R + 2\partial^{\mu}\phi\partial\mu\phi + \tfrac{1}{2}e^{4\phi}\partial^{\mu}a\partial_{\mu}a = 0 \ , \tag{29}$$

$$\nabla_{\mu}(e^{-2\phi}F^{n\,\mu\nu} - ia * F^{n\,\mu\nu}) = \partial_{\mu}\left[\sqrt{-g}(e^{-2\phi}F^{n\,\mu\nu} - ia * F^{n\,\mu\nu})\right] = 0 \, , \tag{30}$$

and the corresponding equations for $\tilde{G}^{n\,\mu\nu}$.

If we use Eqs. (29) and (30) in the action (23) we end up with the following on-shell action, which is a total derivative:

$$I_{\text{on-shell}} = \frac{1}{16\pi}\int d^4x\ \partial_\mu\{\sqrt{-g}\,[\omega^\mu + A^n_\nu(e^{-2\phi}F^{n\,\mu\nu} - ia * F^{n\,\mu\nu}) + \tilde{B}^n_\nu(e^{+2\phi}\tilde{G}^{n\,\mu\nu} - ia * \tilde{G}^{n\,\mu\nu})]\}\ . \tag{31}$$

An equivalent form of the on-shell action can be given in terms of differential forms. The gauge part of the on-shell action was calculated for Einstein-Maxwell theory in [10] and found to be an exact differential form; we generalize that procedure here. In addition we will need the gravitation action as a form. For this purpose one may start with the gravitational action in terms of tetrad and spin connection forms

$$\int \sqrt{-g}\,\mathcal{L}_{\text{grav}} = \int e_a\wedge e_b\wedge *R^{ab} - d\,(e_a\wedge e_b\wedge *\omega^{ab})\ . \tag{32}$$

The on-shell action takes the form

$$I_{\text{on-shell}} = \frac{1}{16\pi}\text{Tr}\int d\left[\ *\omega + A\wedge(e^{-2\phi} * F - iaF) + \tilde{B}\wedge(e^{+2\phi} * \tilde{G} - ia\tilde{G})\right]\ , \tag{33}$$

where Tr on the vector fields means sum over all vector fields and

$$\text{Tr} * \omega = (e_a\wedge e_b\wedge *\omega^{ab})\ . \tag{34}$$

Thus we have shown that the on-shell bosonic part of the SO(4) supergravity action is an integral over the exact differential form $d\Xi$, where

$$\Xi = \frac{1}{16\pi}\text{Tr}\left[*\omega + A\wedge(e^{-2\phi}*F - iaF) + \tilde{B}\wedge(e^{+2\phi}*\tilde{G} - ia\tilde{G})\right]. \tag{35}$$

Thus, using Gauss's theorem, we see that

$$I_{\text{on-shell}} = \int_{\partial M}\Xi\ , \tag{36}$$

where ∂M is the boundary of M.

B. Gibbons-Hawking-type calculation of the Euclidean action

To perform the explicit calculation of the Euclidean action for the $U(1)^2$ black holes described in Ref. [7] we only have to evaluate Eq. (31). However, since in general the answer would be infinite, we have to subtract the K_0 term described in the first section [3]. For convenience we switch from ω to K; the quantity to be evaluated is[1]

$$I^\infty_{E} = I_{E}(\text{gauge}) + I^\infty_{E}([K]) = \frac{1}{16\pi}\int d^4x\ \partial_\mu[\sqrt{g}\,[A_\nu(e^{-2\phi}F^{\mu\nu}) + \tilde{B}_\nu(e^{+2\phi}\tilde{G}^{\mu\nu})]] + \frac{1}{8\pi}\int^{r\to\infty} d^3x\sqrt{h}(K-K_0), \tag{37}$$

where the coordinates x^μ, the metric $g_{\mu\nu}$, etc., are now Euclidean objects and the superscript ∞ means again that the only boundary of these spacetimes is at $r\to\infty$. The reason is the same as in the Schwarzschild case and was explained in the first section.

We start by calculating the extrinsic curvature term. The dilaton black-hole metric is given by [7]

$$ds^2 = e^{2U}dt^2 - e^{-2U}dr^2 - R^2d\Omega^2\ , \tag{38}$$

$$e^{2U} = \frac{(r-r_+)(r-r_-)}{R^2}\ , \tag{39}$$

$$R^2 = r^2 - \Sigma^2\ , \tag{40}$$

$$r_\pm = M \pm r_0\ . \tag{41}$$

For the metric (38) the radius of the local two-sphere is R, rather than r. The metric may be rearranged as

$$ds^2 = e^{2U}dt^2 - e^{-2U}\left(\frac{dr}{dR}\right)^2 dR^2 - R^2d\Omega^2, \tag{42}$$

so that

$$K = \frac{r[(r-r_+)(r-r_-)]^{\frac{1}{2}}}{R^2}\left[\frac{(r-M)R}{r(r-r_+)(r-r_-)} + \frac{1}{R}\right] \tag{43}$$

and

$$\frac{1}{8\pi}\int^r d^3x(K-K_0) = -\frac{\pi}{\kappa}\left[r - M + \frac{r(r-r_+)(r-r_-)}{R^2} - 2[(r-r_+)(r-r_-)]^{\frac{1}{2}}\right]\ . \tag{44}$$

[1]There is no axion, and we have only one F and one $\tilde{G}$ field.

Evaluating the limit of this expression at $r \to \infty$ we get

$$I_E^\infty(K) = +\frac{\pi}{\kappa} M\,. \tag{45}$$

We now proceed to the evaluation of $I_E(\text{gauge})$. The gauge and dilaton fields are given by

$$F = \frac{Qe^{\phi_0}}{(r-\Sigma)^2} dt \wedge dr\ , \qquad A = \frac{Qe^{\phi_0}}{(r-\Sigma)} dt\ , \tag{46}$$

$$\tilde{G} = \frac{Pe^{-\phi_0}}{(r+\Sigma)^2} dt \wedge dr\ , \qquad \tilde{B} = \frac{Pe^{-\phi_0}}{(r+\Sigma)} dt\ , \tag{47}$$

$$e^{2\phi} = e^{2\phi_0} \frac{r+\Sigma}{r-\Sigma}\,. \tag{48}$$

There is a subtlety involved in evaluation of the surface integral of the gauge terms. Gibbons and Hawking argue in their treatment of the Reissner-Nordström black hole that, since the gauge potentials are singular on the event horizon $r_h = r_+$ (due to the vanishing of g_{tt}), one must make a gauge transformation to render them zero there,

$$A'_\mu(r) = A_\mu(r) - A_\mu(r_h)\,. \tag{49}$$

The gauge integrals become

$$I_E(\text{gauge}) = \lim_{r\to\infty}\left[\ \left[Q^2\left(\frac{1}{r-\Sigma} - \frac{1}{r_+-\Sigma}\right) + P^2\left(\frac{1}{r+\Sigma} - \frac{1}{r_++\Sigma}\right)\right] \times \int_0^{\frac{2\pi}{\kappa}} \frac{d\tau}{2} \int \frac{d\Omega}{4\pi}\right]. \tag{50}$$

In fact, we have found that the prescription (49) may be thought of in another way, which ends up producing the same result.

If we do perform the gauge transformation (49), it is clear that there is no contribution from the horizon to $I_E(\text{gauge})$, since the gauge-transformed vector potentials and therefore the integrand vanish on it. If, on the other hand, we do not wish to perform such a gauge transformation, then we must make a careful consideration of Gauss's theorem. The surface integral has to be calculated on the boundary of the region in which the potentials are well behaved and defined. In the case at hand, this boundary includes the horizon, and the integrand no longer vanishes there. Nevertheless, it turns out that the functional dependencies of the gauge and dilaton fields conspire in such a way as to reproduce the previous result (50). To see this, consider the surface integral for the F term:

$$-\frac{1}{8\pi}\int_{\partial M} d\sigma\, n_\mu \left[e^{-2\phi}F^{\mu\nu}A_\nu\right] = \frac{1}{8\pi}\int_{r_+}^{r\to\infty} dt\, d\Omega \left[R^2 e^{-2\phi_0}\left(\frac{r-\Sigma}{r+\Sigma}\right)\frac{Qe^{\phi_0}}{(r-\Sigma)^2}\frac{Qe^{\phi_0}}{(r-\Sigma)}\right]$$
$$= \int_{r_+}^{r\to\infty} \frac{d\tau}{2}\frac{d\Omega}{4\pi}\left[\frac{Q^2}{(r-\Sigma)}\right]. \tag{51}$$

A similar thing happens for the G term. Looking at Eq. (51) we see that the result for the surface integral is simply

$$I_E(\text{gauge}) = -\frac{\pi}{\kappa}\left[\frac{Q^2}{(r_+-\Sigma)} + \frac{P^2}{(r_++\Sigma)}\right], \tag{52}$$

which coincides with Eq. (50) obtained by doing the gauge transformation demanded by Gibbons and Hawking.

Finally, putting together Eqs. (45) and (52) for the extrinsic curvature term at infinity and the gauge terms we get

$$I_E^\infty = \frac{\pi}{\kappa}\left[M - \frac{Q^2}{r_+-\Sigma} - \frac{P^2}{r_++\Sigma}\right], \tag{53}$$

which may be rearranged using the relations

$$\Sigma = \frac{P^2-Q^2}{2M}\,, \qquad r_0^2 = M^2 + \Sigma^2 - P^2 - Q^2, \tag{54}$$

to give

$$I_E^\infty = \frac{\pi}{\kappa} r_0 = \pi(r_+^2 - \Sigma^2) = \frac{1}{4}A(r_h)\,. \tag{55}$$

For extreme dilaton black holes, this expression reduces to

$$I_E^\infty(\text{extreme}) = 2\pi|PQ|\,. \tag{56}$$

Thus the method of Gibbons and Hawking, generalized to dilaton black holes, gives the result that the on-shell action coincides with the entropy and is one quarter of the area of the event horizon.[2]

IV. ON-SHELL LAGRANGIAN FOR EXTREME $N = 2$ BLACK HOLES

In Sec. III, the generalization of the Gibbons-Hawking method of the calculation of the Euclidean action for dila-

[2]For $PQ = 0$, the result of the calculation of the entropy was given previously in [11]. It agrees (at $a = 1$) with our expression (55) taken in the appropriate limit.

ton black holes was presented. One starts with nonextreme black holes characterized by some finite temperature and surface gravity, and performs the calculation of the on-shell action in Euclidean signature by compactifying the Euclidean time coordinate. It turns out that one can express the total action as a surface integral, and care has to be taken to evaluate the contribution from the extrinsic curvature only from spatial infinity. As a final step, the extremal limit can be considered, and the result is

$$S_{\text{stringy}} = \tfrac{1}{4}A = \pi(M^2 - \Sigma^2) = \tfrac{1}{2}\pi|z_1^2 - z_2^2| \;, \qquad (57)$$

where z_1, z_2 are the central charges of extreme black holes defined in [7]. This shows that when $N = 2$ supersymmetry is restored, which takes place when the central charges are equal,

$$|z_1| = |z_2| \;, \qquad (58)$$

the action vanishes. This was never possible for classical extreme Reissner-Nordström black holes. Indeed those black holes are solutions of $N = 2$ supergravity, which are characterized by a super-Poincaré algebra at infinity with only one central charge[3]

$$S_{\text{RN}} = \tfrac{1}{4}A = \pi M^2 = \tfrac{1}{2}\pi|z^2| \;. \qquad (59)$$

The stringy black holes are solutions of $N = 4$ supergravity, and the restoration of $N = 2$ supersymmetry is the restoration of O(2) internal symmetry, which makes the two central charges equal and the action (entropy) vanish in agreement with the fact that the area of the horizon for these solutions in the canonical geometry is zero.

Since the presence of the dilaton has radically changed the properties of extreme black holes, it becomes possible to address the following problem: Could we calculate the partition function for the extreme dilatonic black hole *directly*, avoiding the intermediate step of introducing the concept of a temperature at all? The answer is positive for maximally supersymmetric purely magnetic (electric) extreme black holes, as we will now show.

Our starting point for the calculation of the on-shell action will be the Lagrangian in

$$I = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left[\,-R + 2\partial^\mu\phi\partial_\mu\phi - \left(e^{-2\phi}F^{\mu\nu}F_{\mu\nu} + e^{2\phi}\tilde{G}^{\mu\nu}\tilde{G}_{\mu\nu}\right)\right] , \qquad (60)$$

with the additional K term which removes the second derivatives of the metric from the Lagrangian. The gravitational part of the action is given by Eq. (24) and the vector ω^μ in the total derivative term in the Lagrangian is given by Eq. (25). Equation (25) can be also given in the form

[3]The maximum number of central charges is $N/2$ for even N.

$$\omega^\mu = -\frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}g^{\lambda\mu}\right) - g^{\lambda\mu}\left(\partial_\lambda \ln\sqrt{-g}\right) \;. \qquad (61)$$

For our calculations there will be no need to rewrite the volume integral for the total derivative part in the Lagrangian [second term in Eq. (24)] as a surface integral (K term). Also we will not transform the gauge part of the action to a surface integral as we did in the previous section. It will be sufficient to keep all terms in a volume integral in what follows.

The dilaton part of the Lagrangian is

$$\sqrt{-g}\,\mathcal{L}_{\text{dil}} = \sqrt{-g}\,2\partial^\mu\phi\partial_\mu\phi \;. \qquad (62)$$

The gauge part of the Lagrangian for the purely magnetic solution is

$$\sqrt{-g}\,\mathcal{L}_{\text{gauge}} = \sqrt{-g}\,\mathcal{L}_{\text{magn}} = -\sqrt{-g}\,e^{2\phi}\tilde{G}^{\mu\nu}\tilde{G}_{\mu\nu} \;, \qquad (63)$$

and for the purely electric

$$\sqrt{-g}\,\mathcal{L}_{\text{gauge}} = \sqrt{-g}\,\mathcal{L}_{\text{electr}} = -\sqrt{-g}\,e^{-2\phi}F^{\mu\nu}F_{\mu\nu} \;. \qquad (64)$$

The total action is given by the volume integral

$$16\pi I = \int d^4x\,\sqrt{-g}\,\left(\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{dil}} + \mathcal{L}_{\text{gauge}}\right) \;. \qquad (65)$$

The maximally supersymmetric purely magnetic (electric) extreme black holes are described by the metric [7]

$$ds^2 = e^{2U}dt^2 - e^{-2U}d\mathbf{x}^2 \;. \qquad (66)$$

Before using the field equations let us calculate the total derivative term in the gravitational part of the Lagrangian for the ansatz (66). We find using Eq. (61) that

$$\partial_\mu\left(\sqrt{-g}\,\omega^\mu\right) = -2\,\partial_i\partial_i U \;. \qquad (67)$$

The total gravitational part of the Lagrangian becomes

$$\sqrt{-g}\,\mathcal{L}_{\text{grav}} = \sqrt{-g}\,(-R) - 2\,\partial_i\partial_i U \;. \qquad (68)$$

At this stage we may start taking the equations of motion into account. The dilaton for maximally supersymmetric extreme black holes is related to the metric as

$$\phi = \pm U \;, \qquad (69)$$

where $(-)$ is for magnetic and $(+)$ for electric solution. The first equation of motion which will be used to calculate the on-shell Lagrangian is the one which relates the scalar curvature to the dilaton contribution, see Eq. (29). It follows that, on-shell,

$$\sqrt{-g}\,\left(\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{dil}}\right) = -2\,\partial_i\partial_i U \;. \qquad (70)$$

To treat the gauge action we have to use another equation of motion:

$$\nabla^2\phi - \tfrac{1}{2}e^{-2\phi}F^2 + \tfrac{1}{2}e^{2\phi}\tilde{G}^2 = 0 \;. \qquad (71)$$

For the electric solution with $U = \phi$ it leads to

$$\sqrt{-g}\,\mathcal{L}_{\text{electr}} = 2\,\partial_i\partial_i\phi \qquad (72)$$

and the total on-shell Lagrangian becomes

$$\sqrt{-g}\,\mathcal{L} = -2\,\partial_i\partial_i U + 2\,\partial_i\partial_i\phi = 0 \;. \tag{73}$$

For the magnetic solution with $U = -\phi$

$$\sqrt{-g}\,\mathcal{L}_{\rm magn} = -2\,\partial_i\partial_i\phi \tag{74}$$

and the total on-shell Lagrangian becomes

$$\sqrt{-g}\,\mathcal{L} = -2\,\partial_i\partial_i U - 2\,\partial_i\partial_i\phi = 0 \;. \tag{75}$$

Thus the Lagrangian and therefore the action vanishes for $PQ = 0$ dilaton black holes, in agreement with Eq. (55). In other words, we get the same result by evaluating the action directly as we get by taking the $PQ = 0$ limit of the expression for black holes with regular horizons.

Note that, in the process of calculation of the on-shell Lagrangian for maximally supersymmetric extreme dilatonic black holes, we never faced the problem of going to Euclidean signature, choosing a proper gauge for the vector potentials, and thinking about boundary surfaces (the horizon versus infinity). All of those problems were present in a standard treatment and, as explained in the previous section, can all be solved in quite satisfactory ways. It is therefore encouraging that an independent calculation of the action exists, as given above for the extreme purely magnetic (electric) black holes that is consistent with the general formula that the action is one quarter of the area of the event horizon.

V. DISCUSSION

In this paper we have found that the entropy of general static spherically symmetric black holes with a regular event horizon is given by evaluating only the extrinsic curvature term at the horizon and is one quarter of the area of the event horizon. This generalizes the corresponding result derived for the Schwarzschild black hole by Hawking in [8].

For charged dilaton black holes we have performed the calculation both of the action and of the entropy by following Gibbons and Hawking [3]. In calculating the on-shell bosonic action for the theory in which the dilaton black hole is embedded, we have seen that it is topological and thus may be written as a surface integral. We have found that the entropy coincides with the on-shell action, in agreement with what one might expect from scaling arguments as in [8]. Investigation of the action versus entropy of axion-dilaton black holes [12] is in progress.

A remaining puzzle is the physical origin of the entropy of $\mathrm{U}(1)^2$ dilaton black holes. Extreme dilaton black holes, which could be the stable end points of the evaporation process, may be thought of as "ground states". In the theory we consider, the charges P and Q are central charges [in different U(1) groups], and there are no elementary charged particles to discharge the black hole. These black holes also have zero temperature and unbroken $N = 1$ supersymmetry [7], but the entropy is nonzero. For these extreme black holes, the entropy is given by Eq. (56), $S = 2\pi|PQ|$. In [7] we formulated a supersymmetric nonrenormalization theorem which says that the result (56) remains intact to higher-order (perturbative) corrections in the supersymmetric theory.

In a quantum-mechanical system, entropy at zero temperature usually corresponds to degeneracy of the ground state. However, for the charged dilaton black holes the relation between the entropy and the degeneracy of these configurations is missing: what kind of "internal" degrees of freedom does the degeneracy correspond to? Since the degeneracy of a quantum ground state is an integer,[4] one may then be tempted to speculate that the entropy of extreme black holes is subject to the quantization rule

$$S = \frac{A}{4} = 2\pi|PQ| = \ln(n) \;, \tag{76}$$

where n is an integer and the area is measured in Planck units. The size of the horizon is then not arbitrary, but restricted by the rule (76). Then purely magnetic or purely electric black holes have $n = 1$ and are clearly allowed; they already have zero entropy and area.

Another possibility is to take seriously the fact that the quantity $e^{A/4}$ is generically not an integer, and that we do not know about the existence of any internal degrees of freedom of the extreme electric-magnetic black holes responsible for the degeneracy of the state. Therefore, a possible conclusion is that this state is *not* a ground state of a quantum-mechanical system, having noninteger e^S. Then we are led to a resolution of the problem: quantum mechanically the extreme electric-magnetic black holes have to be unstable under splitting to another configuration of extreme black holes which *is* a ground state and does have an integer value $n = 1$.

The possibility that black holes may quantum mechanically split into other black holes was proposed in [7].[5] A specific example appropriate for the issue of the ground state would be the splitting of the extreme electric-magnetic black hole into a purely magnetic and a purely electric one. Such bifurcation is forbidden classically but could in principle occur in a quantum-mechanical process and may be enforced by quantum-mechanical instability of the zero temperature state with noninteger value of e^S. It can be described by

$$(P, Q) \rightarrow (P, 0) + (0, Q) \;. \tag{77}$$

The extreme electric-magnetic black hole has the following relations between parameters:

$$M = \frac{|P| + |Q|}{\sqrt{2}} \;, \quad \Sigma = \frac{|P| - |Q|}{\sqrt{2}} \;, \tag{78}$$

which gives

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[4] We thank L. Susskind for discussions on this point.

[5] Splitting of black holes is closely related to the possibility of splitting of the universe into many baby universes. A particularly relevant example is splitting of one Robinson-Bertotti (RB) universe into many RB universes, as discussed by Brill [10]. For a recent discussion of splitting of dilaton black holes with massive dilaton fields see [13].

$$M^2 + \Sigma^2 - P^2 - Q^2 = 0 \ . \tag{79}$$

The parameters of the daughter black holes are related to those of the parent as

$$M = M_1 + M_2 \ , \quad M_1 = \Sigma_1 = \frac{|P|}{\sqrt{2}} \ ,$$
$$\Sigma = \Sigma_1 + \Sigma_2 \ , \quad M_2 = -\Sigma_2 = \frac{|Q|}{\sqrt{2}} \ . \tag{80}$$

After splitting the total entropy is equal to zero, and

$$M_1^2 + \Sigma_1^2 - P^2 = 0, \quad M_2^2 + \Sigma_2^2 - Q^2 = 0 \ . \tag{81}$$

These black holes are in an equilibrium with each other, since the attractive force between them vanishes due to supersymmetry [7]. Indeed, let us consider Newtonian, Coulomb, and dilatonic forces. The force between two distant objects of masses and charges $(M_1, Q_1, P_1, \Sigma_1)$ and $(M_2, Q_2, P_2, \Sigma_2)$ is

$$F_{12} = -\frac{M_1 M_2}{r_{12}^2} + \frac{Q_1 Q_2}{r_{12}^2} + \frac{P_1 P_2}{r_{12}^2} - \frac{\Sigma_1 \Sigma_2}{r_{12}^2} \ . \tag{82}$$

The dilatonic force is attractive for charges of the same sign and repulsive for charges of opposite sign. Using the relations (80) for the masses and dilaton charges in terms of the magnetic and electric charges $P_1 = P$, $P_2 = 0$, $Q_1 = 0$, $Q_2 = Q$, we see that F_{12} vanishes.

Thus, after splitting a possible ground state of the quantum-mechanical system is reached which could be a pure state with $S = 0$.[6] The area of the horizon of both black holes is now zero. Equation (80) describes the distribution of masses and charges in a particular example of the general extreme supersymmetric multi-black-hole solution, given in [7]. Will the purely electric and purely magnetic black holes continue splitting to the smallest values of charges? Is the value of the entropy $S = 2\pi|PQ|$ responsible for the degeneracy properties of the ground state? These and many other questions can be asked in connection with the calculated value of the entropy of charged dilatonic black holes.

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[6] Another example of a solution with zero entropy, is the supersymmetric domain wall [14]. The authors argue that it describes a nondegenerate ground state.

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