Gravitational radiation from a particle in circular orbit around a black hole. III. Stability of circular orbits under radiation reaction

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We use the Teukolsky perturbation formalism to show that (i) a particle in circular motion around a nonrotating black hole remains in a circular orbit under the influence of radiation reaction, and (ii) circular orbits are stable only if the orbital radius is greater than a critical radius $r_c \simeq 6.6792M$, where M is the mass of the black hole. A circular orbit is stable if, when slightly perturbed so that it acquires a small eccentricity, radiation reaction decreases the eccentricity; a circular orbit is unstable if radiation reaction increases the eccentricity. Our analysis is restricted by four major assumptions: (i) the black hole is nonrotating, (ii) the eccentricity is always small, (iii) the gravitational perturbations are linear, and (iv) the adiabatic approximation (that radiation reaction takes place over a time scale much larger than the orbital period) is valid. On the other hand, our analysis is not limited to weak-field, slow-motion situations; it is valid for particle motion in strong gravitational fields.

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I. INTRODUCTION AND SUMMARY

A. Motivation

A particle of mass μ, which interacts with the gravitational field of an isolated object of mass M, does not, in general, move on a spacetime geodesic. This is due to the fact that the combined system emits gravitational waves; the problem of radiation reaction—to determine the influence of this emission on the motion of the particle—is a difficult one in general relativity.

Gravitational radiation reaction has a well-known electromagnetic analogue: A charged particle, accelerated by an electric field, does not move according to the Lorentz equations of motion, because of the emission of electromagnetic waves. There are difficult conceptual problems associated with radiation reaction in electromagnetism [1]; however, these conceptual problems are not a serious impediment to computations, at least when radiation reaction is a small effect. The use of half retarded minus half advanced potentials, together with the rejection of runaway solutions on physical grounds, provide a well-defined calculational basis for most applications [2].

In contrast with the electromagnetic case, the problem of gravitational radiation reaction is plagued with conceptual *and* calculational difficulties, which are mostly due to the nonlocal character of the problem. Nonlocality enters in essentially two different ways. (i) As a consequence of the principle of equivalence, a gravitational wave can be identified as such only in a region of spacetime larger than several wavelengths [3], which precludes the construction of a local radiative field. (ii) Because gravitational waves are in general scattered by the curvature of spacetime, waves emitted at one time may influence the motion of the particle at some later time [4]; these *tails* in the waves can produce noticeable effects, most especially if the curvature is large.

In order to gain insight into the general problem of gravitational radiation reaction, it is important to look at simple special cases for which the above problems can be addressed. To study such a simple case is the main purpose of this paper.

The question of radiation reaction is most pressing in the context of the late evolution of compact binary systems [5], since the waves generated by such systems are the most promising for detection by kilometer-size interferometric detectors [6]. Extraction of the information encoded in the waves will require an accurate calculation of the expected wave forms [7]; these theoretical wave forms are used as matched filters through which the detected signal is processed [8]. Radiation reaction governs the rate at which the wave frequency increases with time, as the compact objects spiral together toward coalescence. During the last stages of evolution, when the waves are most interesting for detection, the wave frequency sweeps from approximately 10 Hz to several hundred Hz in just a few minutes, during which the waves oscillate about 10^4 times. It is therefore essential to incorporate radiation reaction, to a fractional accuracy of at least 10^{-4}, into the calculation of the theoretical wave forms [7]. Thus, the practical importance of radiation reaction in the evolution of compact binary systems provides more motivation for the work presented here.

Also of interest are the last stages of evolution, under radiation reaction, of a solar-mass compact object orbiting a galactic, supermassive black hole. Such a binary system could be observed with an eventual space-based

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interferometric detector, which would operate between 10^{-4} Hz and 10^{-1} Hz [9]. Because we consider small mass ratios (see Sec. I B below), the results presented in this paper are directly relevant to these sources.

Most of the work devoted so far to gravitational radiation reaction, in particular for the two-body problem, has been restricted to weak-field, slow-motion situations [3, 10–13]. Lincoln and Will [12] have calculated, using post-Newtonian theory, the orbital motion of a binary system at post$^{5/2}$-Newtonian order, which only incorporates radiation reaction at leading order. These calculations are not accurate enough for the purpose of constructing matched filters for interferometric detectors [7]. Higher-order corrections to the post-Newtonian, radiation-reaction force have recently been calculated by Iyer and Will [13].

By comparison, very little has been done for strong-field situations. Gal'tsov [14] has laid the foundations for strong-field radiation-reaction calculations in the case of particle motion in the field of a Kerr black hole. His formalism is based on the notion of a local, gauge-dependent radiation-reaction force. However, Gal'tsov's only explicit calculation of this force was also restricted to weak-field, slow-motion situations. Anderson and Ori, Finn, Ori, and Thorne [15] have studied the strong-field transition between inspiral and plunge motion in Schwarzchild (and in the equatorial plane of Kerr); however, their analysis does not require a detailed knowledge of radiation-reaction effects. In this paper we present concrete results on radiation reaction in strong-field situations.

B. The problem, method of solution, and approximations

We study the effects of radiation reaction on the bound motion of a particle of mass μ in the geometry of a Schwarzschild black hole of mass M. Two quantities are of fundamental interest: the orbit's averaged radius r_0, and the orbit's eccentricity ε. The radius r_0 denotes the averaged value of the orbit's Schwarzschild radial coordinate; the maximal value of the orbital radius defines the eccentricity: $r_{\max} = r_0(1+\varepsilon)$. More precise definitions of r_0 and ε will be given in Sec. II. We shall suppose that both the eccentricity ε and the mass ratio μ/M are much smaller than unity. However, we do not suppose that the radius r_0 is large, so our analysis includes strong-field situations.

We adopt the Teukolsky perturbation formalism [16], and consider the linear gravitational perturbations associated with the motion of the particle. The perturbations are described in terms of the complex Weyl scalar Ψ_4, which becomes radiative at large distances from the source. The rates of loss of orbital energy E, and orbital angular momentum L, due to gravitational radiation, can be calculated by solving the Teukolsky equation.

The *secular* evolution (the evolution over time scales much larger than the orbital period) of r_0 and ε can be determined from the knowledge of $\dot{E}$ and $\dot{L}$, where an overdot denotes time differentiation followed by an average over several orbital periods. In particular, the following relations can be derived (Sec. II): $\dot{r_0} = \dot{r_0}(r_0, \dot{L})$ and $\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon, r_0, \dot{E}, \dot{L})$. We shall use the perturbation formalism to calculate the rates of loss of energy and angular momentum. These calculations are performed (i) analytically, for the special case of weak fields and slow motions, and (ii) numerically, for the general case.

Our calculations are restricted to small eccentricities, $\varepsilon \ll 1$. The work presented in this paper can therefore be interpreted as a stability analysis: A circular orbit with radius r_0 is slightly perturbed and acquires a small eccentricity ε. The orbit evolves because of radiation reaction; the sign of $\dot{\varepsilon}$ determines whether the perturbed orbit is driven more circular, or more eccentric. Circular orbits are thus *stable* if $\dot{\varepsilon} < 0$, and are *unstable* if $\dot{\varepsilon} > 0$. Previous studies have shown that circular orbits are always stable in weak-field, slow-motion situations [11, 12]; our own study confirms this, and also determines whether this remains true in strong-field situations.

Recently, and independently of us, Tanaka *et al.* [17] numerically calculated the gravitational wave forms, and the fluxes of energy and angular momentum at infinity, for orbits with *arbitrary* eccentricities. The differences between their analysis and ours are significant. Tanaka *et al.* are mostly concerned with what can be observed at infinity, and are not much concerned with radiation reaction. In particular, they do not calculate the fluxes of energy and angular momentum at the black-hole horizon, which we do here, and which is important for radiation reaction. We have become aware of the work by Tanaka *et al.* very shortly before submitting this paper for publication.

Our calculations are also restricted to small mass ratios. This condition comes from two requirements: (i) that the gravitational perturbations be small enough to be linear, which implies $\mu/M \ll 1$ and (ii) that the *adiabatic approximation* be valid. The adiabatic approximation supposes that radiation reaction takes place over a time scale which is much larger than the orbital period. We shall show below (Sec. IV F) that this implies a restriction on μ/M; this restriction is not severe at large distances, but becomes $\mu/M \ll (1-6M/r_0)^{3/2}$ for r_0 approaching $6M$. The adiabatic approximation must therefore break down at $r_0 = 6M$, where, even without radiation reaction, circular orbits become unstable.

The adiabatic approximation is a fundamental feature of our analysis. It allows us to suppose that, over time scales comparable to the orbital period, the motion of the particle is, in fact, geodesic; nongeodesic behavior becomes noticeable only over much larger time scales. Moreover, the motion is also strictly periodic, and, consequently, the gravitational waves have a well-resolved frequency spectrum; the waves' frequencies change appreciably only over time scales much larger than the orbital period. Our problem is therefore one for which we first determine the rates of loss of energy and angular momentum for the slightly eccentric, geodesic motion of a particle around a Schwarzschild black hole, and then use these rates to infer the slow evolution of the orbit.

C. The results

Our analysis first allows us to prove that, if the particle's orbit is strictly circular ($\varepsilon = 0$), then radiation reaction produces a strictly circular evolution. In other words, *circular orbits remain circular under radiation reaction.* Previous proofs of this statement were restricted to weak-field, slow-motion situations [11, 12]; our proof is valid both for weak and strong fields.

If the eccentricity is small but not identically zero, our analysis shows that radiation reaction (i) *decreases* the eccentricity if r_0 is larger than a certain critical value r_c and (ii) *increases* the eccentricity if r_0 is smaller than r_c. (This behavior was also discovered in Ref. [17].) Thus circular orbits are *stable* if $r_0 > r_c$, and *unstable* if $r_0 < r_c$. The point $r_0 = r_c$ corresponds to $\dot{\varepsilon}$ changing sign; we have estimated numerically that

$$r_c/M \simeq 6.6792. \tag{1.1}$$

Our results are most conveniently presented in terms of the dimensionless quantity $c(r_0)$, defined as

$$c(r_0) = \frac{r_0}{\dot{r_0}}\frac{\dot{\varepsilon}}{\varepsilon} = \frac{d\ln\varepsilon}{d\ln r_0}, \tag{1.2}$$

and which can be interpreted as the ratio of the *inspiral time scale* $r_0/|\dot{r_0}|$ (the time scale over which the orbital radius r_0 changes appreciably) over the *circularization time scale* $\varepsilon/|\dot{\varepsilon}|$ (the time scale over which the eccentricity changes appreciably). By virtue of the adiabatic approximation, both time scales are much larger than the orbital period. A plot of $c(r_0)$, obtained numerically, is given in Fig. 1.

For large r_0 (weak-field, slow-motion), $c(r_0)$ can be calculated analytically (Sec. V A) and takes the form

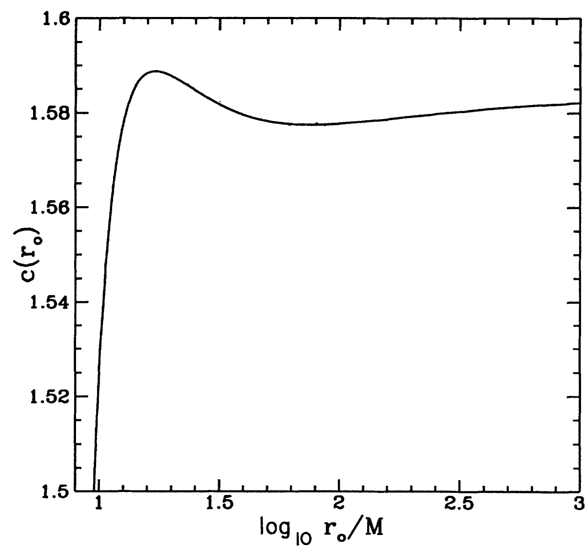

FIG. 1. A plot of $c(r_0)$, as defined in Eq. (1.2), as a function of $\log_{10} r_0/M$. Shown is the range $10 \lesssim r_0/M < 1000$, in which $c(r_0)$ has the most interesting behavior. For $r_0/M > 1000$, $c(r_0)$ is well approximated by Eq. (1.3), and approaches the value $19/12 \simeq 1.5833$ as r_0 tends to infinity. The function $c(r_0)$ changes sign at $r_0 = r_c \simeq 6.6792M$, and approaches minus infinity when $r_0 \to 6M$, in a way well described by Eq. (1.4).

$$c(r_0) = \tfrac{19}{12}[1 - \tfrac{3215}{3192}v^2 + \tfrac{377}{152}\pi v^3 + O(v^4)], \tag{1.3}$$

where $v = (M/r_0)^{1/2} \ll 1$ acts as a post-Newtonian expansion parameter. The leading-order term of Eq. (1.3) corresponds to a Newtonian calculation of the orbit, together with the use of the quadrupole formula to determine $\dot{r_0}$ and $\dot{\varepsilon}$ [11]. The first-order correction (at post-Newtonian, v^2, order) corresponds to post-Newtonian corrections to the orbital motion. The second-order correction (at post$^{3/2}$-Newtonian, v^3, order) corresponds to effects due to the propagation of the gravitational waves in the field of the black hole—effects associated with the tails of the waves.

For values of r_0 approaching $6M$ (highly relativistic situation; Sec. V B), $c(r_0)$ behaves according to

$$c(r_0 \to 6M) \sim -\tfrac{1}{4}(1 - 6M/r_0)^{-1}, \tag{1.4}$$

and therefore grows to arbitrarily large, negative values. This behavior is a consequence of the fact that circular orbits, even without radiation reaction, become unstable at $r_0 = 6M$. We recall that the limit $r_0 \to 6M$ must be taken with care, in view of the adiabatic approximation; orbits arbitrarily close to $r_0 = 6M$ can be considered at the price of taking μ/M sufficiently small.

Equations (1.3) and (1.4) are derived analytically, and imply that $c(r_0)$ must change sign at some radius $r_0 = r_c$. We have therefore provided an *analytical proof* that circular orbits are stable in the range $r_0 > r_c > 6M$ only. However, a numerical calculation is necessary to show that $c(r_0)$ changes sign only once, and to determine the value of r_c, Eq. (1.1).

The complete evolution of the eccentricity, so long as it remains small, can be obtained by integrating Eq. (1.2). It is most convenient to parametrize the evolution with r_0, and to express the eccentricity in terms of the function $\gamma(r_0; r_i)$, defined as

$$\gamma(r_0; r_i) = \ln\frac{\varepsilon(r_0)}{\varepsilon(r_i)} = \int_{r_i}^{r_0} \frac{c(r_0')}{r_0'}dr_0', \tag{1.5}$$

where r_i is some initial radius. If r_0 and r_i are both much larger than $6M$, then Eqs. (1.3) and (1.5) imply

$$\gamma(r_0; r_i) \simeq \alpha(r_0/M) - \alpha(r_i/M), \tag{1.6}$$

where

$$\alpha(x) = \tfrac{19}{12}\Big(\ln x + \tfrac{3215}{3192}x^{-1} - \tfrac{377}{1288}\pi x^{-3/2}\Big). \tag{1.7}$$

If, on the other hand, r_0 is very close to $6M$, but $r_i \gg 6M$, then Eqs. (1.4) and (1.5) imply

$$\gamma(r_0; r_i) \simeq -\tfrac{1}{4}\ln(r_0/6M - 1). \tag{1.8}$$

The behavior of $\gamma(r_0; r_i)$, for r_i arbitrarily fixed to $100M$, is depicted in Fig. 2. From this curve one can easily infer the corresponding $\gamma(r_0; r_i)$ for any $r_i < 100M$.

As one sees from Fig. 2, during the weak-field, slow-motion phase of the orbital evolution, the eccentricity is reduced by many orders of magnitude—the orbit becomes essentially circular. The eccentricity reaches a minimum value when $r_0 = r_c$, and then starts increasing. Eventually, if the mass ratio μ/M is arbitrarily small and

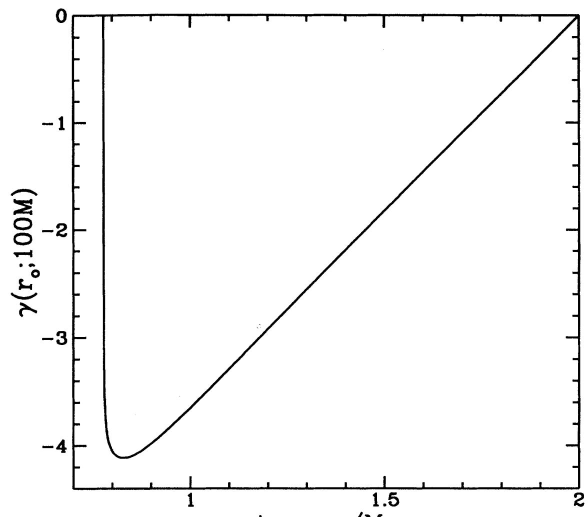

FIG. 2. A plot of $\gamma(r_0, r_i)$, as defined in Eq. (1.5), for $r_i = 100M$, as a function of $\log_{10} r_0/M$. The curve may be continued, both to the left and to the right, using the analytical estimates (1.6)–(1.8). For example, $\gamma(1000M; 100M) \simeq 3.6392$. The function $\gamma(r_0; r_i)$ has a minimum at $r_0 = r_c \simeq 6.6792M$, and grows to plus infinity when $r_0 \to 6M$. Horizontal lines intersect the curve at two distinct points ($r_0 = r_1$ and $r_0 = r_2$) for which the eccentricity is equal, $\varepsilon(r_1) = \varepsilon(r_2)$.

the adiabatic approximation holds, the orbit shrinks to a radius r_0 for which the eccentricity becomes equal to its initial value; in general this occurs very close to $6M$, as is indicated on the graph. For reasonable mass ratios, however, the eccentricity has not increased by much by the time the adiabatic approximation breaks down. As an example, consider a solar-mass object spiraling around a $10^6 M_\odot$ galactic black hole; this example is particularly relevant to space-based gravitational-wave detectors [9]. For $\mu/M = 10^{-6}$, the adiabatic approximation becomes invalid in the vicinity of $r_0 = r_1$, where $r_1/M = 6.002$; our numerical results then imply $\varepsilon(r_1)/\varepsilon(r_c) \simeq 4.0$. For such binary systems, the inspiral time from $r_0 = r_c$ to $r_0 = r_1$ is of the order of 2 yr. For $\mu/M = 10^{-8}$, the ratio of the eccentricities is only increased by a factor of 2.

D. Organization of the paper

The remainder of this paper is devoted to deriving the results quoted in the preceding subsection. We begin with a precise formulation of the problem in Sec. II. We first provide definitions for the quantities r_0 and ε, and then derive the evolution equations $\dot{r_0} = \dot{r_0}(r_0, \dot{L})$, $\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon, r_0, \dot{E}, \dot{L})$. Two conditions which ensure that $\dot{\varepsilon} \propto \varepsilon$ are imposed, and are justified in later sections. The first condition is that, for circular motion, gravitational waves carry energy and angular momentum in such a way that $\dot{E}/\dot{L} = \Omega = (M/{r_0}^3)^{1/2}$; the second condition is that corrections to $\dot{E}$ and $\dot{L}$, due to nonvanishing eccentricity, are second order in ε. The fact that $\dot{\varepsilon} \propto \varepsilon$ implies that circular orbits remain circular under radiation reaction; the stability of circular orbits depends on the sign of the proportionality factor.

We present a brief summary of the Teukolsky perturbation formalism [16] in Sec. III. First, the inhomogeneous Teukolsky equation, and its formal solution, are described in detail. Then we explain the method for extracting, from the solution, the gravitational wave forms, and the rates at which the waves carry energy and angular momentum. The section is concluded with a proof, valid for arbitrarily strong fields, that $\dot{E}/\dot{L} = \Omega$ for circular orbits.

The calculations relevant for slightly eccentric motion are presented in Sec. IV. The first step consists of integrating the radial and azimuthal geodesic equations; the integration is carried out to second order in the eccentricity. This calculation is presented in Sec. IV A and Sec. IV B offers an overview of the remaining steps. The form of the results obtained for $r(t)$ and $\phi(t)$ allows us, in Sec. IV C to (i) identify the frequency spectrum of the gravitational waves, (ii) witness important simplifications, and (iii) prove that corrections to $\dot{E}$ and $\dot{L}$ are second order in the eccentricity. All of this may be achieved without performing detailed calculations; instead, all computations are kept at a schematic level. These schematic calculations are pushed even further, in Sec. IV D, to derive expressions for $\dot{r_0}$ and $\mu\dot{\varepsilon}/\varepsilon$; this allows us to witness more cancellations, which greatly simplify the problem. The detail of the remaining calculations are presented in Sec. IV E. Conditions on μ/M, which ensure the validity of the adiabatic approximation, are formulated in Sec. IV F.

We present our analytical and numerical results in Sec. V. We first consider the weak-field, slow-motion ($r_0 \gg 6M$) limit of our formalism, and derive post-Newtonian expansions for the quantities of interest. This analysis yields Eq. (1.3) above. We then consider the highly relativistic ($r_0 \to 6M$) limit of the formalism, which is also tractable analytically. This analysis yields Eq. (1.4) above. In situations where r_0 is neither very large nor very close to $6M$, our equations must be integrated numerically, which we describe next. Our numerical analysis yields Eq. (1.1) above, as well as the graphs presented in the figures.

We conclude in Sec. VI with a recapitulation of our fundamental results, and a discussion of our approximations.

Throughout the paper we use geometrized units in which the speed of light and the gravitational constant are set equal to unity. Most of the paper is essentially self-contained, except for Sec. V, which relies heavily on previous papers in this series. These previous papers are concerned with purely circular orbits; paper I [18] is devoted to analytical methods, while paper II [19] is devoted to numerical methods. Both analytical and numerical methods are utilized in this paper.

II. FORMULATION OF THE PROBLEM

A. Definition of r_0 and ε

Timelike geodesics in the Schwarzschild geometry obey the equations

$$\begin{aligned} dt/d\tau &= \tilde{E}/f, \\ d\phi/d\tau &= \tilde{L}/r^2, \\ (dr/d\tau)^2 + V(\tilde{L}, r) &= \tilde{E}^2, \end{aligned} \tag{2.1}$$

where τ is the particle's proper time; $\tilde{E} = E/\mu$ and $\tilde{L} = L/\mu$ are, respectively, the specific orbital energy and angular momentum. We have also introduced $f = 1 - 2M/r$ and $V(\tilde{L}, r)$ is the effective potential for radial motion:

$$V(\tilde{L}, r) = f(1 + \tilde{L}^2/r^2). \tag{2.2}$$

We suppose that the motion takes place in the equatorial plane, $\theta = \pi/2$, and near a minimum of the potential $V(\tilde{L}, r)$. We *define* the radius $r = r_0$ to be the position of this minimum; since $\partial V/\partial r|_{r=r_0} = 0$, we have

$$\tilde{L}^2 = M^2 \left[v^2(1 - 3v^2)\right]^{-1}, \tag{2.3}$$

where $v = (M/r_0)^{1/2}$. Radial motion corresponds to small oscillations about $r = r_0$.

We *define* the eccentricity ε so that $r = r_0(1 + \varepsilon)$ is a turning point of the radial motion, at which $\tilde{E}^2 = V(\tilde{L}, r)$. This equation can be expanded in powers of ε, which yields

$$\begin{aligned} (1 - 3v^2)\tilde{E}^2 = {} & (1 - 2v^2)^2 + v^2(1 - 6v^2)\varepsilon^2 \\ & -2v^2(1 - 7v^2)\varepsilon^3 + O(\varepsilon^4). \end{aligned} \tag{2.4}$$

Equation (2.3) implies $r_0 = r_0(\tilde{L})$, while Eq. (2.4) implies $\varepsilon = \varepsilon(\tilde{L}, \tilde{E})$.

B. Radiation reaction—evolution of r_0 and ε

The results of Sec. II A imply that the knowledge of the rates of loss of energy and angular momentum, due to gravitational radiation, is sufficient to determine the evolution of both r_0 and ε. We are interested in the *secular* evolution of these quantities—the evolution over time scales much larger than the orbital period. The secular evolution is well defined, and can be unambiguously calculated. In contrast, the short-term evolution is not so well defined, because gravitational waves cannot be localized in a region of spacetime smaller than a few wavelengths [3]. To perform a time averaging over several orbital periods is therefore a fundamental feature of our calculations. We shall henceforth denote by an overdot the operation of time differentiation followed by an average over several orbital periods; thus $\dot{\psi} = \langle d\psi/dt \rangle$, for any quantity ψ.

An evolution equation for r_0 is obtained by using Eq. (2.3) to calculate $\mu\dot{r_0} = (dr_0/d\tilde{L})\dot{L}$, which yields

$$\mu\dot{r_0} = 2(1 - 3v^2)^{3/2}\left[v(1 - 6v^2)\right]^{-1}\dot{L}. \tag{2.5}$$

Similarly, one may use Eqs. (2.3) and (2.4) to calculate $\mu\dot{\varepsilon} = (\partial\varepsilon/\partial\tilde{E})\dot{E} + (\partial\varepsilon/\partial\tilde{L})\dot{L}$, which yields

$$\mu\dot{\varepsilon} = \frac{1}{\varepsilon}\frac{(1 - 2v^2)(1 - 3v^2)^{1/2}}{v^2(1 - 6v^2)}\left\{\left[1 + \frac{v^2(1 - 6v^2)}{2(1 - 2v^2)^2}\varepsilon^2 + O(\varepsilon^3)\right]\dot{E} - \left[1 - \frac{1 - 12v^2 + 18v^4}{(1 - 2v^2)(1 - 6v^2)}\varepsilon^2 + O(\varepsilon^3)\right]\Omega\dot{L}\right\}, \tag{2.6}$$

where $\Omega = v/r_0 = (M/r_0{}^3)^{1/2}$.

The rate of loss of orbital energy is equal to minus the rate at which gravitational waves carry energy. We therefore write $\dot{E} = -\dot{E}^{(\mathrm{GW})}$, and expand $\dot{E}^{(\mathrm{GW})}$ in powers of the eccentricity:

$$\dot{E}^{(\mathrm{GW})} = \dot{E}^{(0)} + \varepsilon\dot{E}^{(1)} + \varepsilon^2\dot{E}^{(2)} + O(\varepsilon^3), \tag{2.7}$$

$\dot{E}^{(0)}$ corresponding to circular motion. Similarly, we write $\dot{L} = -\dot{L}^{(\mathrm{GW})}$ and

$$\dot{L}^{(\mathrm{GW})} = \dot{L}^{(0)} + \varepsilon\dot{L}^{(1)} + \varepsilon^2\dot{L}^{(2)} + O(\varepsilon^3). \tag{2.8}$$

In Secs. III C and IV C below, we will show that

$$\dot{E}^{(0)} = \Omega\dot{L}^{(0)}, \quad \dot{E}^{(1)} = \dot{L}^{(1)} = 0, \tag{2.9}$$

which implies that the lowest-order corrections to $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$ are second order in the eccentricity.

Substitution of Eqs. (2.8) and (2.9) into (2.5) implies

$$\begin{aligned} \mu\dot{r_0} = {} & -2M(1 - 3v^2)^{3/2}\left[v^4(1 - 6v^2)\right]^{-1}\dot{E}^{(0)} \\ & +O(\varepsilon^2); \end{aligned} \tag{2.10}$$

the evolution of r_0 is therefore dominated by the circular limit of Eq. (2.5), and corrections due to the small eccentricity can be ignored.

Substitution of Eqs. (2.7), (2.8), and (2.9) into (2.6) yields important cancellations, and the final answer is

$$\begin{aligned} \mu\dot{\varepsilon} = {} & -\varepsilon(1 - 2v^2)(1 - 3v^2)^{1/2}\left[v^2(1 - 6v^2)\right]^{-1} \\ & \times\left[g(v)\dot{E}^{(0)} + \dot{E}^{(2)} - \Omega\dot{L}^{(2)}\right] + O(\varepsilon^2), \end{aligned} \tag{2.11}$$

where

$$g(v) = \frac{2 - 27v^2 + 72v^4 - 36v^6}{2(1 - 2v^2)^2(1 - 6v^2)}. \tag{2.12}$$

Thus the calculation of $\mu\dot{\varepsilon}$ requires the computation of $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$, accurately to second order in the eccentricity. Because of the crucial relations (2.9), $\mu\dot{\varepsilon}$ is itself *linear* in the eccentricity.

Equations (2.9) are therefore the key to the proof that circular orbits remain circular under radiation reaction, since Eq. (2.11) implies $\dot{\varepsilon}(\varepsilon = 0) = 0$. The problem of determining the evolution of r_0 and ε is now equivalent to that of calculating $\dot{E}^{(0)}$, and the pieces of $\dot{E}^{(2)}$ and $\dot{L}^{(2)}$ which do not cancel out when the combination $\dot{E}^{(2)} - \Omega\dot{L}^{(2)}$ is constructed.

III. THE PERTURBATION FORMALISM

This section contains a brief summary of the relevant equations. More detail can be found in paper I [18], and in the references quoted herein.

A. The Teukolsky equation

The stress-energy tensor associated with the motion of a particle perturbs the gravitational field of a Schwarzschild black hole. The gravitational perturbations are described by the Weyl scalar $\Psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$, where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $n^\alpha = \frac{1}{2}(1,-f,0,0)$, and $\bar{m}^\alpha = (0,0,1,-i\csc\theta)/\sqrt{2}r$; throughout we denote complex conjugation with an overbar. At large distances, Ψ_4 describes outgoing gravitational waves; at the black-hole horizon, Ψ_4 describes ingoing waves.

The Weyl scalar can be decomposed into Fourier-harmonic components according to

$$r^4\Psi_4 = \int_{-\infty}^{\infty} d\omega \sum_{\ell m} R_{\omega\ell m}(r)\,{}_{-2}Y_{\ell m}(\theta,\phi)e^{-i\omega t}, \tag{3.1}$$

where ${}_sY_{\ell m}(\theta,\phi)$ are spin-weighted spherical harmonics [20]; the sums over ℓ and m are restricted to $-\ell \le m \le \ell$ and $\ell \ge 2$. The radial function $R_{\omega\ell m}(r)$ satisfies the inhomogeneous Teukolsky equation [16]

$$\left[r^2 f\frac{d^2}{dr^2} - 2(r-M)\frac{d}{dr} + U(r)\right]R_{\omega\ell m}(r) = -T_{\omega\ell m}(r), \tag{3.2}$$

with

$$U(r) = f^{-1}\left[(\omega r)^2 - 4i\omega(r-3M)\right] - \lambda, \tag{3.3}$$

where $\lambda = (\ell-1)(\ell+2)$.

The source term in Eq. (3.2) is calculated from the particle's stress-energy tensor $T^{\alpha\beta}(x) = \mu\int d\tau\, u^\alpha u^\beta \delta^{(4)}[x - z(\tau)]$, where x is the spacetime point, $z(\tau)$ the particle's trajectory with tangent $u^\alpha = dz^\alpha/d\tau$, and τ is the particle's proper time. The first step is to construct the projections ${}_0T = T_{\alpha\beta}n^\alpha n^\beta$, ${}_{-1}T = T_{\alpha\beta}n^\alpha \bar{m}^\beta$, and ${}_{-2}T = T_{\alpha\beta}\bar{m}^\alpha \bar{m}^\beta$. Then one calculates the Fourier-harmonic components ${}_sT_{\omega\ell m}(r)$ according to

$${}_sT_{\omega\ell m}(r) = \frac{1}{2\pi}\int dt\, d\Omega\, {}_sT\, {}_s\bar{Y}_{\ell m}(\theta,\phi)e^{i\omega t}, \tag{3.4}$$

where $d\Omega$ is the element of solid angle. The source is

$$\begin{aligned} T_{\omega\ell m}(r) = 2\pi\{\, & 2[\lambda(\lambda+2)]^{1/2} r^4\, {}_0T_{\omega\ell m}(r) \\ & + 2(2\lambda)^{1/2} r^2 f \mathcal{L}\, r^3 f^{-1}\, {}_{-1}T_{\omega\ell m}(r) \\ & + r f \mathcal{L} r^4 f^{-1} \mathcal{L}\, r\, {}_{-2}T_{\omega\ell m}(r)\}, \end{aligned} \tag{3.5}$$

where $\mathcal{L} = f d/dr + i\omega$.

The inhomogeneous Teukolsky equation (3.2) can be integrated by means of a Green's function [21]. The solution at large radii is

$$R_{\omega\ell m}(r\to\infty) \sim \mu\omega^2 Z^H_{\omega\ell m} r^3 e^{i\omega r^*}, \tag{3.6}$$

where $r^* = r + 2M\ln(r/2M - 1)$, and the solution near the black-hole horizon is

$$R_{\omega\ell m}(r\to 2M) \sim \mu\omega^3 Z^\infty_{\omega\ell m} r^4 f^2 e^{-i\omega r^*}. \tag{3.7}$$

The amplitudes $Z^{H,\infty}_{\omega\ell m}$ are defined by

$$Z^{H,\infty}_{\omega\ell m} = \frac{1}{2i\mu\omega^2 Q^{\rm in}_{\omega\ell}}\int_{2M}^{\infty} dr\, \frac{R^{H,\infty}_{\omega\ell}(r)T_{\omega\ell m}(r)}{r^4 f^2}, \tag{3.8}$$

where the functions $R^H_{\omega\ell}(r)$ and $R^\infty_{\omega\ell}(r)$ are solutions of the *homogeneous* Teukolsky equation. $R^H_{\omega\ell}(r)$ is the solution with boundary conditions corresponding to ingoing waves at the black-hole horizon, $R^H_{\omega\ell}(r\to 2M) \sim (\omega r)^4 f^2 e^{-i\omega r^*}$; $R^H_{\omega\ell}(r)$ represents a superposition of ingoing and outgoing waves at large radii, $R^H_{\omega\ell}(r\to\infty) \sim Q^{\rm in}_{\omega\ell}(\omega r)^{-1}e^{-i\omega r^*} + Q^{\rm out}_{\omega\ell}(\omega r)^3 e^{i\omega r^*}$. $R^\infty_{\omega\ell}(r)$ is the solution with boundary conditions corresponding to outgoing waves at infinity, $R^\infty_{\omega\ell}(r\to\infty) \sim (\omega r)^3 e^{i\omega r^*}$; $R^\infty_{\omega\ell}(r)$ represents a superposition of ingoing and outgoing waves at the horizon.

The amplitudes $Z^{H,\infty}_{\omega\ell m}$ satisfy the identities

$$\bar{Z}^{H,\infty}_{-\omega,\ell,-m} = (-1)^\ell Z^{H,\infty}_{\omega\ell m}, \tag{3.9}$$

which we now derive. We use the fact that $u^\theta = 0$, which implies ${}_s\bar{T} = (-1)^s\,{}_sT$; substitution into Eq. (3.4), using ${}_sY_{\ell,-m}(\theta,\phi) = (-1)^{s+\ell}\,{}_s\bar{Y}_{\ell m}(\theta,\phi)$, then yields ${}_s\bar{T}_{-\omega,\ell,-m}(r) = (-1)^\ell\,{}_sT_{\omega\ell m}(r)$. It follows from this and Eq. (3.5) that $\bar{T}_{-\omega,\ell,-m}(r) = (-1)^\ell T_{\omega\ell m}(r)$. The homogeneous Teukolsky equation is invariant under complex conjugation followed by $\omega \to -\omega$, so $\bar{R}^{H,\infty}_{-\omega,\ell}(r) = R^{H,\infty}_{\omega\ell}(r)$ and $\bar{Q}^{\rm in}_{-\omega,\ell} = Q^{\rm in}_{\omega\ell}$. Equation (3.9) finally follows from Eq. (3.8).

B. Wave forms; energy and angular momentum fluxes

At large distances, the two fundamental polarizations of the gravitational waves, h_+ and $h_\times$, can be obtained from Eqs. (3.1) and (3.6); they are

$$h_+ - ih_\times = \frac{2\mu}{r}\sum_{\ell m} {}_{-2}Y_{\ell m}\int_{-\infty}^{\infty} d\omega\, Z^H_{\omega\ell m}e^{-i\omega u}, \tag{3.10}$$

where $u = t - r^*$ represents retarded time. The transverse traceless gravitational-wave tensor is

$$h^{\rm TT}_{ab} = (h_+ - ih_\times)m_a m_b + (h_+ + ih_\times)\bar{m}_a\bar{m}_b. \tag{3.11}$$

The rates at which gravitational waves carry energy and angular momentum to infinity can be calculated from the Isaacson stress-energy tensor [22], which is constructed from $h^{\rm TT}_{ab}$. An alternative but equivalent method involves reading off the multipole moments of the radiative field, as defined by Thorne [23], and using the relevant equations of Ref. [23] to calculate $\dot{E}^\infty$ and $\dot{L}^\infty$. To present the results we now specialize to the case considered in this paper, in which the frequency spectrum of the waves is characterized by a discrete set of distinct frequencies ω_k. Then

$$Z^H_{\omega\ell m} = \sum_k Z^{Hk}_{\ell m}\delta(\omega - \omega_k), \tag{3.12}$$

and

$$\dot{E}^\infty = \frac{\mu^2}{4\pi}\sum_{\ell m k} {\omega_k}^2 |Z^{Hk}_{\ell m}|^2, \tag{3.13}$$

$$\dot{L}^{\infty} = \frac{\mu^2}{4\pi} \sum_{\ell m k} \frac{m}{\omega_k} \omega_k{}^2 |Z_{\ell m}^{Hk}|^2. \tag{3.14}$$

The rates at which the black hole absorbs energy and angular momentum can be calculated along similar lines [24]. From $\Psi_4(r \to 2M)$ one recovers the gravitational-wave tensor, from which the Isaacson stress-energy tensor is calculated. The calculation of the fluxes then reproduces the results of Teukolsky and Press [25], which were derived in a completely different manner:

$$\dot{E}^{H} = \frac{\mu^2}{4\pi M^2} \sum_{\ell m k} \alpha_\ell^k |Z_{\ell m}^{\infty k}|^2, \tag{3.15}$$

$$\dot{L}^{H} = \frac{\mu^2}{4\pi M^2} \sum_{\ell m k} \frac{m}{\omega_k} \alpha_\ell^k |Z_{\ell m}^{\infty k}|^2, \tag{3.16}$$

for

$$Z_{\omega \ell m}^{\infty} = \sum_k Z_{\ell m}^{\infty k} \delta(\omega - \omega_k). \tag{3.17}$$

We have introduced

$$\alpha_\ell^k = \frac{2^{12}\left[1 + 4(M\omega_k)^2\right]\left[1 + 16(M\omega_k)^2\right]}{\left[\lambda(\lambda+2)\right]^2 + 144(M\omega_k)^2} (M\omega_k)^8. \tag{3.18}$$

The total rates of loss of energy and angular momentum are then $\dot{E}^{(\mathrm{GW})} = \dot{E}^{\infty} + \dot{E}^{H}$ and $\dot{L}^{(\mathrm{GW})} = \dot{L}^{\infty} + \dot{L}^{H}$.

C. Proof that $\dot{E}^{(0)} = \Omega \dot{L}^{(0)}$

For circular motion, the particle's stress-energy tensor is proportional to $\delta(\phi - \Omega t)$. Equations (3.4) and (3.5) then imply $T_{\omega \ell m} \propto \delta(\omega - m\Omega)$—the wave frequency ω is a harmonic of the orbital frequency Ω. Equation (3.8) further implies $Z_{\omega \ell m}^{H,\infty} \propto \delta(\omega - m\Omega)$, so that we can write

$$Z_{\omega \ell m}^{H,\infty} = A_{\ell m}^{H,\infty} \delta(\omega - m\Omega), \tag{3.19}$$

which is a special case of Eqs. (3.12) and (3.17), with $\omega_k = m\Omega$. Equations (3.13)–(3.16) then yield

$$\dot{E}^{\infty} = \Omega \dot{L}^{\infty} = \frac{\mu^2}{4\pi} \sum_{\ell m} (m\Omega)^2 |A_{\ell m}^{H}|^2 \tag{3.20}$$

and

$$\dot{E}^{H} = \Omega \dot{L}^{H} = \frac{\mu^2}{4\pi M^2} \sum_{\ell m} \alpha_\ell |A_{\ell m}^{\infty}|^2, \tag{3.21}$$

where $\alpha_\ell = \alpha_\ell^k(\omega_k = m\Omega)$. Finally, Eqs. (3.20) and (3.21) imply $\dot{E}^{(0)} = \Omega \dot{L}^{(0)}$. Notice that the proof does not require the explicit calculation of $A_{\ell m}^{H,\infty}$. The key to the proof is the observation that for a mode of given m and ω_k, $\dot{E}^{\infty,H}/\dot{L}^{\infty,H} = \omega_k/m$. This property is very general and holds for arbitrary fields; cf. Ref. [26].

IV. GRAVITATIONAL WAVES FROM SLIGHTLY ECCENTRIC MOTION

A. First step—slightly eccentric motion

The first step of the calculation consists of solving the geodesic equations for slightly eccentric orbits. We begin with the radial equation. Equations (2.1) imply

$$(dr/dt)^2 + U(\tilde{E}, \tilde{L}, r) = 0, \tag{4.1}$$

where

$$U(\tilde{E}, \tilde{L}, r) = (f/\tilde{E})^2 [V(\tilde{L}, r) - \tilde{E}^2]. \tag{4.2}$$

Our strategy is to expand $r(t)$ according to

$$r(t) = r_0\left[1 + \varepsilon \xi^{(1)}(t) + \varepsilon^2 \xi^{(2)}(t) + O(\varepsilon^3)\right], \tag{4.3}$$

and to similarly expand $U(\tilde{E}, \tilde{L}, r)$, using Eqs. (2.3) and (2.4). Collecting terms of equal order in ε yields (i) a differential equation for $\xi^{(1)}(t)$,

$$(d\xi^{(1)}/dt)^2 = \Omega_r{}^2 (1 - \xi^{(1)\,2}), \tag{4.4}$$

where

$$\Omega_r = \Omega(1 - 6v^2)^{1/2} \tag{4.5}$$

is the *radial frequency*, the fundamental frequency of radial motion, and (ii) a linear differential equation for $\xi^{(2)}(t)$,

$$\frac{1}{\Omega_r{}^2} \frac{d\xi^{(1)}}{dt} \frac{d\xi^{(2)}}{dt} + \xi^{(1)} \xi^{(2)} = -\frac{1 - 7v^2}{1 - 6v^2} + \frac{2v^2}{1 - 2v^2} \xi^{(1)} + \frac{1 - 11v^2 + 26v^4}{(1 - 2v^2)(1 - 6v^2)} \xi^{(1)\,3}. \tag{4.6}$$

Equation (4.4) can be integrated to give

$$\xi^{(1)}(t) = \cos \Omega_r t, \tag{4.7}$$

where the time origin is chosen so that $r(t=0) = r_0(1 + \varepsilon)$. Substitution of Eq. (4.7) into (4.6) then yields, after integration,

$$\xi^{(2)}(t) = q_1(v)(1 - \cos \Omega_r t) + q_2(v)(1 - \cos 2\Omega_r t), \tag{4.8}$$

where $q_1(v) = (1 - 7v^2)(1 - 6v^2)^{-1}$ and $q_2(v) = (1 - 11v^2 + 26v^4)[2(1 - 2v^2)(1 - 6v^2)]^{-1}$.

Integration of the azimuthal equation proceeds along similar lines. Equations (2.1) imply

$$d\phi/dt = (\tilde{L}/\tilde{E})(f/r^2), \tag{4.9}$$

which may be expanded in powers of ε using Eqs. (2.3), (2.4), (4.3), (4.7), and (4.8). Integration then yields

$$\phi(t) = \Omega_\phi t - \varepsilon p_1(v) \sin \Omega_r t + \varepsilon^2 p_2(v) \sin \Omega_r t + \varepsilon^2 p_3(v) \sin 2\Omega_r t + O(\varepsilon^3), \tag{4.10}$$

where

$$p_1(v) = 2(1 - 3v^2)[(1 - 2v^2)(1 - 6v^2)^{1/2}]^{-1},$$

$$p_2(v) = 2\,(1 - 3v^2)(1 - 7v^2) \times [(1 - 2v^2)(1 - 6v^2)^{3/2}]^{-1},$$

$$p_3(v) = (5 - 64v^2 + 250v^4 - 300v^6) \times [4(1 - 2v^2)^2(1 - 6v^2)^{3/2}]^{-1},$$

and

$$\Omega_\phi = \left[1 - \frac{3(1-3v^2)(1-8v^2)}{2(1-2v^2)(1-6v^2)}\varepsilon^2\right]\Omega \qquad (4.11)$$

is the *azimuthal frequency*, the fundamental frequency of azimuthal motion. That $\Omega_\phi \neq \Omega_r$ reflects the fact that eccentric orbits in Schwarzschild are not closed.

B. The remaining steps—an overview

The next steps of the calculation consist of (i) substituting the results of the preceding subsection into the expression for the particle's stress-energy tensor,

$$T^{\alpha\beta} = \mu\frac{u^\alpha u^\beta}{r^2 u^t}\delta[r - r(t)]\delta(\cos\theta)\delta[\phi - \phi(t)], \qquad (4.12)$$

(ii) constructing the projections ${}_sT$, and (iii) expanding to second order in the eccentricity. In particular, we must expand $r - r(t)$ about $r - r_0$, thereby introducing derivatives of the radial δ function; and expand $\phi - \phi(t)$ about $\phi - \Omega_\phi t$, which introduces derivatives of the azimuthal δ function.

The next task is to obtain the Fourier-harmonic components ${}_sT_{\omega\ell m}(r)$, using Eq. (3.4). The integration over ϕ implies that the derivatives of $\delta(\phi - \Omega_\phi t)$ are integrated by parts, and the nth derivative of $\delta(\phi - \Omega_\phi t)$ is therefore equivalent to $(im)^n\delta(\phi - \Omega_\phi t)$.

Once the source to the Teukolsky equation has been evaluated using Eq. (3.5) we calculate $Z^{H,\infty}_{\omega\ell m}$ using Eq. (3.8). Since the source has support only at $r = r_0$, the integration can be performed analytically, and involves several integrations by parts. As a result, $Z^{H,\infty}_{\omega\ell m}$ can be expressed as a function of (i) r_0, (ii) the functions $R^{H,\infty}_{\omega\ell}(r)$ and their derivatives at $r = r_0$, and (iii) the coefficient $Q^{\rm in}_{\omega\ell}$.

In weak-field, slow-motion situations (r_0 large), the analytical techniques developed in paper I [18] may be used to calculate, approximately, $R^H_{\omega\ell}(r)$ and $Q^{\rm in}_{\omega\ell}$. The result is an analytical expression for $Z^H_{\omega\ell m}$, valid for $r_0 \gg 6M$. Since $\dot{E}^H/\dot{E}^\infty$ and $\dot{L}^H/\dot{L}^\infty$ are of order v^8 and hence very small [14, 27], the weak-field, slow-motion calculation does not require the computation of $Z^\infty_{\omega\ell m}$.

In a strong-field situation, $R^{H,\infty}_{\omega\ell}(r)$ and $Q^{\rm in}_{\omega\ell}$ must be obtained, for a given value of r_0, by numerically integrating the homogeneous Teukolsky equation. The result is then a numerical expression for $Z^{H,\infty}_{\omega\ell m}$, valid for that value of r_0.

Once $Z^{H,\infty}_{\omega\ell m}$ has been obtained we observe that the continuous sum over ω reduces to a discrete sum, as in Eqs. (3.12) and (3.17). We then calculate $\dot{E}^{(\rm GW)}$ and $\dot{L}^{(\rm GW)}$ with the help of Eqs. (3.13)–(3.16). Finally, Eqs. (2.10) and (2.11) are used to calculate $\dot{r_0}$ and $\dot{\varepsilon}/\varepsilon$.

C. Frequency spectrum, simplifications, and proof that $\dot{E}^{(1)} = \dot{L}^{(1)} = 0$

Each step of the calculation, as outlined in the preceding subsection, would require an extremely long and tedious computation if some remarkable simplifications did not occur along the way. These simplifications arise because of the following. (i) The gravitational waves possess a frequency spectrum characterized by a discrete set of frequencies. As in the circular case, the waves have frequencies equal to the harmonics of the azimuthal frequency, $\omega = m\Omega_\phi$. However, a small eccentricity also implies the existence of *side bands* [28], at $\omega = m\Omega_\phi \pm \Omega_r$ and $\omega = m\Omega_\phi \pm 2\Omega_r$. (ii) The calculation of $\dot{E}^{(\rm GW)}$ and $\dot{L}^{(\rm GW)}$ includes a time averaging, which causes a large number of terms to vanish. In particular, all $O(\varepsilon)$ terms average out, as do most $O(\varepsilon^2)$ terms. And (iii) the calculation of $\dot{\varepsilon}/\varepsilon$ only requires the computation of $\dot{E}^{(2)} - \Omega\dot{L}^{(2)}$, which also generates important cancellations.

We now look more closely into the nature of the waves' frequency spectrum. The calculation of ${}_sT_{\omega\ell m}(r)$ was outlined in Sec. IV B. After the angular integration has been performed, it is clear from Eqs. (4.7), (4.8), (4.10), and (4.12) that the next step is to integrate over time terms which are proportional to (i) $e^{i(\omega - m\Omega_\phi)t}$, (ii) $e^{\pm i\Omega_r t}e^{i(\omega - m\Omega_\phi)t}$, and (iii) $e^{\pm 2i\Omega_r t}e^{i(\omega - m\Omega_\phi)t}$. It is also clear that the terms with dependence (i) are dominantly $O(\varepsilon^0)$, while the terms with dependence (ii) are dominantly $O(\varepsilon)$, and the terms with dependence (iii) are dominantly $O(\varepsilon^2)$. Correspondingly, time integration yields terms which are proportional to (i) $\delta(\omega - m\Omega_\phi)$, with magnitude $O(\varepsilon^0)$, (ii) $\delta(\omega - m\Omega_\phi \pm \Omega_r)$, with magnitude $O(\varepsilon)$, and (iii) $\delta(\omega - m\Omega_\phi \pm 2\Omega_r)$, with magnitude $O(\varepsilon^2)$. Finally, Eqs. (3.5), (3.8), and (3.10) imply that the gravitational waves possess the frequency spectrum described previously.

Our schematic considerations can be pushed further. It is indeed clear from the results obtained thus far that $Z^{H,\infty}_{\omega\ell m}$ must have the structure (we momentarily remove the H,∞ subscripts for the sake of clarity)

$$Z_{\omega\ell m} = A_{\ell m}\delta(\omega - \omega_m) - \tfrac{1}{2}B^-_{\ell m}\delta(\omega - \omega_-)\varepsilon - \tfrac{1}{2}B^+_{\ell m}\delta(\omega - \omega_+)\varepsilon + C_{\ell m}\delta(\omega - \omega_m)\varepsilon^2 + D^-_{\ell m}\delta(\omega - \omega_-)\varepsilon^2$$
$$+ D^+_{\ell m}\delta(\omega - \omega_+)\varepsilon^2 + E^{-2}_{\ell m}\delta(\omega - \omega_{-2})\varepsilon^2 + E^{+2}_{\ell m}\delta(\omega - \omega_{+2})\varepsilon^2 + O(\varepsilon^3), \qquad (4.13)$$

where $\omega_m = m\Omega_\phi$, $\omega_\pm = m\Omega_\phi \pm \Omega_r$, and $\omega_{\pm 2} = m\Omega_\phi \pm 2\Omega_r$. The various coefficients of the δ functions are expected to be complicated functions of (i) r_0, (ii) $R^{H,\infty}_{\omega\ell}(r)$ and their derivatives at $r = r_0$, and (iii) $Q^{\rm in}_{\omega\ell}$. All these coefficients *can* be calculated with the help of the equations presented in this and the preceding section; however, we shall now show that only a small number actually *need* be calculated.

Substitution of Eq. (4.13) into (3.13)–(3.16), using (3.12) and (3.17), yields

$$\dot{E}^{\infty} = \frac{\mu^2}{4\pi}\sum_{\ell m}\omega_m{}^2\Big[|A^H_{\ell m}+\varepsilon^2 C^H_{\ell m}|^2 + \Big(\frac{\omega_-}{\omega_m}\Big)^2|\tfrac{1}{2}B^{H-}_{\ell m}|^2\varepsilon^2 + \Big(\frac{\omega_+}{\omega_m}\Big)^2|\tfrac{1}{2}B^{H+}_{\ell m}|^2\varepsilon^2 + O(\varepsilon^3)\Big], \tag{4.14}$$

$$\Omega_\phi \dot{L}^{\infty} = \frac{\mu^2}{4\pi}\sum_{\ell m}\omega_m{}^2\Big[|A^H_{\ell m}+\varepsilon^2 C^H_{\ell m}|^2 + \frac{\omega_-}{\omega_m}|\tfrac{1}{2}B^{H-}_{\ell m}|^2\varepsilon^2 + \frac{\omega_+}{\omega_m}|\tfrac{1}{2}B^{H+}_{\ell m}|^2\varepsilon^2 + O(\varepsilon^3)\Big], \tag{4.15}$$

$$\dot{E}^{H} = \frac{\mu^2}{4\pi M^2}\sum_{\ell m}\alpha_\ell\Big[|A^\infty_{\ell m}+\varepsilon^2 C^\infty_{\ell m}|^2 + \frac{\alpha^-_\ell}{\alpha_\ell}|\tfrac{1}{2}B^{\infty-}_{\ell m}|^2\varepsilon^2 + \frac{\alpha^+_\ell}{\alpha_\ell}|\tfrac{1}{2}B^{\infty+}_{\ell m}|^2\varepsilon^2 + O(\varepsilon^3)\Big], \tag{4.16}$$

$$\Omega_\phi \dot{L}^{H} = \frac{\mu^2}{4\pi M^2}\sum_{\ell m}\alpha_\ell\Big[|A^\infty_{\ell m}+\varepsilon^2 C^\infty_{\ell m}|^2 + \frac{\alpha^-_\ell}{\alpha_\ell}\frac{\omega_m}{\omega_-}|\tfrac{1}{2}B^{\infty-}_{\ell m}|^2\varepsilon^2 + \frac{\alpha^+_\ell}{\alpha_\ell}\frac{\omega_m}{\omega_+}|\tfrac{1}{2}B^{\infty+}_{\ell m}|^2\varepsilon^2 + O(\varepsilon^3)\Big], \tag{4.17}$$

where $\alpha_\ell = \alpha^k_\ell(\omega_k = \omega_m)$ and $\alpha^\pm_\ell = \alpha^k_\ell(\omega_k = \omega_\pm)$. These results teach us that the coefficients $D^{H,\infty\pm}_{\ell m}$ and $E^{H,\infty\pm 2}_{\ell m}$ are irrelevant to our calculation; their contributions vanish after the time averaging has been carried out. More simplifications arise below.

Equations (4.14)–(4.17) imply that corrections to $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$, due to nonvanishing eccentricity, are second order in ε. Thus $\dot{E}^{(1)} = \dot{L}^{(1)} = 0$, as was first written in Eq. (2.9). The proof that circular orbits remain circular under radiation reaction is now complete.

D. Calculation of $\dot{r_0}$ and $\mu\dot{e}/\varepsilon$

The calculation of $\dot{r_0}$ is almost complete. Explicit expressions for $A^{H,\infty}_{\ell m}$ will be given in Sec. IV E; these may be used together with Eqs. (3.20) and (3.21) to calculate $\dot{E}^{(0)}$, which is then substituted in Eq. (2.10).

The calculation of $\mu\dot{e}/\varepsilon$ requires the computation of $\dot{E}^{(0)}$ and $\dot{E}^{(2)} - \Omega\dot{L}^{(2)}$. In Eqs. (4.14)–(4.17), a number of terms are *explicitly* second order in the eccentricity; others are $O(\varepsilon^2)$ only *implicitly*, by virtue of the fact that $\Omega_\phi = \Omega(1-\Delta\Omega\varepsilon^2)$, where $\Delta\Omega$ can be read off from Eq. (4.11). To make all dependence on ε explicit we now adapt our notation so that $\omega_\pm = m\Omega \pm \Omega_r$ and write $\omega_m{}^2|A^{H,\infty}_{\ell m}|^2 = (m\Omega)^2|A^{H,\infty}_{\ell m}|^2 + O(\varepsilon^2)$. It follows that the quantity $\dot{E}^{(2)} - \Omega\dot{L}^{(2)} + \Delta\Omega\dot{E}^{(0)}$ only requires the calculation of the coefficients $B^{H,\infty\pm}_{\ell m}$. With the help of Eq. (2.11) we finally obtain

$$\mu\dot{e}/\varepsilon = -\frac{(1-2v^2)(1-3v^2)^{1/2}}{v^2(1-6v^2)}\big[\Gamma - h(v)\dot{E}^{(0)}\big], \tag{4.18}$$

where $\Gamma = \Gamma^\infty + \Gamma^H$, with

$$\Gamma^\infty = \frac{\mu^2}{16\pi}\Omega_r\sum_{\ell m}(\omega_+|B^{H+}_{\ell m}|^2 - \omega_-|B^{H-}_{\ell m}|^2), \tag{4.19}$$

where $\omega_\pm = m\Omega \pm \Omega_r$ and

$$\Gamma^H = \frac{\mu^2}{16\pi M^2}\Omega_r\sum_{\ell m}\Big(\frac{\alpha^+_\ell}{\omega_+}|B^{\infty+}_{\ell m}|^2 - \frac{\alpha^-_\ell}{\omega_-}|B^{\infty-}_{\ell m}|^2\Big). \tag{4.20}$$

We also have

$$h(v) = \frac{1-12v^2+66v^4-108v^6}{2(1-2v^2)^2(1-6v^2)}, \tag{4.21}$$

with $v = (M/r_0)^{1/2}$.

Equations (4.18)–(4.21) imply that the calculation of $\mu\dot{e}/\varepsilon$ is much simpler than the individual computations, to second order in the eccentricity, of $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$. Because of the occurrence of important cancellations, the calculation only requires the computation of $B^{H,\infty\pm}_{\ell m}$, and the leading-order part of $A^{H,\infty}_{\ell m}$. Computation of all other coefficients, as well as the $O(\varepsilon^2)$ part of $A^{H,\infty}_{\ell m}$, is superfluous.

Because of those various cancellations, the calculation of $\mu\dot{e}/\varepsilon$ may now proceed in complete ignorance of the $O(\varepsilon^2)$ corrections to the motion of the particle. The only essential correction, the $O(\varepsilon^2)$ part of Ω_ϕ, has already been incorporated into Eq. (4.18). The computation of $B^{H,\infty\pm}_{\ell m}$ only requires a calculation accurate to first order in the eccentricity.

E. Calculation of $A^{H,\infty}_{\ell m}$ and $B^{H,\infty\pm}_{\ell m}$

The calculation follows the lines of Sec. IV B above. We find

$$A^{H,\infty}_{\ell m} = \frac{\pi}{i(\omega_m r_0)^2 Q^{\mathrm{in}}_{\omega_m \ell}}({}_0A^{H,\infty}_{\ell m} + {}_{-1}A^{H,\infty}_{\ell m} + {}_{-2}A^{H,\infty}_{\ell m}), \tag{4.22}$$

where (we momentarily remove all unnecessary indices for the sake of clarity)

$$\begin{aligned}{}_0A &= {}_0a f_0 R,\\ {}_{-1}A &= {}_{-1}a f_0\big[(2f_0 + i\omega_m r_0)R - f_0 r_0 R'\big],\end{aligned} \tag{4.23}$$

$$\begin{aligned}{}_{-2}A = {}_{-2}a f_0[\, & i\omega_m r_0(2-2v^2+i\omega_m r_0)R\\ & -2(f_0+i\omega_m r_0)f_0 r_0 R' + (f_0 r_0)^2 R''].\end{aligned}$$

Here, $\omega_m = m\Omega$, $f_0 = 1-2M/r_0 = 1-2v^2$, $R = R^{H,\infty}_{\omega_m \ell}(r_0)$, and a prime denotes differentiation with respect to r_0. Also

$$B^{H,\infty\pm}_{\ell m} = \frac{\pi}{i(\omega_\pm r_0)^2 Q^{\mathrm{in}}_{\omega_\pm \ell}}\Big({}_0B^{H,\infty\pm}_{\ell m} + {}_{-1}B^{H,\infty\pm}_{\ell m} + {}_{-2}B^{H,\infty\pm}_{\ell m}\Big), \tag{4.24}$$

where

$$
\begin{aligned}
{}_0B^{\pm} &= {}_0c^{\pm}R_{\pm} - {}_0a(f_0 r_0 R'_{\pm} - 4v^2 R_{\pm}), \\
{}_{-1}B^{\pm} &= {}_{-1}c^{\pm}\big[(2-4v^2+i\omega_{\pm}r_0)R_{\pm} - f_0 r_0 R'_{\pm}\big] \\
&\quad + {}_{-1}a\big[(4v^2-8v^4+6iM\omega_{\pm}-i\omega_{\pm}r_0)R_{\pm} - f_0(1+i\omega_{\pm}r_0)r_0R'_{\pm} + (f_0r_0)^2R''_{\pm}\big], \\
{}_{-2}B^{\pm} &= {}_{-2}c_{\pm}\big[(f_0r_0)^2R''_{\pm} - 2f_0(1-2v^2+i\omega_{\pm}r_0)r_0R'_{\pm} + i\omega_{\pm}r_0(2-2v^2+i\omega_{\pm}r_0)R_{\pm}\big] \\
&\quad - {}_{-2}a\big[(f_0r_0)^3R'''_{\pm} - 2i\omega_{\pm}r_0(f_0r_0)^2R''_{\pm} - f_0(2-8v^2+8v^4-10iM\omega_{\pm}+2i\omega_{\pm}r_0+\omega_{\pm}{}^2r_0{}^2)r_0R'_{\pm} \\
&\qquad + 2i\omega_{\pm}r_0(1-6v^2+4v^4-4iM\omega_{\pm}+i\omega_{\pm}r_0)R_{\pm}\big],
\end{aligned}
\tag{4.25}
$$

with $\omega_{\pm} = m\Omega \pm \Omega_r$ and $R_{\pm} = R^{H,\infty}_{\omega_{\pm}\ell}(r_0)$. We have introduced

$$
\begin{aligned}
{}_0a_{\ell m} &= \big[\lambda(\lambda+2)\big]^{1/2}\,{}_0Y_{\ell m}(\tfrac{\pi}{2},0)\,\big[2(1-2v^2)(1-3v^2)^{1/2}\big]^{-1}, \\
{}_{-1}a_{\ell m} &= i\lambda^{1/2}\,{}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)\,v\big[(1-2v^2)^2(1-3v^2)^{1/2}\big]^{-1}, \\
{}_{-2}a_{\ell m} &= -{}_{-2}Y_{\ell m}(\tfrac{\pi}{2},0)\,v^2\big[2(1-2v^2)^3(1-3v^2)^{1/2}\big]^{-1},
\end{aligned}
\tag{4.26}
$$

where $\lambda = (\ell-1)(\ell+2)$ and

$$
{}_sc^{\pm}_{\ell m} = {}_sa_{\ell m}\Big[2 - s - 2(3-s)v^2 \pm i(2+s)v(1-6v^2)^{1/2} \pm 2m(1-3v^2)(1-6v^2)^{-1/2}\Big]. \tag{4.27}
$$

The previous equations imply the following symmetry properties: $\bar{A}^{H,\infty}_{\ell,-m} = (-1)^{\ell}A^{H,\infty}_{\ell m}$, $\bar{B}^{H,\infty\mp}_{\ell,-m} = (-1)^{\ell}B^{H,\infty\pm}_{\ell m}$, and for $m=0$, $\bar{B}^{H,\infty -}_{\ell,0} = B^{H,\infty +}_{\ell,0}$.

F. The adiabatic approximation

We conclude this section by formulating the conditions under which the adiabatic approximation holds. The results of this subsection were summarized in Sec. I B.

We require that the *inspiral time scale* $r_0/|\dot{r}_0|$ always be much larger than the *orbital period* $2\pi/\Omega_r$. Using Eq. (2.10) this requirement becomes

$$
\mu/M \ll \frac{1}{4\pi}\frac{v^5(1-6v^2)^{3/2}}{(1-3v^2)^{3/2}}\frac{1}{(M/\mu)^2\dot{E}^{(0)}}. \tag{4.28}
$$

At large radii, $r_0 \gg 6M$, $(M/\mu)^2\dot{E}^{(0)} \simeq 32v^{10}/5$, and the adiabatic condition (4.28) becomes $\mu/M \ll (5/128\pi)v^{-5}$. This is superseded by a wide margin by the condition $\mu/M \ll 1$, which ensures that the gravitational perturbations are linear. Near $r_0 = 6M$ we may use the numerical results of Sec. V C and put $(M/\mu)^2\dot{E}^{(0)} \simeq 9\times10^{-4}$, and Eq. (4.28) becomes $\mu/M \ll 2.8(1-6v^2)^{3/2}$. This condition is far more restrictive than $\mu/M \ll 1$.

V. ANALYTICAL AND NUMERICAL RESULTS

A. Weak-field, slow-motion case

For $r_0 \gg 6M$ and $v = (M/r_0)^{1/2} \ll 1$, the analytical techniques developed in paper I may be used to calculate, approximately, $R^H_{\omega_{\pm}\ell}(r)$ and $Q^{\rm in}_{\omega_{\pm}\ell}$. The expressions for these quantities may then be substituted into the equations of Secs. IV D and E, to obtain $\mu\dot{\varepsilon}/\varepsilon$ in the form of a post-Newtonian expansion. As was mentioned previously, there is no need to calculate $\dot{E}^H$ and $\dot{L}^H$, because they contribute only at order v^8 to the post-Newtonian expansion [27]. The calculations are straightforward and will be presented without much detail.

The calculation of $\mu\dot{\varepsilon}/\varepsilon$ up through order v^3 beyond Newtonian requires the computation of $B^{H\pm}_{\ell m}$ for $\ell=2$ and $\ell=3$. We may use the symmetry properties of $B^{H\pm}_{\ell m}$ and only consider non-negative values of m; for $m=0$, only $B^{H+}_{\ell,0}$ is required. We find

$$
\begin{aligned}
B^{H+}_{2,2} &= (\pi/5)^{1/2}v^2\big[-18 + 27v^2 - 54\pi v^3 + O(iv^3,v^4)\big], \\
B^{H-}_{2,2} &= (\pi/5)^{1/2}v^2\big[6 + \tfrac{221}{7}v^2 + 6\pi v^3 + O(iv^3,v^4)\big], \\
B^{H+}_{2,1} &= (\pi/5)^{1/2}v^2\big[-\tfrac{16}{3}iv + \tfrac{176}{21}iv^3 + O(v^4)\big], \\
B^{H-}_{2,1} &= O(v^6), \\
B^{H+}_{2,0} &= (\pi/30)^{1/2}v^2\big[-4 + \tfrac{206}{7}v^2 - 4\pi v^3 + O(iv^3,v^4)\big],
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
B^{H+}_{3,3} &= (\pi/42)^{1/2}v^2\big[64iv + O(v^3)\big], \\
B^{H-}_{3,3} &= (\pi/42)^{1/2}v^2\big[-24iv + O(v^3)\big], \\
B^{H+}_{3,1} &= (\pi/70)^{1/2}v^2\big[\tfrac{8}{3}iv + O(v^3)\big], \\
B^{H-}_{3,1} &= O(v^6),
\end{aligned}
$$

and $B^{H\pm}_{3,m} = O(v^5)$ for $m = \{0,2\}$. In the above, the notation $O(iv^3)$ signifies that those terms of order v^3, which are purely imaginary, do not contribute, at order v^3, to $|B^{H\pm}_{\ell m}|^2$. That the coefficients $B^{H-}_{\ell,1}$ are so small is due to the fact that, for $m=1$, $\omega_- = \Omega - \Omega(1-6v^2)^{1/2} = 3v^2\Omega + O(v^4)$; since ω_- is suppressed by a factor v^2 with respect to ω_+, the resulting $B^{H-}_{\ell,1}$ is much smaller than $B^{H+}_{\ell,1}$.

We now substitute Eqs. (5.1) into (4.19) and (4.18), and use the post-Newtonian expansion

$$
\dot{E}^{(0)} = \dot{E}_N\Big[1 - \tfrac{1247}{336}v^2 + 4\pi v^3 + O(v^4)\Big] \tag{5.2}
$$

derived in paper I [$\dot{E}_N = \frac{32}{5}(\mu/M)^2v^{10}$ is the leading-order Newtonian expression]; this yields

$$\dot{\varepsilon} = \dot{\varepsilon}_N \Big[1 - \tfrac{6849}{2128} v^2 + \tfrac{985}{152} \pi v^3 + O(v^4) \Big], \tag{5.3}$$

where $\dot{\varepsilon}_N$ is the leading-order Newtonian expression,

$$\mu \dot{\varepsilon}_N / \varepsilon = -\tfrac{304}{15} (\mu/M)^2 v^8. \tag{5.4}$$

Throughout the post-Newtonian regime, $v \ll 1$, $\dot{\varepsilon}$ is negative—radiation reaction therefore reduces the eccentricity.

Substitution of Eq. (5.2) into (2.10) and use of Eqs. (5.3) and (5.4) yields Eq. (1.4).

B. Highly relativistic case

Analytical calculations may also be carried out in the case where r_0 approaches $6M$. Because $h(v)$ diverges when $v^2 \to 1/6$, cf. Eq. (4.21), and because both $\dot{E}^{(0)}$ and Γ have well-defined limits when $r_0 \to 6M$, $\mu\dot{\varepsilon}/\varepsilon$ is dominated by the second term on the right-hand side of Eq. (4.18).

Our claim that $\dot{E}^{(0)}$ is well behaved in the vicinity of $r_0 = 6M$ can be substantiated by (i) an inspection of the perturbation formalism, which shows no sign of a singularity at $r_0 = 6M$; in particular, $R^{H,\infty}_{\omega\ell}(r)$ and $Q^{\rm in}_{\omega\ell}$, for $M\omega = mM\Omega = 6^{-3/2}m$, are well behaved. And (ii) with numerical calculations, which confirm the proper behavior of $\dot{E}^{(0)}$ in the vicinity of $r_0 = 6M$.

The proper behavior of Γ can be established as follows. Writing $\delta = (1-6M/r_0)^{1/2} \ll 1$ we first infer the various δ dependence of the relevant quantities. Using the equations of Sec. IV E we find that the ${}_s a_{\ell m}$ are independent of δ, while ${}_s c^{\pm}_{\ell m} = \pm m \, {}_s a_{\ell m} \delta^{-1} + O(\delta^0)$. Using the fact that $R^{H,\infty}_{\omega_\pm \ell}(r)$ and $Q^{\rm in}_{\omega_\pm \ell}$ are properly behaved, Eq. (4.24) then implies $B^{\pm}_{\ell m} = \pm k_{\ell m}\delta^{-1} + k^{\pm}_{\ell m} + O(\delta)$, where $k_{\ell m}$ and $k^{\pm}_{\ell m}$ are independent of δ. The fact that, at leading order in δ, $B^{+}_{\ell m}$ and $B^{-}_{\ell m}$ differ only by a sign is an important aspect of this discussion. [The case $m = 0$ requires special thought, since then $\omega_\pm = \pm\delta\Omega$, and Eq. (4.24) suggests that $B^{\pm}_{\ell,0}$ might be more singular than $O(\delta^{-1})$. However, a careful study of the Teukolsky equation reveals that this does not happen.] The final step is to substitute our result for $B^{\pm}_{\ell m}$ into Eq. (4.19), and notice a remarkable cancellation of the leading-order, $O(\delta^{-2})$ terms. Multiplication by $\Omega_r = \delta\Omega$ then ensures that each term in the sum over ℓ and m is $O(\delta^0)$. That Γ has a well-defined limit follows from the fact that the sum converges for every $r_0 \geq 6M$, which was verified numerically.

Having established that Γ and $\dot{E}^{(0)}$ have well-defined limits when r_0 approaches $6M$, Eq. (4.18) reduces to

$$\mu\dot{\varepsilon}/\varepsilon \sim \frac{3}{2\sqrt{2}} \dot{E}^{(0)}\big|_{r_0=6M} (1 - 6M/r_0)^{-2}, \tag{5.5}$$

for $r_0 \to 6M$.

Substitution of Eqs. (5.5) and (2.10) into (1.2) yields Eq. (1.4).

C. General case—numerical integration

When r_0 is neither very large nor very close to $6M$, $R^{H,\infty}_{\omega_\pm \ell}(r)$ and $Q^{\rm in}_{\omega_\pm \ell}$ must be calculated numerically. By performing the integration for a wide range of orbital radii we obtain $\mu\dot{\varepsilon}/\varepsilon$ as a function of r_0. The numerical results may then be checked against the limiting cases (5.3) and (5.5).

We have carried out the numerical integration using a straightforward generalization of the algorithm presented in paper II [19] (we shall not repeat the discussion of paper II here). We have constructed our integrator upon the Bulirsh-Stoer method, using FORTRAN subroutines given in Ref. [29]; all operations were performed with double precision. We have verified that our numerical results are in agreement with the limiting cases of Secs. V A and B; this agreement gives us great confidence in our results, which are summarized in Fig. 1.

It is easy to obtain high numerical accuracy by adjusting the tolerance of our integrator to a very small value; we have typically chosen a tolerance of 10^{-6}. Although it is hard to *prove*, we *believe* our numbers to be accurate to at least six significant digits. Our estimate of the critical radius r_c (at which $\dot{\varepsilon}$ changes sign) should also be accurate to six significant digits; we have chosen to quote only five digits in Eq. (1.1).

The accuracy of our numerical results is also subject to errors of non-numerical origin, which arise because the infinite sum over ℓ must be truncated. The magnitude of the error thus induced can be controlled by requiring that the terms ignored contribute to a fractional error no greater than a certain value ζ. Since a multipole of order ℓ contributes a fractional amount of order $(M/r_0)^{\ell-2}$ to $\dot{E}$ and $\dot{L}$ [18] we arrive at the following criterion on the maximal value of ℓ which needs be included in the sum:

$$\ell_{\rm max} \geq 2 - \ln\zeta / \ln(r_0/M). \tag{5.6}$$

For example, choosing $\zeta = 10^{-6}$ yields $\ell_{\rm max} = 10$ for $r_0/M = 6$ and $\ell_{\rm max} = 3$ for $r_0/M = 10^6$.

The graph of Fig. 2 was obtained by numerically integrating Eq. (1.5), in the range between $r_0/M = 6 + 10^{-8}$ and $r_0/M = 100$. The integration was performed using the extended trapezoidal rule, which is accurate enough for our purposes.

VI. CONCLUSION

We have established in this paper that a particle in circular motion around a nonrotating black hole remains on a circular orbit under the influence of radiation reaction. Furthermore, we have shown that circular orbits are *stable* only if the orbital radius is greater than a critical radius $r_c \simeq 6.6792M$, where M is the mass of the black hole.

Also, our analysis permits us to follow the evolution, under radiation reaction, of an orbit's eccentricity, so long as it remains small. We find that the eccentricity is reduced by many orders of magnitude during the post-Newtonian phase of the inspiral, but that it starts increasing once the orbit's radius is smaller than r_c. For reasonable values of μ/M, the eccentricity increases by at most an order of magnitude before the adiabatic approximation breaks down and the particle begins its plunge toward the black hole.

Our analysis is restricted by four major assumptions: (i) the black hole is nonrotating, (ii) the eccentricity is always small, (iii) the gravitational perturbations are linear, and (iv) the adiabatic approximation is valid. On the other hand, our analysis is not limited to weak-field, slow-motion situations; it is valid for particle motion in strong gravitational fields.

We now examine whether any of our four assumptions could be relaxed, and at what cost, in future work.

Assumption (i) could be removed without much effort, that is, our analysis could be extended to the case of a rotating black hole, if and only if the orbit lies in the hole's equatorial plane. In the more general and more interesting situation of nonequatorial orbits, the formulation of the problem of radiation reaction would take a significantly different form. In such cases, the motion possesses a nonvanishing value of the Carter constant, whose rate of change cannot be simply (if at all) related to the rates of change of energy and (vectorial) angular momentum. The general analysis would therefore require techniques more sophisticated than the ones utilized here; for example, a numerical implementation of Gal'tsov's formalism [14].

Assumption (ii) is one of simplicity and could be removed without introducing additional conceptual difficulties. For example, a calculation valid to higher order in the eccentricity could be carried out, at the price of a modest effort. A calculation valid to all orders in ε could also be performed by numerical integration of the geodesic equations; see Ref. [17].

Assumption (iii) cannot be removed easily. Strong-field analyses valid for arbitrary mass ratios would require either the formulation of a higher-order perturbation theory, or the complete numerical solution of Einstein's equations for the two-body problem. Both approaches are still a long way into the future. A recent analysis by Kidder, Will, and Wiseman [30] suggests that the value of the critical radius r_c should increase with the mass ratio μ/M.

Assumption (iv) could be removed (at least partially) by incorporating, at the very beginning, radiation-reaction effects into the motion of the particle. Thus the motion would be nongeodesic to begin with, and higher-order radiation-reaction effects could then be calculated. These higher-order effects would be quite small at large orbital radii; but for a given mass ratio, there exists an orbital radius r_0 at which the adiabatic approximation breaks down, and at which higher-order effects would become important. The breakdown of the adiabatic approximation, and the transition from slow inspiral to fast plunge, is discussed in Ref. [15].

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