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Naked singularities in spherically symmetric inhomogeneous Tolman-Bondi dust cloud collapse

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We investigate here the occurrence and nature of a naked singularity for the inhomogeneous gravitational collapse of Tolman-Bondi dust clouds. It is shown that the naked singularities form at the center of the collapsing cloud in a wide class of collapse models, which includes the earlier cases considered by Eardley and Smarr and Christodoulou. This class also contains self-similar as well as non-self-similar models. The structure and strength of this singularity are examined, and the question is investigated as to when a nonzero measure set of nonspacelike trajectories could be emitted from the singularity, as opposed to isolated trajectories coming out. It is seen that the weak energy condition and positivity of energy density ensures that the families of nonspacelike trajectories come out of the singularity. The curvature strength of the naked singularity is examined, which provides an important test for its physical significance. This is done in terms of the strong curvature condition, which ensures that all the volume forms must be crushed to zero size in the limit of approach to the singularity, and, also, the divergence of the Kretschmann scalar $\mathcal{K}=R^{abcd}R_{abcd}$ is pointed out. We show that the class considered here contains subclasses of solutions which admit strong curvature naked singularities in either of the senses stated above. The conditions are discussed for the naked singularity to be globally naked. An implication for the fundamental issue of the final fate of gravitational collapse is that naked singularities need not be considered as artifacts of geometric symmetries of space-time such as self-similarity, but arise in a wide range of gravitational collapse scenarios once the inhomogeneities in the matter distribution are taken into account. It is argued that a physical formulation for the cosmic censorship may be evolved which avoids the features above. Possibilities in this direction are suggested while indicating that the analysis presented here should be useful for any possible rigorous formulation of the cosmic censorship hypothesis.

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I. INTRODUCTION

It is generally believed that a generic gravitational collapse would commence from a highly inhomogeneous initial state. This will be described in terms of an inhomogeneous energy density distribution given as a regular initial data on a spacelike hypersurface. The general class of solutions of Einstein's field equations describing spherically symmetric dust clouds, independent of the homogeneity assumption, was given by Tolman [1], which was further developed and studied by Bondi [2]. This class could be used to model the gravitational collapse of matter from general inhomogeneous initial conditions and one can study the fundamentally important issue of the final fate of gravitational collapse of a massive star which has exhausted its nuclear fuel within this framework. The assumptions involved here are the vanishing pressure and the spherical symmetry of the matter distribution which is in the form of dust. One could argue that, in the final stages of collapse, the matter distribution would become almost spherically symmetric, and that the pressures should play a minor role to justify the dust approximation. From our view point, however, the main advantage is that subject to these conditions these models allow us to describe the evolution of inhomogeneous distributions of matter, which offers a very general class for the study of the gravitational collapse phenomena.

A special case of the these Tolman-Bondi classes of solutions is the Oppenheimer-Snyder [3] study of a completely homogeneous dust cloud collapse with zero pressure. This example has been studied in great detail and has provided much insight towards understanding the final fate of a continually collapsing massive body such as a star, which could achieve no equilibrium state because of the dominance of gravitational forces. This case provides the basic motivation for the idea of formation of black holes as the final state of collapse, and the related cosmic censorship hypothesis [4] which broadly states that the singularities forming in gravitational collapse must necessarily be hidden behind the event horizons of gravity and hence permanently invisible to the outside observers. This cosmic censorship hypothesis plays a fundamental role in both the theory and applications of the black-hole physics and has been recognized as one of the most important open problems in the general theory of relativity and gravitation physics today.

As it turns out, despite several attempts no proof or any precise mathematical formulation of the cosmic censorship has been available so far. Further, the completely homogeneous dust collapse mentioned above could also be viewed as a special case which forms a set of zero measure in the general inhomogeneous class represented by

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the Tolman-Bondi solutions. It thus becomes imperative to study the general class of Tolman-Bondi models in greater detail in order to understand the final fate of a gravitationally collapsing massive body when the effects of inhomogeneities are taken into account. In fact, it was pointed out by the numerical simulations of Eardley and Smarr [5] that naked singularities not covered by event horizons arise in the marginally bound Tolman-Bondi collapse, and subsequently a class of such models was studied in detail analytically by Christodoulou [6] to draw the same conclusion. However, these singularities were shown to be gravitationally weak by Newman [7], who studied the curvature strengths of such naked singularities and conjectured that nature avoids strong curvature naked singularities.

Our purpose here is to study the Tolman-Bondi inhomogeneous collapse for a rather general class of models, which includes the above classes, and to study the formation and structure of the naked singularity occurring at the center of the collapsing cloud. We show that the formation of a naked singularity is a generic feature for a very wide range of solutions considered here. We have recently shown [8] such a result for the general class of self-similar models describing the gravitational collapse of a perfect fluid, where it is shown that a powerfully strong curvature naked singularity forms from which families of nonspacelike geodesics escape in the space-time. Further to this, the class considered here is shown to include all the self-similar Tolman-Bondi models as well as a wide range of non-self-similar models. This indicates that the naked singularity may not be regarded as the consequence of the geometric property of self-similarity only [9]. A naked singularity may not be treated as a serious enough situation if only a single nonspacelike trajectory escaped from it. Thus, we examine the sufficient conditions when families of nonspacelike geodesics could escape from the naked singularity. Interestingly, it turns out that the validity of the weak energy condition in the space-time ensures the existence of such families. This is analogous to the results of Ref. [8] for the self-similar class. We also discuss the issue as to when the naked singularity will be globally naked, i.e., visible to far away observers.

The organization of the paper is as below. In Sec. II, the basic parameters of the Tolman-Bondi models describing the inhomogeneous dust collapse are specified. The existence and structure of the naked singularity is analyzed in Sec. III. We also characterize here the conditions that ensure that families of nonspacelike geodesics, rather than a single isolated trajectory, are emitted from the naked singularity. In particular, it is shown that the weak energy condition, together with the positivity of energy density, implies that a nonzero measure set of nonspacelike geodesics comes out from the naked singularity. The global versus local nakedness of the singularity is also discussed here. The curvature strength of the naked singularity provides an important test of the physical significance for the same. This issue is examined in Sec. IV, where it is shown that the models considered here include both self-similar as well as non-self-similar classes admitting a strong curvature singularity in a powerful sense. The final Sec. V summarizes some of the implications and conclusions.

II. TOLMAN-BONDI SPACE-TIMES

The Tolman-Bondi metric representing collapse of a spherically symmetric inhomogeneous dust cloud in the comoving coordinates (i.e., $u^i=\delta^i_t$) is given by

$$ds^2=-dt^2+\frac{R'^2}{1+f}dr^2+R^2(d\theta^2+\sin^2\theta d\phi^2) , \qquad (1)$$

$$T^{ij}=\epsilon\delta^i_t\delta^j_t, \quad \epsilon=\epsilon(t,r)=\frac{F'}{R^2R'} , \qquad (2)$$

where T^{ij} is the stress-energy tensor, ϵ is the energy density, and R is a function of both t and r given by

$$\dot{R}^2=\frac{F}{R}+f . \qquad (3)$$

Here the dot and the prime denote partial derivatives with respect to the parameters t and r, respectively, and, as we are only concerned with the gravitational collapse of dust, we require $\dot{R}(t,r)<0$. The quantities F and f are arbitrary functions of r. The quantity $4\pi R^2(t,r)$ gives the proper area of the mass shells and the area of such a shell at $r=$const goes to zero when $R(t,r)=0$. Integration of Eq. (3) gives

$$t-t_0(r)=-\frac{R^{3/2}G(-fR/F)}{\sqrt{F}} , \qquad (4)$$

where $G(y)$ is a strictly real positive and bound function which has the range $1\geq y\geq -\infty$ and is given by

$$G(y)=\left[\frac{\arcsin\sqrt{y}}{y^{3/2}}-\frac{\sqrt{1-y}}{y}\right] \quad \text{for } 1\geq y>0 ,$$

$$G(y)=\tfrac{2}{3} \quad \text{for } y=0 , \qquad (5)$$

$$G(y)=\left[\frac{-\operatorname{arcsinh}\sqrt{-y}}{(-y)^{3/2}}-\frac{\sqrt{1-y}}{y}\right] \quad \text{for } 0>y\geq -\infty ,$$

and $t_0(r)$ is a constant of integration. We thus have in all three arbitrary functions of r, namely, $f(r)$, $F(r)$, and $t_0(r)$. One could, however, use the remaining coordinate freedom left in the choice of scaling of r in order to reduce the number of such arbitrary functions to two. We therefore rescale R using this coordinate freedom such that

$$R(0,r)=r . \qquad (6)$$

Then $t_0(r)$ is evaluated by using the equation above and (4) to give

$$t_0(r)=\frac{r^{3/2}G(-fr/F)}{\sqrt{F}} . \qquad (7)$$

The time $t=t_0(r)$ corresponds to the value $R=0$ where the area of the shell of matter at a constant value of the coordinate r vanishes. It follows that the singularity curve $t=t_0(r)$ corresponds to the time when the matter shells meet the physical singularity. Thus, the range of the coordinates is given by

$$0 \le r < \infty, \quad -\infty < t < t_0(r) \ . \tag{8}$$

It follows that unlike the collapsing Friedmann case, where the physical singularity occurs at a constant epoch of time (say, at $t=0$), the singular epoch now is a function of r as a result of inhomogeneity in the matter distribution. One could recover the Friedmann case from the above equations if we set $t_0(r)=t_0'(r)=0$.

The function $f(r)$ classifies the space-time as bound, marginally bound, or unbound depending on the range of its values which are

$$f(r)<0, \quad f(r)=0, \quad \text{and } f(r)>0 \ ,$$

respectively. The function $F(r)$ can be interpreted as the weighted mass (weighted by the factor $\sqrt{1+f}$) within the dust ball $\mathcal{B}$ of coordinate radius r which is conserved in the following sense:

$$m(r)=\frac{F(r)}{2}=\int_{\mathcal{B}}(1+f)^{1/2}\epsilon(t,r)dv$$
$$=4\pi\int_0^r \rho(r)r^2 dr \ , \tag{9}$$

where $\epsilon(0,r)=\rho(r)$. For physical reasonableness the weak energy condition would be assumed throughout the space-time, i.e., $T_{ij}V^iV^j \ge 0$ for all nonspacelike vectors V^i. This implies that the energy density ϵ is everywhere positive ($\epsilon \ge 0$), including the region near $r=0$. Partial derivatives of R such as R' and $\dot{R}'$ are of importance in our analysis. We get, from the Eqs. (3)–(7),

$$R'=r^{\alpha-1}\left[(\eta-\beta)X+[\Theta-(\eta-\tfrac{3}{2}\beta)X^{3/2}G(-PX)]\times\left[P+\frac{1}{X}\right]^{1/2}\right]$$
$$\equiv r^{\alpha-1}H(X,r) \ , \tag{10}$$

$$\dot{R}'=\frac{\Lambda^{1/2}}{2rX^2}\left[-\beta X^2\left[\frac{1}{X}+P\right]^{1/2}+\Theta-(\eta-\tfrac{3}{2}\beta)X^{3/2}G(-PX)\right]$$
$$\equiv\frac{-N(X,r)}{r} \ , \tag{11}$$

where we have put

$$X=(R/r^\alpha) \ ,$$
$$\eta=\eta(r)=r\frac{F'}{F} \ ,$$
$$\beta=\beta(r)=r\frac{f'}{f} \ , \tag{12}$$
$$p=p(r)=rf/F \ ,$$

$$P=pr^{\alpha-1} \ ,$$
$$\Lambda=\frac{F}{r^\alpha} \ ,$$
$$\Theta\equiv\frac{t_0'\sqrt{\Lambda}}{r^{\alpha-1}} \tag{13}$$
$$=\frac{1+\beta-\eta}{(1+p)^{1/2}r^{3(\alpha-1)/2}}+\frac{(\eta-\frac{3}{2}\beta)G(-p)}{r^{3(\alpha-1)/2}} \ .$$

The function $\beta(r)$ is defined to be zero when f is constant and zero. The factor r^α has been introduced here for the sake of convenience in examining the structure of the naked singularity. The exact value of the positive constant $\alpha \ge 1$ is to be determined and will depend on the different models of the space-time which allow naked singularities. Functions $H(X,r)$ and $N(X,r)$ are defined by Eqs. (10) and (11). Using the scaling given by (6), the energy density ϵ on the hypersurface $t=0$ is written as $\epsilon=F'/r^2$. Since the weak energy conditions are satisfied and F is a function of r only, it follows that $F' \ge 0$ throughout the space-time. One can write the energy density as

$$\epsilon=\frac{\eta\Lambda}{R^2H} \ . \tag{14}$$

Since $F'=\eta\Lambda r^{\alpha-1}$, it follows from the above that everywhere $H(X,r)\ge 0$ and $\eta\Lambda \ge 0$ as a consequence of the weak energy condition.

Singularities are the boundary points of the space-time where the normal differentiability and manifold structures break down. In other words, these are the points where the energy density given by Eq. (2), or the curvature quantities such as the scalar polynomials constructed out of the metric tensor and the Riemann tensor, diverge. One example of such a quantity is the Kretschmann scalar $\mathcal{K}=R_{abcd}R^{abcd}$, which is given in the Tolman-Bondi case by

$$\mathcal{K}=12\frac{F'^2}{R^4R'^2}-32\frac{FF'}{R^5R'}+48\frac{F^2}{R^6} \ . \tag{15}$$

Such singularities are indicated by the existence of incomplete future- or past-directed nonspacelike geodesics in the space-time which terminate at the singularity. Then one requires that the curvature quantities stated above assume unboundedly large values in the limit of approach to the singularity along the nonspacelike geodesics terminating there. If such a condition is satisfied, then one would like to consider the singularity to be a physically significant curvature singularity.

In Tolman-Bondi space-times singularities occur, as one can see from Eqs. (2) and (15), at points where $R=0$, which are called shell-focusing singularities, and also at points where $R'=0$. At the points where $R'=0$ the Tolman-Bondi metric is degenerate and these are called shell crossings. In the context of Tolman-Bondi space-times the points $R>0$, $F'>0$, where $R'=0$, are called the shell-crossing singularities [7]. Such shell-crossing singularities in Tolman-Bondi space-times have been analyzed in detail in the literature [10,11], and their nature appears to be fairly well understood. Even though such shell-crossing singularities could be locally naked, the important point is they have been shown to be gravitationally weak [7]. Thus, it is generally believed that such shell-crossing singularities need not be taken seriously as far as the cosmic censorship conjecture is concerned. The absence of shell-crossing singularities in a space-time turns out to be related to the condition that the function $t_0(r)$ giving the proper time for the shells to fall into the physical singularity should be a monotonically increasing function. The dust density and certain components of the curvature blow up near such a singularity. However, the causal structure of the space-time can be extended through such a singularity and the space-time metric can also be defined in the neighborhood of such a point in a distributional sense [12]. In the context of such a situation, we do not consider here such shell crossings, and assume that there are no shell-crossing singularities in the space-time (except probably right at the center $r=0$ [10]). This does not involve any loss of generality as our basic purpose here is to examine the formation and local structure of the shell-focusing naked singularity at the center of the collapsing cloud. Whereas the existence of shell crossings will not affect the qualitative nature of these general conclusions, the above assumption allows the calculations to be presented in a more transparent manner.

Unlike the shell crossings, the space-time metric, however, admits no extension through a shell-focusing singularity occurring at $R=0$ which is more difficult to ignore. A shell-focusing singularity can be avoided only by rejecting the forms of matter such as dust as the fundamental forms of matter (see, e.g., [5]). Hence, we investigate here the occurrence of such shell-focusing singularities at the center of the collapsing dust cloud and examine their nature and structure for the Tolman-Bondi space-times. It has been shown earlier [6] that a shell-focusing singularity occurring at $r>0$, $R=0$ is totally spacelike and therefore our discussion would be confined to the singularity at $r=0$.

The points (t_0,r_0), where a shell-focusing singularity $R(t_0,r_0)=0$ occurs, are related by Eq. (4). The singularity here occurs at $r=r_0$ at the coordinate time $t=t_0$ and we would call the singularity a *central singularity* if it occurs at $r=0$. Earlier work [3,4] has shown that this central shell-focusing singularity is naked, though gravitationally weak, for a class of Tolman-Bondi space-times for which the energy density (which is assumed to be positive everywhere and is taken to be nonzero at $r=0$) and the metric are even smooth functions of t and r. Translated in terms of parameters defined above, this corresponds to the class for which $\eta(0)=3$, $\beta(0)=2$, and $p(r)$ is an even smooth function of r. In terms of functions $F(r)$ and $f(r)$ it amounts to the conditions

$$F(r)=r^3\mathcal{F}(r), \quad \infty>\mathcal{F}(0)>0, \quad 0<p(r)\le 1 \ . \tag{16}$$

It was, however, pointed out by Waugh and Lake [13] and Ori and Piran [14] that this class of gravitationally weak naked singularities excludes the self-similar Tolman-Bondi models, where they showed the singularity to be gravitationally strong along the Cauchy horizon, which is a null geodesic coming out of the singularity. Further, Grillo [15] pointed out an example in the case of bound Tolman-Bondi models which are non-self-similar and the naked singularity is gravitationally strong. In fact, we have pointed out recently [16] that the naked singularity exists and is gravitationally strong for a wide class of Tolman-Bondi models which are non-self-similar in general and include all the self-similar models as a special subclass. In the notation used here, these models are characterized by the conditions $\eta(0)=1$ with $F(r)$ and $f(r)$ being analytic at $r=0$.

Throughout the present consideration we would require rather general differentiability conditions on the functions $F(r)$ and $f(r)$ in that they will be assumed to be at least C^1 at the center $r=0$, $\infty>\eta(0)>0$, and $\beta(0)$ is finite [17]. We note that the function f, and also its first derivatives (through R'), enter the metric potentials. One might actually argue that the above condition is a more general condition than should be required because it is often a customary practice to assume that the metric is C^2 differentiable (which ensures again the above requirement), so that the metric transformations and other functions connected with the metric are well defined to do regular physics. Hence, such a condition may be considered to be physically reasonable and a rather general differentiability requirement which includes practically all the inhomogeneous collapse Tolman-Bondi models of interest. In fact, one could argue that if the metric is not C^2 differentiable, but say only C^1 on initial surface, it may be considered as being already naked singular and not defining a regular initial data on an initial spacelike hypersurface.

In order to represent the gravitational collapse scenario, we assume the energy density ϵ to have a compact support on an initial spacelike hypersurface and the Tolman-Bondi space-times given by (1) can be matched at some $r=\text{const}=r_c$ to the exterior Schwarzschild field

$$ds^2=-\left[1-\frac{2M}{r_S}\right]dT^2+\frac{dr_S^2}{1-2M/r_S}+r_S^2d\Omega^2 \ , \tag{17}$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$. The value of the Schwarzschild radial coordinate is $r_S=R(t,r_c)$ at the boundary $r=r_c$. We have $m(r_c)=M$, where M is the total Schwarzschild mass enclosed within the dust ball of coordinate radius of $r=r_c$. Without going into further details of the matching conditions we would like to say a few words regarding the apparent horizon. The apparent horizon in the interior dust ball lies at $R=F(r)$. From (4) and (7) one can see that the corresponding time $t=t_{\rm AH}(r)$ is given by

$$t=t_{\rm AH}(r)=\frac{r^{3/2}G(-p)}{\sqrt{F}}-FG(-f) \ . \tag{18}$$

It has been shown earlier [6,7] that emissions from the shell-focusing singularity $R(t_0,r_0)=0$ for all $r_0>0$ would lie in the region above $t=t_{\rm AH}$, i.e., $t_0>t_{\rm AH}$ for all $r_0>0$, t_0 being the time when singularity at $r=r_0$ occurs. Hence all radiations would be future trapped from shell-focusing singularities at $r>0$. At $r=0$, however, $t(0)=t_{\rm AH}(0)$ and the singularity could be at least locally naked. Any light ray terminating at this singularity in the past goes to the future infinity if it reaches the surface of the cloud $r=r_c$ earlier than the apparent horizon at $r=r_c$. In such a case the singularity would be globally naked. We now examine this central singularity in the section below.

III. THE EXISTENCE AND STRUCTURE OF NAKED SINGULARITY

In this section we investigate the existence of the naked central singularity for the general class of Tolman-Bondi space-times under consideration. The singularity is naked if there are future directed nonspacelike curves in the space-time with their past end point at the singularity. The existence of such curves implies that either photons or timelike particles can be emitted from the singularity. In particular, we will examine the future directed nonspacelike geodesics for their past end point at the singularity. Other related issues examined here are when a nonzero measure set of nonspacelike trajectories will meet the singularity in the past, rather than a single isolated geodesic, and when such a singularity will be globally naked.

A. The existence

The tangents $K^a=dx^a/dk$ for the outgoing nonspacelike geodesics in the Tolman-Bondi space-time given by (1) can be written as

$$K^t=\frac{dt}{dk}=\frac{\mathcal{P}}{R}\ , \tag{19}$$

$$K^r=\frac{dr}{dk}=\frac{\sqrt{1+f}\,\sqrt{\mathcal{P}^2-l^2+BR^2}}{RR'}\ , \tag{20}$$

$$(K^\theta)^2+\sin^2\theta(K^\phi)^2=l^2/R^4\ . \tag{21}$$

Here l is an impact parameter that labels different geodesics and vanishes ($l=0$) for radial trajectories, B characterizes the type of geodesics, i.e., $B=0$ for null and $B=-1$ for timelike curves, and the function $\mathcal{P}=\mathcal{P}(t,r)$ satisfies the differential equation

$$\frac{d\mathcal{P}}{dk}+(\mathcal{P}^2-l^2+BR^2)\left[\frac{\sqrt{1+f}\,\dot{R}'}{RR'}-\frac{\dot{R}}{R^2}\right] -(\mathcal{P}^2-l^2+BR^2)^{1/2}\mathcal{P}\frac{\sqrt{1+f}}{R}+B\dot{R}=0\ . \tag{22}$$

The parameter k is an affine parameter along the geodesics. For future-directed nonradial trajectories that meet the central singularity at $R=0$ in past, it follows from Eq. (20) that $\mathcal{P}\geq l$ near the singularity.

If the outgoing nonspacelike geodesics are to terminate in the past at the central singularity $r=0$, which occurs at some time $t=t_0$ at which $R(t_0,0)=0$, then along such geodesics we have $R\to 0$ as $r\to 0$. The following is satisfied along nonspacelike geodesics:

$$\frac{dR}{du}=\frac{1}{\alpha r^{\alpha-1}}\left[\dot{R}\frac{dt}{dr}+R'\right] =\left[1-\frac{\mathcal{P}(f+\Lambda/X)^{1/2}}{\sqrt{1+f}\,\sqrt{\mathcal{P}^2-l^2+BR^2}}\right]\frac{H(x,u)}{\alpha} \equiv U(X,u)\ , \tag{23}$$

where we have put $u=r^\alpha$. The function $H(X,r)$ in the above equation is strictly positive and nonzero for all $r>0$ as a consequence of (10) and (14). For an outgoing geodesic, (dR/du) must be positive while the negative value for this quantity means the geodesic is ingoing. The point $u=r^\alpha=0$, $R=0$ is a singularity of the above differential equation.

It is now essential to understand the exact nature of this singularity. If the functions appearing in the numerator and denominator of (23) are expandable and contain linear terms, then one can apply the standard analysis on the classification of singular points of first-order differential equations [18] to understand the nature of this singularity. However, in the case otherwise, the same could be understood only by means of studying the detailed behavior of the characteristic curves in the vicinity of the singularity. If these characteristics terminate at the singularity in past with a definite tangent, this is determined by the limiting value of $X=R/r^\alpha=R/u$ at $R=0$, $u=0$. If the nonspacelike geodesics meet the singularity with a definite value of tangent, then using Eq. (23) and l'Hospital rule we get, for the value of X_0,

$$X_0=\lim_{R\to 0,u\to 0}\frac{R}{u}=\lim_{R\to 0,u\to 0}\frac{dR}{du} =\lim_{R\to 0,u\to 0}U(X,u)=U(X_0,0)\ . \tag{24}$$

If a real and positive value of X_0 satisfies the above equation then the singularity could be naked. Real and positive roots of the above equation gives the possible values of tangents the outgoing geodesics can have at the singularity. Thus, if a real and positive value of $X=X_0$ satisfying the above equation exists, then integral curves of the differential equation (23), i.e., outgoing nonspacelike geodesics, can terminate in the past at the singularity with a definite value of the tangent given by $X=X_0$. Clearly if no real positive root of the above exists then the singularity is not naked.

In order to make the discussion transparent, at this point we would limit ourselves to radial null geodesics only. Similar consideration can be given for the nonradial nonspacelike geodesics as well in terms of (23) and (24), which will be more complicated in view of the assumed generality of the functions involved. Equations (23) and (24) could then be written as

$$\frac{dR}{du}=\left[1-\frac{\sqrt{f+\Lambda/X}}{\sqrt{1+f}}\right]\frac{H(X,u)}{\alpha}\equiv U(X,u)\ , \tag{25}$$

$$V(X_0)=0\ , \tag{26}$$

where

$$V(X)=U(X,0)-X$$
$$=\left[1-\frac{\sqrt{f_0+\Lambda_0/X}}{\sqrt{1+f_0}}\right]\frac{H(X,0)}{\alpha}-X \ , \qquad (27)$$

where we have introduced the notation

$$\begin{aligned}&\beta_0=\beta(0)\ ,\\&\eta_0=\eta(0)\ ,\\&f_0=f(0)\ ,\\&P_0=P(0)\ ,\\&\Lambda_0=\Lambda(0)\ ,\\&\Theta_0=\Theta(0)\ .\end{aligned} \qquad (28)$$

Along an outgoing null geodesic from the singularity, r increases and so does the area coordinate R. A point to note is that dR/du is positive for $X>\Lambda$, which implies $R>F$, and the geodesic is outgoing. If the geodesics cross and get inside the curve $R=F$, which represents the apparent horizon, dR/du becomes negative and hence the geodesics are ingoing (in the sense that area coordinate R starts decreasing). Since the apparent horizon $R=F$ is the boundary of all trapped surfaces, if a null geodesic terminating at the singularity is to be outgoing it must have $R>F$ at the singularity along the geodesic. The null geodesic would also reach the future infinity if it does not get inside the apparent horizon (i.e., $R<F$) within the boundary of the dust cloud and reaches this boundary at $r=r_c$ with $R>F$ along the same geodesic.

In the description given here the constant α actually represents the behavior of singular geodesics near the singularity, i.e., $R\propto r^{\alpha}$ near the singularity. In fact, we can write $R=X_0r^{\alpha}$ in the neighborhood of singularity, X_0 being the real and positive root of Eq. (26). Thus, the determination of α really means determining the behavior of possible singular geodesics terminating at the singularity. The algorithm for evaluation of the value of α is as follows: Given the functions $F(r)$ and $f(r)$ (which specify the Tolman-Bondi model), the unique value of α is determined by the condition that $\Theta(r)\sqrt{P+1/X}$ does not vanish or goes to infinity identically as $r\rightarrow 0$ in the limit of approach to the central singularity along any $X=$const direction. This condition ensures that the quantity $H(X,0)$ will not be identically zero or infinite in (26) and (27). Note the $\Theta(r)$ vanishes identically only for the case of Friedmann models [$\eta(r)=3$, $\beta(r)=2$] where the singularity is spacelike. Once such a value of α is determined, the values of positive roots of Eq. (22) are then determined if there are any. There remains a possibility when such a value of α cannot be found. Such a case can arise only in some of the situations where $\beta(0)=2, \eta(0)>3$. In this case, actually one has $\Theta(r)\propto(r^{\eta_0-2-\alpha}\ln r)$ near the singularity at $r=0$. However, in this situation one can use a suitable change of the variable R, namely, $\bar{R}=R+ar^{\eta_0-2}(\ln r+b)$ and $\bar{X}=X+ar^{\eta_0-2-\alpha}(\ln r+b)$ (a and b are some constants). This reduces Eqs. (25) and (26) in the desired form and the value of α can then again be determined. Once the value of α is known in this manner, one can easily establish whether the singularity could possibly be naked. That is, if for this value of α the quantity Λ_0 diverges, then clearly the space-time does not permit a naked singularity as $X_0=-\infty$. In fact, this puts an upper bound on the possible values of α if the singularity is to be naked, which is given by $\alpha\leq\eta_0$. It follows from Eq. (26) that $V(0)\neq 0$; hence, $X=0$ cannot be the root of $V(X)=0$ and this implies that $H_0=H(X_0,0)\neq 0$. We specify the values of α for some specific classes. If, for example, $\eta_0=1$, then $\alpha=1$. It should be noted that, for the case when $\eta_0=3$, $\beta_0=2$ (the cases that have been discussed in [5–7]), and F and f are even functions of r, the value of α turns out to be

$$\alpha=\tfrac{7}{3},\quad X_0=(\tfrac{3}{4}\Theta_0)^{2/3}\ . \qquad (29)$$

In these cases a shell-crossing singularity also occurs at the central singularity (i.e., $R'=0$) along with the shell-focusing singularity. Again, α determines the occurrence of a shell-crossing singularity at the central singularity. It follows from Eqs. (10) and (26) that, near the central singularity at $r=0$, $R'=r^{\alpha-1}H(X_0,0)$. Hence, for $\alpha>1$ the shell-crossing singularity would also occur along with a shell-focusing one. This actually happens in the cases already discussed by Refs. [5,6]. On the other hand, if $\alpha=1$ no shell-crossing singularity occurs at the central singularity as the cases discussed in Ref. [15].

This determination of the value of α allows one to determine the existence of real and positive roots of Eq. (26). If the equation $V(X)=0$ has a real and positive root, the singularity could be naked and the geodesics could terminate at the singularity in past with the tangent $X=X_0$ in the (u,R) plane. Therefore, existence of at least one real positive root of (26) is the necessary condition for the space-time to admit naked singularity. The positive root $X=X_0$ actually represents the value of the tangent to null geodesics at the singularity, and it follows from Eq. (26) that $X_0>\Lambda_0$. Since Λ_0 is the value of the tangent of the apparent horizon $R=F$ [19] at the singularity, it is clear that the geodesics in such cases could be at least locally naked. Clearly if no real positive root of the above is found, the singularity $R=0$, $r=0$ is not naked. It should be noted that many real positive roots of Eq. (26) may exist which give the possible values of tangents to the null geodesics at the singularity. It is possible, however, that the integral curves may or may not realize a given value X_0 at the singularity.

To determine whether a value X_0 is realized at the naked singularity along any outgoing singular geodesic, which establishes the nakedness of the singularity, consider the equation of radial null geodesics in the form $u=r^{\alpha}=u(X)$. From Eq. (25) we have

$$\frac{dX}{du}=\frac{1}{u}\left[\frac{dR}{du}-X\right]=\frac{U(X,u)-X}{u}\ . \qquad (30)$$

The solution of the above gives trajectories of radial null geodesics in the form $u=u(X)$. The necessary condition for a null geodesic to terminate at the singularity at

$R=0$, $u=0$ is that $V(X)=0$ must have a real positive root $X=X_0$. In such a case, nonspacelike curves could terminate at the singularity with the tangent X_0. Therefore, if the null geodesics do terminate at the singularity then $u\to 0$ as $X\to X_0$ along the same. Let $X=X_0$ be a simple root of Eq. (26). We could then write

$$V(X)\equiv(X-X_0)(h_0-1)+h(X)\ , \tag{31}$$

where h_0 is a constant, the value of which is determined in terms of the quantities defined earlier as

$$h_0=\frac{1}{H_0}\left[\frac{\Lambda_0 H_0^2}{2\alpha X_0^2\sqrt{f_0+\Lambda_0/X_0}\sqrt{f_0+1}}+\frac{X_0N_0}{\sqrt{f_0+\Lambda_0/X_0}}\right]. \tag{32}$$

The function $h(X)$ is so chosen that

$$h(X_0)=\left[\frac{dh}{dX}\right]_{X=X_0}=0\ ,$$

i.e., $h(X)$ contains higher-order terms in $(X-X_0)$ and $H_0=H(X_0,0)$, $N_0=N(X_0,0)$. Equation (30) could then be written as

$$\frac{dX}{du}-(X-X_0)\frac{h_0-1}{u}=\frac{S}{u}\ , \tag{33}$$

where $S=S(X,u)=U(X,u)-U(X,0)+h(X)$ is such that $S(X_0,0)=0$; i.e., in the limit as $u=0$, $X=X_0$, we have $S\to 0$. Integration of (33) gives the equation of geodesics as $u=u(X)$. Multiplying (33) by u^{-h_0+1} and integrating gives

$$X-X_0=Du^{h_0-1}+u^{h_0-1}\int Su^{-h_0+1}du\ , \tag{34}$$

where D is a constant of integration that labels different geodesics. If the singularity is the end point of these geodesics with tangent $X=X_0$, we must have $X\to X_0$ as $u\to 0$ in (34). Note that as $X\to X_0$, $u\to 0$, the last term in Eq. (34) always vanishes near the singularity regardless of the value of the constant h_0 (this is due to the reason that, as $u\to 0$, $X\to X_0$, we have $S\to 0$). The first term on the right-hand side of the equation, namely, Du^{h_0-1}, however, vanishes only if $h_0>1$. It follows therefore that the single null geodesic described by $D=0$ always terminates at the singularity $R=0$, $u=0$, with $X=X_0$ as tangent. On the other hand, if $h_0>1$, a family of outgoing singular geodesics terminates at the singularity with each curve being labeled by different values of constant D.

Therefore, if a real and positive root of Eq. (26) exists, then singularity will always be at least locally naked. It follows that the existence of a real and positive root of Eq. (26) is both the necessary and sufficient condition for the singularity to be locally naked.

The above analysis implies that a very wide class of Tolman-Bondi space-times would, in fact, allow the existence of a naked singularity. In the following we consider a few examples which illustrate this point and provide insight into the formalism. Let us consider first the marginally bound Tolman-Bondi space-times characterized by the functions $F(r)$ and $f(r)$ as

$$f(r)=0,\quad F(r)=F_0r^n,\quad n\neq 3, n\geq 1\ . \tag{35}$$

In the above F_0 is to be treated as a constant. In this case, the relevant functions and Eq. (26) are

$$\alpha=1,\quad H(X,r)=\frac{nX}{3}+\frac{3-n}{3\sqrt{X}},\quad \Lambda(r)=F_0r^{n-1}\ ,$$
$$V(X)=(3-n)X+n\sqrt{\Lambda(0)}\sqrt{X}-\frac{3-n}{\sqrt{X}}+\frac{(3-n)\sqrt{\Lambda(0)}}{X}=0\ . \tag{36}$$

In the case $n>1$, where $\Lambda(0)=0$, the above equation has only one positive root $X=1$ which satisfies the equation $V(X)=0$ for all $n>1$, thus establishing the existence of naked singularity for all these space-times. These results agree with the earlier numerical calculations of [5] for the cases $n=\frac{9}{5}$ and $\frac{9}{7}$. In case $n=1$, the space-time is self-similar with $\Lambda(0)=F_0$ and Eq. (36) becomes

$$2x^4+x^3\sqrt{F_0}-2x+2\sqrt{F_0}=0\ , \tag{37}$$

where we have put $x^2=X$. The above has real and positive roots if

$$(F_0)^{3/2}<4(26-15\sqrt{3})\ . \tag{38}$$

For example, for $\sqrt{F_0}=\frac{7}{17}$ there are two positive roots $x=0.5$ and 0.658. Hence, for all such values given by Eq. (38), the singularity is naked.

Next, consider the Tolman-Bondi space-times defined by the values of F and f given by

$$f(r)=f_0r^2(1+f_1r^3)\ ,\quad F(r)=F_0r^3\ ,\quad \frac{f_0}{F_0}=p_0>-1\ . \tag{39}$$

Here f_0, F_0, and f_1 are to be treated as some constants. For this second example the relevant quantities are written as

$$\beta_0=2,\quad \eta(r)=3,\quad p(r)=p_0(1+f_1r^3)\ ,$$
$$\alpha=3,\quad \Theta_0=f_1\left[\frac{1}{\sqrt{1+p_0}}-\frac{3}{2}G(-p_0)\right],\quad \Lambda(r)=F_0\ ,$$
$$V(X)=0\Longrightarrow 2x^4+x^3\sqrt{F_0}-\Theta_0x+\Theta_0\sqrt{F_0}=0\ , \tag{40}$$

where we have again put $X=x^2$ and we see that, for a wide range of constants f_0, F_0, f_1, the positive root of the above would exist and the singularity would be naked. In fact, for

$$\frac{\Theta_0}{(F_0)^{3/2}}>13+\tfrac{15}{2}\sqrt{3}\ , \tag{41}$$

the above equation always has two real positive roots establishing the nakedness of the singularity. These space-times are effectively of the type as those considered by

Newman [7], however, a condition on the evenness of functions was assumed there which we have relaxed here.

B. The structure of singularity

We have shown above that if a real positive root of $V(X)=0$ exists then at least one single outgoing geodesic would terminate at the singularity in the past and thus the singularity would be naked. If a single ray in the (u,R) plane escapes from the singularity, it amounts to a single wave front being emitted, and thus the singularity appears naked only instantaneously to a distant observer. If the singularity is to be naked for a finite period of time, a nonzero measure set of null geodesics (i.e., families of null geodesics) must have the singularity as their past end point. In earlier examples of a naked singularity occurring in Vaidya space-times [20] and in self-similar space-times, families of nonspacelike geodesics terminate at the naked singularity in the past. In fact, an analysis of self-similar gravitational collapse of a perfect fluid in order to examine the nature and structure of naked singularity has shown [8] that a nonzero measure of nonspacelike geodesics terminate at the singularity in past provided the weak energy condition and positivity of energy are not violated in the near regions of the singularity. This results into the exposure of the singularity to a distant observer for an infinite period of time. We therefore examine this issue of termination of families of nonspacelike geodesics at the singularity below.

It follows from Eq. (34) that when only one simple real positive root $X=X_0$ for $V(X)=0$ exists, no families of geodesics would terminate at the singularity if $h_0 \leq 0$. On the other hand, if $h_0>1$ it is seen that an infinity of integral curves will meet the singularity in the past with tangent $X=X_0$, different curves being characterized by different values of the constant D. Thus, one sufficient condition for the families of nonspacelike curves to meet the naked singularity in past is $h_0>1$, when $V(X)=0$ admits only one simple real positive root. Such a condition corresponds to the requirement that $h_0-1 =(dV/dX)_{X=X_0}$ must be positive; i.e., $V(X)$ must be an increasing function at $X=X_0$. The interpretation of such a condition is seen very clearly in the case of self-similar models [13,8], where this derivative of V is determined directly by the Einstein field equations in terms of the energy density and the components of the metric tensor. It turns out in that case that this derivative will be positive with $h_0>1$ provided the weak energy condition is satisfied and the energy density is always greater than a certain lower bound in the neighborhood of the singularity, which gives a sufficient condition for families to meet the naked singularity in the past.

Suppose now that Eq. (26) has two simple positive roots X_1 and X_2. In such a case at least one singular geodesic would always terminate along each of the tangents $X=X_1$ and $X=X_2$ at the singularity. Furthermore, since $V(X)=0$ has two simple roots it follows that the value of its derivative h_0-1 would be negative along one of the roots and positive along the other. Therefore, at least along one of the roots $h_0>1$. Hence the situation that emerges is that in such a case families of geodesics will always terminate along one of the roots for which $h_0>1$ while along the other only a single geodesic would escape. The conclusions are the same if $V(X)=0$ has more than two simple positive roots. Thus, existence of two positive roots is a sufficient condition for a nonzero set of geodesics to terminate at the singularity.

This situation is similar to the scenario arising in the gravitational collapse of radiation shells which we have analyzed in detail for the case of a linear mass function in Vaidya space-times [20], where the full structure of families of all the nonspacelike geodesics terminating at the naked singularity in the past has been worked out. It is seen there that when the corresponding quantity there has two roots, they provide the tangent values for the escaping geodesics. The families of nonspacelike geodesics meet along one of the roots as the tangent at the naked singularity, where as there is a single null trajectory escaping from the singularity at the second root. In fact, Lemos [21] has pointed out recently several parallels between the self-similar Tolman-Bondi models and the self-similar radiation collapse described by the linear mass Vaidya space-time background, showing that this radiation collapse picture can be taken as a limiting case of Tolman-Bondi space-times when viewed in an appropriate sense.

It was shown in Ref. [8] that if the positivity of energy was respected in the near regions of the singularity (i.e., $\epsilon+P>0$ in the neighborhood of the singularity), then infinite many integral curves terminate at the singularity which was naked. We show here that a similar conclusion holds in the Tolman-Bondi case as well.

Let the energy density ϵ be positive in the collapsing region near the central singularity at $r=0$, i.e.,

$$\epsilon=\frac{\eta\Lambda}{R^2H}>0 \tag{42}$$

This implies that $\Lambda_0>0$ and then the definition of η implies that $\alpha=\eta(0)$. Let one simple positive root $X=X_0$ exist for the equation $V(X)=0$. Note that in the (X,u) plane Eq. (30) has a singular point at $X=X_0$, $u=0$. Therefore, in order to analyze the behavior of the integral curves in the (X,u) plane near this singular point we integrate Eq. (30) near the singularity to get

$$X-X_0=Du^{(h_0-1)} . \tag{43}$$

Hence in the case $h_0<1$ integral curves move away from the singular point $X=X_0$, $u=0$ in the (u,X) plane. However, in the (R,u) plane the above equation transforms to

$$R-X_0u=Du^{h_0} . \tag{44}$$

Therefore, if $h_0 \leq 0$ integral curves approaching $R-X_0u$ in the (R,u) plane would move further and further from the point $R=0$, $u=0$ and would not terminate there. On the other hand, if $h_0>0$ the integral curves in the (R,u) plane move into the point $R=0$, $u=0$ with either $R=X_0u$ or with the R axis as their ultimate tangent. In fact, the equation of these integral curves terminating at the singularity is given by

$$\frac{R}{u^{h_0}} - X_0 u^{1-h_0} = D + \int S(Y,u) u^{-h_0+1} du \ , \tag{45}$$

where we have put

$$Y = R/u^{h_0} = X u^{1-h_0}$$

and note that, in the limit,

$$\lim_{u\to 0} S(X,u) u^{1-h_0} = S(Y,u) u^{1-h_0} \to \text{const} \times Y \ .$$

Thus, we see that infinite many integral curves (each characterized by a different value of the constant D) would terminate at the singularity provided $h_0 > 0$. Hence, we deduce that future-directed null geodesics would terminate at the singularity in the past, as long as

$$\infty > h_0 = h(X_0) > 0 \ . \tag{46}$$

If the positivity of energy in the near regions of singularity is respected as stated in Eq. (42), i.e., $\Lambda_0 > 0$, then using Eqs. (32) and (26) and the fact that if $f(0) \neq 0$ then $\beta(0)=0$ we get for the value of h_0, when $\Lambda_0 \neq 0$,

$$h_0 = \frac{\Lambda_0 H_0}{2\alpha X_0^2 (f_0 + 1)} \ . \tag{47}$$

Hence, we conclude that $h_0 > 0$ as long as the positivity of energy holds in the near regions of the singularity. Therefore, families of geodesics would always terminate at the singularity when it is naked and provided the positivity of energy holds.

It is illustrative at this point to note the examples given in the earlier section in the context of families meeting the singularity. Note that, for the first example given by Eq. (35), in the case $n=2$, for example, $\Lambda_0 = 0$ and $V(X)=0$ has only one root given by $X=1$ and $h_0 = \frac{1}{2}$. Therefore, no families of integral curves terminate at the singularity with the tangent $X=1$. On the other hand, for $n=1$, $\Lambda_0 \neq 0$ the space-time is self-similar and the families or infinitely many nonspacelike curves terminate at the singularity. The same is the case with the second example in which $\Lambda_0 \neq 0$ where families would terminate at the singularity when it is naked.

C. Global visibility

It is seen that the existence of a real positive root $V(X)=0$ establishes that the singularity would be at least locally naked. Such a locally naked singularity could be globally naked as well. To examine this issue note that the apparent horizon lies at $R(t,r)=F(r)$, and therefore if a geodesic gets inside the apparent horizon it becomes ingoing (i.e., $R < F$ along geodesics and dR/dr is negative). Eventually this trajectory falls back to the singularity. Therefore, if a light ray is to reach future infinity in order for the singularity to be globally naked, it must cross $r = r_c$, which is the boundary of the dust cloud before the apparent horizon. Hence all escaping nonspacelike geodesics that reach the boundary $r = r_c$ with $R(r_c) > F(r_c)$ would reach future infinity. Since geodesics emerge from the singularity with the tangent value X_0 and the apparent horizon has the tangent at the singularity Λ_0, it follows from Eq. (26) that $X_0 > \Lambda_0$. As a result, because of the generality of the function $F(r)$ one can always choose suitably r_c and $F(r_c)=2M$ (M being the Schwarzschild mass of the cloud) such that geodesics reach the boundary of the cloud $r=r_c$ with $R(r_c) > F(r_c)$ making the singularity globally naked. However, given a boundary $r=r_c$ and $F(r_c)=2M$, which and whether any singular geodesics would reach future infinity depends on the global properties of the functions $F(r)$ and $f(r)$.

At this point we first discuss an explicit class of Tolman-Bondi models where we show the singularity to be globally naked, before discussing the general scenario for global nakedness. Because of the complicated nature of the equations, exact solutions to geodesics are virtually nonexistent in these models even in cases of simple forms of functions $F(r)$ and $f(r)$, except in the cases of Friedmann models corresponding to complete homogeneity. We consider the first example given in the earlier section by Eq. (35) for $n=1$. This situation represents a self-similar marginally bound collapse ($f=0$) and illustrates the formalism discussed here giving a comparison with the results already obtained. Earlier, this example has been analyzed using a special null trajectory which is the Cauchy horizon [13] which is given by $X=$const. We show below, however, that actually one can integrate the geodesic equations completely for this self-similar case to obtain radial null families. As it was pointed out earlier, in this case if condition (38) is satisfied, then $V(X)=0$ has two real positive and two complex roots. Let x_1, x_2 ($x_1 > x_2$) be two such positive roots of this equation. The equation of geodesics, in the form $r=r(x)$, $X=x^2$, is given by

$$r = r(X) \equiv r(x) = D\frac{(x-x_2)^{n_2}}{(x-x_1)^{n_1}} f_1(x) \ , \tag{48}$$

where

$$f_1(x) = \exp\left[-\int \frac{Ax+B}{x^2+D_1 x + D_2} dx\right] \ . \tag{49}$$

Here n_1, n_2, A, B, D_1, D_2 are constants given by

$$x^4 + \frac{\sqrt{\Lambda_0}}{2}x^3 - x + \sqrt{\Lambda_0} = (x-x_1)(x-x_2)(x^2+D_1 x + D_2) \ , \tag{50}$$

$$\frac{3x^3}{x^4 + (\sqrt{\Lambda_0}/2)x^3 - x + \sqrt{\Lambda_0}} = \frac{n_1}{x-x_1} - \frac{n_2}{x-x_2} + \frac{Ax+B}{x^2+D_1 x + D_2} \ , \tag{51}$$

and D is the constant which labels the different geodesics. The constants n_1, n_2 are positive. In fact, for the case $\Lambda_0 = \frac{7}{17}$, they are given by

$$\begin{aligned} x_1 &= 0.658\,303\ , \\ x_2 &= 0.5\ , \\ n_1 &= 2.093\,56\ , \\ n_2 &= 1.085\,11\ , \\ D_1 &= 1.364\,19\ , \\ D_2 &= 1.2509\ , \\ A &= -1.991\,54\ , \\ B &= -1.263\,54\ . \end{aligned} \tag{52}$$

It is clear from Eq. (48) that geodesics reach $r=0$ at $x=x_2$ and $r=\infty$ at $x=x_1$, making the singularity globally naked. Note that $\eta(r)\Lambda(r)=F_0<x_2$ and therefore all the trajectories that are emitted in the region $x_1>x>x_2$ reach the future infinity. In fact, $x=x_1$ and x_2 are also geodesics which cross the boundary of the cloud and escape to future infinity.

We now discuss the conditions which ensure the global nakedness of the singularity in general. At this point we assume that the functions η and β are at least C^0 in the interval $r_c \geq r > 0$. Since the later two functions involve the first derivatives of f and F in the form f'/f and F'/F, this requirement implies that f and F have at least first continuous derivatives existing. As discussed in Sec. II, the C^2 differentiability of the metric in the concerned interval will ensure the above requirement.

Consider now the situation that $V(X)=0$ has only one simple root $X=X_0$ and that a family of curves terminates at the singularity (i.e., $h_0>1$) with this value of tangent. Let $\eta(r)\Lambda(r)<\alpha X_0$ for $r_c \geq r>0$. In such a situation the singularity would be globally naked. To see this consider now the equation of geodesics given by Eq. (34) where the constant D labels different geodesics terminating at the singularity and is determined by the boundary conditions. For a singular geodesic that reaches the boundary of the dust cloud $u=u_c=r^\alpha=r_c^\alpha$ with $X=(R_c/r_c^\alpha)=X_c$ we have

$$X_c - X_0 = D u_c^{h_0-1} + u_c^{h_0-1}\int_{u_c} S u^{-h_0+1} du \tag{53}$$

and hence the equation of such a geodesic can be written as

$$X - X_0 = (X_c - X_0)\left[\frac{u}{u_c}\right]^{h_0-1} + u^{h_0-1}\int_{u_c}^{u} S u^{-h_0+1} du\ . \tag{54}$$

The event horizon is represented by the geodesic for which $X_c=\Lambda(r_c)$. Since it is outgoing $dR/d(r^\alpha)$ is positive at $r=0$ and ejected into the region $R>F$, where dR/dr is positive. Therefore, all the geodesics that reach the line $r=r_c$ [the line at which the metric (1) is matched with the Schwarzschild exterior] with $X_c>\Lambda(r_c)$ would escape to infinity, while others would become ingoing. It follows that the geodesics that reach future infinity with their past end point at the singularity are given by Eq. (54) with $X_c>\Lambda_c$. Hence, in case when a family of geodesics terminates at the singularity with tangent $X=X_0$ and $\eta(r)\Lambda(r)<\alpha X_0$, for $r_c \geq r>0$, the singularity would be globally naked as there would always be some geodesics that would escape to infinity.

Consider the case now when the equation $V(X)=0$ has two positive roots X_1 and X_2 $(X_1>X_2)$. In such a case, as shown earlier, families of curves would emerge from the singularity with the tangent either X_1 or X_2. Let $\eta(r)\Lambda(r)<\alpha X_2$ for $r_c \geq r>0$, then it ensures that some geodesics would cross the boundary of the cloud with $X_c>\Lambda(r_c)$ making the singularity globally naked. The same holds even in the case when more than two positive roots exist. Thus, if the family of geodesics does terminate at the singularity with tangent X_0, then the condition $\eta(r)\Lambda(r)<\alpha X_0$ for $r_c \geq r>0$ implies the global nakedness of the singularity.

IV. CURVATURE STRENGTH

Consider the case when naked singularities occurred in the gravitational collapse of matter with a reasonable equation of state and in a space-time where desirable conditions such as the energy conditions, etc., are satisfied. Even such a situation may not be considered as a problem from the point of view of cosmic censorship if the naked singularities forming were gravitationally weak in a suitable sense. In fact, it was shown [7] that the naked singularities forming in the classes of Tolman-Bondi models considered by Eardley and Smarr and Christodoulou are gravitationally weak. This is a useful result because, if true in general, it would have important implications for the cosmic censorship hypothesis. Thus, it was conjectured that nature avoids naked singularities where nonspacelike trajectories end in a strong curvature singularity [7,22].

The gravitational strength and physical seriousness of a space-time singularity have been discussed in detail and characterized precisely in the literature. In particular, Clarke and Krolak [23] have provided a sufficient condition for a singularity to be strong in the sense of Tipler [24], which is that at least along one null geodesic with the affine parameter k, with $k=0$ at the singularity, the following should be satisfied in the limit of approach to the singularity:

$$\lim_{k\to 0} k^2 R_{ab} K^a K^b > 0\ . \tag{55}$$

This provides a sufficient condition for all the two-forms $\mu(k)$ defined along the singular null geodesic to vanish as singularity is approached and implies a very powerful curvature growth establishing a strong curvature singularity. For the timelike geodesics this will imply that all the volume forms defined by the Jacobi fields along these trajectories must vanish in the limit of approach to the singularity or they must vanish infinitely many times in this limit.

The criteria on the strength of a singularity are, of course, subject to further refinement. However, the important physical consequences of the existence of a singularity are related to its strength. The point is if the singularity is gravitationally weak, it may be possible to extend the space-time through the same classically. On the other hand, when there is a strong curvature singularity forming in the above sense, the gravitational tidal forces associated with this singularity are so strong that any object trying to cross it gets destroyed. Thus, as argued by Ori [25], the extension of space-time becomes meaningless for such a strong singularity which destroys to zero size all the objects terminating at the singularity. From this point of view, the strength of singularity may be considered crucial to the issue of classically extending the space-time and thus avoiding the singularity because, for a strong curvature singularity defined in the above sense, no continuous extension of the space-time may be possible.

For the general class of Tolman-Bondi models under consideration, using (2) we get

$$\Psi = R_{ab}K^aK^b = \frac{F'(K^t)^2}{R^2R'} = \frac{F'(K^t)^2}{R^2R'} , \tag{56}$$

where K^a is tangent to null geodesics. For radial null geodesics, using L'hospital rule and Eqs. (4)–(14) and (19)–(22) and the fact that at the singularity $r \to 0$, $X \to X_0$ we get

$$\lim_{k\to 0} k^2\Psi = \eta_0 \lim_{k\to 0}\left[\frac{k\sqrt{F}\mathcal{P}}{R^2\sqrt{rR'}}\right]^2 = \frac{4\eta_0\Lambda_0}{H_0X_0^2[2\sqrt{1+f_0}(3\alpha-\eta_0)-N_0]^2} . \tag{57}$$

Hence it is seen from the definition of Λ in (12) that

$$\lim_{k\to 0} k^2\Psi = 0 \quad \text{for } \alpha < \eta_0 , \tag{58}$$

$$\lim_{k\to 0} k^2\Psi \neq 0 \quad \text{for } \alpha \geq \eta_0 . \tag{59}$$

However, from our earlier conclusions, naked singularity occurs only when $\alpha \leq \eta_0$, therefore the strong curvature condition is satisfied along singular geodesics only for the classes where $\alpha = \eta_0$. As noted earlier, for the special class considered by Newman and Christodoulou, $\alpha = \frac{7}{3}$ and $\eta = 3$ and hence the naked singularity turns out to be gravitationally weak as concluded earlier. On the other hand, it is clear from the above that for a wide variety of Tolman-Bondi solutions satisfying the condition $\alpha = \eta_0$, the singularity will be a strong curvature singularity in the above sense. In general, it is also possible that nonradial null or timelike curves could terminate at the naked singularity. Then, a similar calculation along nonspacelike geodesics in general gives

$$\lim_{k\to 0} k^2\Psi \propto (r^{\eta_0-\alpha})_{r=0} . \tag{60}$$

Hence, as discussed above, one concludes that the condition for strong curvature is satisfied along nonspacelike geodesics as well if $\alpha = \eta_0$ and if such families meet the naked singularity in the past.

The Kretschmann scalar $R_{abcd}R^{abcd}$ along the geodesics goes in the Tolman-Bondi space-times as

$$\mathcal{K} \propto r^{2(\eta_0-3\alpha)} . \tag{61}$$

Hence, the singularity is a scalar polynomial singularity as long as $\alpha > \eta_0/3$.

The self-similar Tolman-Bondi models are defined by the conditions $f(r) = \text{const}$ and $\eta(r) = 1 = \eta(0) = \alpha$. It follows from the above that the naked singularity forming in this class will be a strong curvature singularity along all the families of radial null geodesics. As shown in Ref. [8], other families of nonspacelike geodesics also terminate at the naked singularity along which as well the strong curvature condition is satisfied.

V. CONCLUDING REMARKS

We have analyzed here the Tolman-Bondi models for the existence and structure of the naked singularities. As stated earlier, these are dust models assuming the pressure $p = 0$, and also the exact spherical symmetry of the space-time. Would the introduction of pressure change the qualitative nature of the conclusions obtained here? This does not seem to be the case at least for the self-similar gravitational collapse of a perfect fluid incorporating pressure as indicated by the analysis of [14,8]. It is possible, on the other hand, that in the final stages of collapse, the dust equation of state could be relevant (see, e.g., Penrose [26] and Hagerdorn [27]) and at higher and higher densities the matter may behave more and more like dust. Again, there is some case for the argument that eventually in the final stages of collapse the matter distribution should become almost spherically symmetric (see, e.g., Nakamura and Sato [28]). Hence, it is clearly useful to examine the inhomogeneous dust collapse as modeled by the Tolman-Bondi space-times. Further, a situation analogous to the singularity theorems might develop here where the conclusions derived under the assumption of spherical symmetry are preserved when small perturbations are taken into account. Thus, spherical symmetry may be a good model to represent a certain class of gravitational collapse.

Also, we have not addressed the issue of the stability of naked singularity. If these are not stable (in a sense to be defined suitably) such naked singularities need not be considered as counterexamples to the cosmic censorship hypothesis. As far as the issue of stability is concerned, one needs to develop a precise criterion for stability in general relativity. In this connection it may be noted, however, that for self-similar Tolman-Bondi models the Cauchy horizon is stable at least against the blueshift mode of instability [21].

Subject to these reservations, it is seen here that Tolman-Bondi space-times admit naked singularities under fairly general conditions, from which a nonzero measure set of nonspacelike trajectories emanate in the future direction. Certain examples of particular classes where nonradial nonspacelike geodesics terminate at the naked singularity in the past are also explicitly worked out. An interesting point is that, in the case of the strong curvature condition being satisfied along radial null trajectories, the same conclusion also holds along all other nonspacelike geodesics. For various other classes of naked singularity space-times, even though the strong curvature condition may not be satisfied along radial curves, they could still be regarded as strong curvature singularity in the sense that the Kretschmann scalar diverges.

Another feature one would like to note here is that while strong curvature naked singularities have been found to occur in self-similar gravitational collapse as indicated earlier, the present consideration gives a wide class of inhomogeneous collapse models which need not be self-similar in general. A wide class of space-times has been pointed out, namely, the ones for which $\alpha=\eta_0$, which gives a set of solutions of the field equations which admit a strong curvature naked singularity. The suggestion that seems to be coming is that the phenomena of naked singularity is probably not related to the space-times with any particular geometric properties such as the self-similarity of the models. It may be that the existence of naked singularity is not just a geometric phenomena and the answer to cosmic censorship conjecture could lie in the dynamics of the Einstein equations. Of course, if one rules out the matter fields such as the dust and perfect fluid, etc., from consideration because they may create singularity even without gravity, then such naked singularities are ruled out (see, however, [29] where the occurrence of naked singularity is pointed out for a wide range of matter satisfying the weak energy condition in self-similar gravitational collapse).

To summarize, the conclusions on the final fate of gravitational collapse are rather different in the generally inhomogeneous Tolman-Bondi models as compared to the Oppenheimer-Snyder case of a completely homogeneous dust collapse, which forms a set of zero measure in the general Tolman-Bondi class considered here. In fact, the similarity in conclusions concerning the nature and structure of the naked singularity for the radiation collapse [20], the general self-similar collapse [8] of perfect fluid, and the results here appear suggestive of a certain general property of Einstein equations. It would be worthwhile to isolate and study this feature as that might help towards a definite mathematical formulation of the cosmic censorship by pointing out the precise features one wants to rule out. Such a study would be of independent interest anyway because not much is understood on the global properties of the Einstein equations except the results on the existence of space-time singularities as predicted by the singularity theorems.

While the analysis we have presented here should be useful towards arriving at any rigorous formulation of cosmic censorship in a provable form as pointed out above, we would like to argue here that a physical formulation of the cosmic censorship may be evolved which avoids features such as above. For example, an interesting feature that emerges from the presently available examples is the role of energy conditions in determining the escape of families of nonspacelike trajectories from the naked singularity, which is an important criteria for the physical significance of the same. In all the presently available collapse scenarios, it is the weak energy condition together with the positivity of energy which leads to the existence of families of nonspacelike geodesics terminating at the naked singularity in the past. Could one then argue that somehow the energy conditions must be violated in the very final stages of gravitational collapse so as to avoid the formation of naked singularity? In fact, in the case of self-similar collapse [8], it can be shown that the violation of the energy condition near the singularity no longer allows the families of nonspacelike geodesics to come out but only an isolated trajectory can emerge. Hence, for all practical purposes, the singularity is no longer naked preserving the effective censorship. Again, as emphasized by Israel [30], many of the naked singularities arising in the spherically symmetric collapse are massless (with the mass being defined in a suitable manner, see also Lake [18]); and as a consequence these may not violate the basic physical spirit of the cosmic censorship. Such possibilities need a serious investigation.

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[17] If one requires the metric to be C^2 from the point of view of physical reasonableness, then the differentiability conditions imposed here on the functions f and F should be considered reasonable. Even when one requires the weaker condition that the metric is finite and continuous at the center, the functions f and F must at least be C^1 at $r=0$ (though the converse is not true). Further, from the physical reasonableness one would require $F(0)=0$ (otherwise there will be a massive singularity present already at $r=0$), which implies $\eta(0)>0$. The condition F being C^1 corresponds to the energy density ϵ not diverging at the center $r=0$ at all times.

[18] See, e.g., the discussion on the nature of the naked singularity in such a case by K. Lake, Phys. Rev. Lett. **68**, 3129 (1992). One could reduce the right-hand side of Eq. (23) for the marginally bound case $f=0$ and for radial null geodesics ($l=0$, $B=0$) to the form P/Q, where both P and Q vanish at $r=0$, $R=0$. However, even in this simplified case P and Q have no linear terms and so the analysis on the nature of the singularity by the usual techniques [see, e.g., T. Davies and E. James, *Non-Linear Differential Equations* (Addison-Wesley, New York, 1966) or F. Verhulst, *Non-Linear Differential Equations and Dynamical Systems* (Springer-Verlag, Berlin, 1990)] do not apply. The only way then to understand the nature of singularity is a direct analysis of the behavior of the characteristic curves in the neighborhood of the singularity, which we have done here.

[19] The tangent to the apparent horizon $R=F$ in (R,u) plane is $(\eta\Lambda/\alpha)$ and at the singularity it becomes $[\eta(0)\Lambda(0)/\alpha]$, which is effectively given by $\Lambda(0)$. This is because when $\alpha<\eta(0)$ then $\Lambda(0)=0$, and when $\alpha=\eta(0)$ the value of tangent becomes $\Lambda(0)$.

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