Scalar-tensor cosmologies

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(Received 9 December 1992)

A method is investigated which enables exact solutions to be found for vacuum and radiation-dominated Friedmann universes of all curvatures in scalar-tensor theories with an arbitrary form for the coupling, $\omega(\phi)$, of the scalar field which determines the strength of the gravitational field. Particular classes of solution are presented for specific representative choices of $\omega(\phi)$, including the theories of Brans and Dicke, Barker, and Bekenstein, to illustrate the range of cosmological behaviors that are possible.

PACS number(s): 98.80.Hw, 12.10.Gq, 04.50.+h, 04.60.+n

I. INTRODUCTION

Renewed interest in the structure of scalar-tensor gravity theories has been created by their possible relevance to a period of inflationary expansion during the early Universe. It has been shown that there are simple conformal interrelationships between general relativistic cosmological models containing self-interacting scalar fields and scalar-tensor gravity theories in vacuum [1]. In addition, there have been proposals to revive the original "old" inflationary universe models in which a phase transition is effected by the nucleation of bubbles of true vacuum by the use of the Brans-Dicke scalar-tensor theory [2]. These models allow the phase transition to complete because inflation is power law rather than exponential in time (although any power-law inflationary universe would allow this to happen [3] and there is no need to depart from general relativity and adopt the Brans-Dicke model). There have also been suggestions that the dark matter problem might be resolved in the context of a scalar-tensor cosmological model in which some forms of matter couple with a different strength to gravitation than does ordinary matter [4]. The author has also argued that primordial black holes formed during the first 10^{-25} s of a universe in which the strength of gravity varies with time would evaporate today with different Hawking temperatures to those predicted by general relativity [5].

The principal observational constraint upon scalar-tensor theories is derived from considering effects upon the synthesis of light elements during the early Universe [6] and their manifestations in the standard solar system test of general relativity [7]. The only exact solutions giving exact scalar-tensor cosmological models are for the case of the Brans-Dicke theory, where $\omega(\phi)=\text{const}$ or, in the trivial cases, $\phi=\text{const}$, where the solutions are identical to those of general relativity [5]. In this paper we show how to obtain exact solutions for homogeneous and isotropic cosmological models in vacuum and with radiation as the matter content for all values of the curvature. The solutions can be given in terms of a single integral over the coupling function $\omega(\phi)$ which can be performed exactly in many cases and numerically in all cases. The vacuum solutions are of physical interest because, in general, the fluid-filled Friedmann universes approach the vacuum solution asymptotically as $t\to 0$ for a wide range of $\omega(\phi)$. We shall give solutions for the specific scalar-tensor theories that have appeared in the literature as well as other cases which cover a wide range of possible behaviors for $\omega(\phi)$. The general field equations are set up in Sec. II; the Friedmann models are introduced in Sec. III together with the most expedient choice of variables; the vacuum solutions are given in Sec. IV, and the radiation solutions are given in Sec. V, and the results are discussed in Sec. VI.

II. SCALAR-TENSOR GRAVITATION THEORIES

Scalar-tensor gravity theories have been formulated in two different ways. Steinhardt and Accetta [8] take the Lagrangian of the theory in the form

$$L_\Phi=-f(\Phi)R+\tfrac{1}{2}\partial_a\Phi\,\partial^a\Phi+16\pi L_m\ , \tag{1}$$

where Φ is a scalar field, $f(\Phi)$ is the coupling to the four-curvature, and L_m is the Lagrangian of the remaining matter fields. If we define a new scalar field $\phi=f(\Phi)$ with a coupling

$$\omega(\phi)=\tfrac{1}{2}f(f'^2)^{-1}\ , \tag{2}$$

then (1) becomes

$$L_\phi=-\phi R+\phi^{-1}\omega(\phi)\partial_a\phi\,\partial^a\phi+16\pi L_m\ . \tag{3}$$

The theory proposed by Brans and Dicke [7] arises in the special case that $\omega=\text{const}$ and $f(\Phi)\propto\Phi^2$. The relative merits of adopting (1), as do La and Steinhardt [2], or (3), as do Barrow and Maeda [1], have been discussed by Liddle and Wands [10].

By varying the action associated with (3) with respect to the space-time metric and the scalar field ϕ, respectively, we obtain the generalized Einstein equations and the wave equation for ϕ:

$$R_{ab}-\tfrac{1}{2}g_{ab}R=-8\pi\phi^{-1}T_{ab}-\omega(\phi)\phi^{-2}[\phi_a\phi_b-\tfrac{1}{2}g_{ab}\phi_i\phi^i]-\phi^{-1}[\phi_{a;b}-g_{ab}\Box\phi]\ , \tag{4}$$

$$[3+2\omega(\phi)]\Box\phi=8\pi T-\omega'(\phi)\phi_i\phi^i\ , \tag{5}$$

$$T^{ab}{}_{;b}=0\ , \tag{6}$$

where T^{ab} is the energy-momentum tensor of the matter content of the theory.

Clearly if T, the trace of the energy-momentum tensor, vanishes, and ϕ is a constant, then (4)–(6) reduce to the standard Einstein equations with a gravitational constant $G=\phi^{-1}$. Hence, any exact solution of Einstein's equations with a trace-free matter source will also be a particular exact solution of the scalar-tensor theory with ϕ and, hence $\omega(\phi)$ constant. However, these particular solutions will not necessarily constitute the general solution for the prescribed matter content. We shall show how to find classes of exact cosmological solutions in which ϕ is not constant for a given $\omega(\phi)$.

III. FRIEDMANN UNIVERSES

The Friedmann metric is given by

$$ds^2=dt^2-a^2(t)[(1-kr^2)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2]\ , \tag{7}$$

where $a(t)$ is the expansion scale factor and k is the curvature constant which can be set equal to 0, +1, or −1 without loss of generality. The choice of k defines flat, closed, and open universes, respectively. We shall assume that the material content of the Universe is blackbody radiation with the equation of state relating the pressure p to the density ρ, as

$$3p=\rho\ . \tag{8}$$

Equations (6) and (7) then reduce to

$$\dot\rho+3\dot a a^{-1}(\rho+p)=0\ . \tag{9}$$

Hence with (8) we have

$$8\pi\rho=3\Gamma a^{-4}\ , \tag{10}$$

where $\Gamma\geq 0$ is a constant. The case $\Gamma=0$ will define the vacuum model in which $p=\rho=0$.

The metric (7) reduces (4)–(6) to the two equations

$$\frac{\dot a^2}{a^2}=\frac{\Gamma}{\phi a^4}-\frac{k}{a^2}-\frac{\dot\phi}{\phi}\frac{\dot a}{a}+\frac{\omega(\phi)}{6}\frac{\dot\phi^2}{\phi^2}\ , \tag{11}$$

$$\ddot\phi+\frac{3\dot a}{a}\dot\phi=-\frac{\dot\phi^2\omega'(\phi)}{3+2\omega(\phi)}\ , \tag{12}$$

where an overdot denotes d/dt.

If we introduce a conformal time η through

$$a\,d\eta=dt\ , \tag{13}$$

then, denoting $d/d\eta$ by a prime, (12) becomes

$$\phi''+\frac{2a'}{a}\phi'=-\frac{\phi'^2\omega'(\phi)}{3+2\omega(\phi)}\ . \tag{14}$$

This integrates to give

$$\phi'a^2=3^{1/2}A(2\omega+3)^{-1/2},\quad A=\text{const}\ . \tag{15}$$

If we introduce the new variable employed by Lorenz-Petzold [9],

$$y=\phi a^2\ , \tag{16}$$

then this transforms the generalized Friedmann equation (11) into the form

$$y'^2=-4ky^2+4\Gamma y+\phi'^2a^4(3+2\omega)\ , \tag{17}$$

which, upon using the integral (15), becomes

$$y'^2=-4ky^2+4\Gamma y+A^2\ . \tag{18}$$

To complete the solution of the problem we integrate (18) to obtain $y(\eta)$, divide (15) by y to obtain ϕ'/ϕ, integrate to obtain $\phi(\eta)$, and, hence, $a(\eta)$ from (16) and $a(t)$ from (13). We now present a number of specific examples.

IV. VACUUM SOLUTIONS

It is most transparent to consider the vacuum and radiation cases separately. The vacuum case is obtained by solving the system with $\Gamma=0$. Three types of solution exist according as $k=0$, +1, or −1. Solving (18) we have

$$y(\eta)=A(\eta+\eta_0),\quad k=0\ , \tag{19}$$

$$y(\eta)=\tfrac{1}{2}A\sinh[2(\eta+\eta_0)],\quad k=-1\ , \tag{20}$$

$$y(\eta)=\tfrac{1}{2}A\sin[2(\eta+\eta_0],\quad k=+1\ , \tag{21}$$

where we take the positive root without loss of generality since A is arbitrary and η_0 is a constant. To complete the solution we need to determine $\phi(\eta)$ from (15), which reduces to

$$\int\frac{[2\omega(\phi)+3]^{1/2}}{\phi}d\phi=\begin{cases}\sqrt{3}\ln(\eta+\eta_0), & k=0, \quad (22)\\ \sqrt{3}\ln[\tan(\eta+\eta_0)], & k=+1, \quad (23)\\ \sqrt{3}\ln[\tanh(\eta+\eta_0)], & k=-1\ . \quad (24)\end{cases}$$

To complete the solutions we need to specify the form of $\omega(\phi)$ so that the left-hand side of (22) can be calculated explicitly. Some examples will be given here to illustrate the general method.

A. $\omega=\omega_0=$const

This is the case of the Brans-Dicke theory and (22)–(24) integrate to yield [9]

$$\phi(\eta)=\begin{cases} A(\eta+\eta_0)^\beta, & k=0, \quad (25)\\ \phi_0\tanh^\beta(\eta+\eta_0), & k=-1, \quad (26)\\ \phi_0\tan^\beta(\eta+\eta_0), & k=+1\ , \quad (27)\end{cases}$$

where the constant β is defined by

$$\beta=[3/(2\omega_0+3)]^{1/2}\ . \tag{28}$$

We shall restrict our attention to the case where $\beta>0$.

Hence, since $a^2=y\phi^{-1}$ we have, for the scale factor $a(\eta)$, for $k=0$,

$$a^2(\eta)=A\phi_0^{-1}(\eta+\eta_0)^{(1-\beta)}\ , \tag{29}$$

$$t=\int a(\eta)d\eta\propto\begin{cases} A^{1/2}\phi_0^{-1/2}(\eta+\eta_0)^{(3-\beta)/2}, & \beta\neq 3, \quad (30)\\ \ln(\eta+\eta_0), & \beta=3\ . \quad (31)\end{cases}$$

Hence,

$$a(t)\propto\begin{cases} t^{(1-\beta)/(3-\beta)}, & \beta\neq 3,\\ e^t, & \beta=3\ .\end{cases} \tag{32}$$

For $k=-1$,

$$a^2(\eta)=A\phi_0^{-1}\sinh^{1-\beta}(\eta+\eta_0)\cosh^{1+\beta}(\eta+\eta_0)\ . \tag{33}$$

For $k=+1$,

$$a^2(\eta)=A\phi_0^{-1}\sin^{1-\beta}(\eta+\eta_0)\cos^{1+\beta}(\eta+\eta_0)\ . \tag{34}$$

We could choose $\eta_0=0$ if we wish to fix the origin of time so that $a=0$ at $\eta=0$ (we will have $\beta<1$ in practice).

B. $2\omega(\phi)+3=\mu^2\phi^{2(n+1)}$, $\mu>0$, constant

We choose this functional form for ω to encompass a wide range of power-law variations with ϕ. Asymptotically, it resembles the power-law form used by Barrow and Maeda [1] to study extended inflation in scalar-tensor theories. In this case (22)–(24) give

$$\phi(\eta)=[(n+1)\mu^{-1}\sqrt{3}]^{1/(n+1)}\ln^{1/(n+1)}f(\eta)\ , \tag{35}$$

where

$$f(\eta)=\begin{cases} \eta+\eta_0, & k=0, \quad (36)\\ \tanh(\eta+\eta_0), & k=-1, \quad (37)\\ \tan(\eta+\eta_0), & k=+1, \quad (38)\end{cases}$$

and hence the evolution of the expansion scale factor in each of these cases is given by, for $k=0$,

$$a^2(\eta)=\frac{A(\eta+\eta_0)}{[(n+1)\mu^{-1}\sqrt{3}]^{1/(n+1)}\ln^{1/(n+1)}(\eta+\eta_0)}\ , \tag{39}$$

for $k=-1$,

$$a^2(\eta)=\frac{A\sinh(\eta+\eta_0)\cosh(\eta+\eta_0)}{[(n+1)\mu^{-1}\sqrt{3}]^{1/(n+1)}\ln^{1/(n+1)}[\tanh(\eta+\eta_0]}\ , \tag{40}$$

and, for $k=+1$,

$$a^2(\eta)=\frac{A\sin(\eta+\eta_0)\cos(\eta+\eta_0)}{[(n+1)\mu^{-1}\sqrt{3}]^{1/(n+1)}\ln^{1/(n+1)}[\tan(\eta+\eta_0)]}\ . \tag{41}$$

If we fix $a=0$ at $\eta=0$ then we can set $\eta_0=0$. In the $n=-\frac{1}{2}$, $k=0$ case we can integrate (13) to obtain $t(\eta)$ in terms of the logarithmic integral function $\mathrm{Li}[x]\equiv\int_0^x(\ln t)^{-1}dt$. We have

$$a^2(\eta)\propto(\eta+\eta_0)/\ln(\eta+\eta_0)$$

and $t\propto\mathrm{Li}[(\eta+\eta_0)^{3/2}]$.

C. $\omega(\phi)=\frac{1}{2}(4-3\phi)/(\phi-1)$

This particular form of $\omega(\phi)$ defines Barker's theory of gravitation [11] and is chosen in order to ensure that there is no time variation of the Newtonian gravitational "constant" G. In the weak-field limit an $\omega(\phi)$ theory will give a time variation in G equal to [12]

$$\frac{\dot G}{G}=-\left[\frac{3+2\omega}{4+3\omega}\right]\left[\frac{G(t)}{G_0}+\frac{2\omega'}{(3+2\omega)^2}\right]\dot\phi\ , \tag{42}$$

where G_0 is the present value of $G(t)$. Hence, to ensure $\dot G=0$ and $G(t)=G_0$ we require

$$\omega(\phi)=(4-3\phi)/(2\phi-2) \tag{43}$$

and this also ensures that the Nordtvedt parameter [7,12] is zero. With this choice of $\omega(\phi)$ Eqs. (22)–(24) yield

$$2\arctan[(\phi-1)^{1/2}]=\begin{cases} \sqrt{3}\ln[c(\eta+\eta_0)], & k=0, \quad (44)\\ \sqrt{3}\ln[c\tan(\eta+\eta_0)], & k=+1, \quad (45)\\ \sqrt{3}\ln[c\tanh(\eta+\eta_0)], & k=-1, \quad (46)\end{cases}$$

and so

$$\phi(\eta)=\begin{cases} \sec^2[\frac{1}{2}\sqrt{3}\ln c(\eta+\eta_0)], & k=0, \quad (47)\\ \sec^2\{\frac{1}{2}\sqrt{3}\ln[c\tan(\eta+\eta_0)]\}, & k=+1, \quad (48)\\ \sec^2\{\frac{1}{2}\sqrt{3}\ln[c\tanh(\eta+\eta_0)]\}, & k=-1, \quad (49)\end{cases}$$

where c is a constant. Fixing $a(0)=0$ the expansion scale factor is given by

$$a^2(\eta)=\begin{cases}A\eta\cos^2(\tfrac{1}{2}\sqrt{3}\ln c\eta), & k=0, \quad (50)\\ \tfrac{1}{2}A\sin(2\eta)\cos^2[\tfrac{1}{2}\sqrt{3}\ln(c\tan\eta)], & k=+1, \quad (51)\\ \tfrac{1}{2}A\sinh(2\eta)\cos^2[\tfrac{1}{2}\sqrt{3}\ln(c\tanh\eta)], & k=-1\ . \quad (52)\end{cases}$$

In the $k=0$ case it is possible to determine $t(\eta)$ explicitly as

$$t=cA^{1/2}\eta^{3/2}[\tfrac{1}{2}\cos(\tfrac{1}{2}\sqrt{3}\ln c\eta)+12^{-1/2}\sin(\tfrac{1}{2}\sqrt{3}\ln c\eta)]\ . \quad (53)$$

D. Bekenstein's scalar-tensor theory

This proposed modification of general relativity due to Bekenstein [13,14] reduces to the study of a scalar-tensor theory defined by

$$2\omega(\phi)+3=-\tfrac{1}{2}f(\phi)[(1-6q)qf-1][r+(1-r)qf]^{-2}\ , \quad (54)$$

where the scalar field ϕ is defined in terms of f by

$$\phi=f^{-r}(1-q^f)\ , \quad (55)$$

and where q and r are two undetermined constants which define the theory. The predictions of the theory are close to those of general relativity when $q>0$ and $r<0$. We see that

$$\int\frac{[2\omega(\phi)+3]^{1/2}}{\phi}d\phi=\begin{cases}2^{-1/2}q^{-1/2}\int x^{-1/2}(x-1)^{-1}[(6q-1)x+1]^{1/2}dx \quad (56)\\ 2^{-1/2}q^{-1/2}[(6q-1)J(f)-6qI(f)]\ , \quad (57)\end{cases}$$

where we have set

$$x=qf \quad (58)$$

and

$$J(f)=\begin{cases}-(1-6q)^{-1/2}\arcsin[2(6q-1)qf+1] \quad \text{if } 6q<1, \quad (59)\\ (6q-1)^{-1/2}\ln\{2(6q-1)^{1/2}[(6q-1)q^2f^2+qf]^{1/2}+2(qf-1)+12q-1\} \quad \text{if } 6q>1\ , \quad (60)\end{cases}$$

$$I(f)=(6q)^{-1/2}\ln\left[\frac{2(6q)^{1/2}\{(6q-1)q^2f^2+qf\}^{1/2}+(12q-1)qf+1}{qf-1}\right]\ . \quad (61)$$

These forms are rather cumbersome and so we just outline the final form of the solution. The scale factor $a(\eta)$ is given by

$$a^2(\eta)=y(\eta)f^r(1-qf)\ , \quad (62)$$

where $y(\eta)$ is given for the required value of $k=0,\pm1$ by (19)–(21) and f is given implicitly in terms of η by (57), (60), and (61) using (22)–(24).

Clearly, other choices for $\omega(\phi)$ could be made for which the left-hand side of (20) is integrable. In fact, we see that it is most efficient for cosmological studies to define scalar tensor theories of gravitation in terms of the behavior of $\phi^{-1}[2\omega(\phi)+3]^{1/2}$ rather than by $\omega(\phi)$ or the $f(\Phi)$ of Eq. (1). Once the integral in Eq. (20) is performed the solutions can be completed as in Secs. A and B *mutatis mutandis.*

V. RADIATION SOLUTIONS

A similar strategy enables the radiation-dominated Friedmann models to be found for all values of k. Integrating (18), we may determine $y(\eta)$ prior to specifying the form of $\omega(\phi)$:

$$y(\eta)=\begin{cases}\Gamma(\eta+\eta_0)^2-A^2/4\Gamma, & k=0, \quad (63)\\ \tfrac{1}{2}\Gamma+\tfrac{1}{2}(\Gamma^2+A^2)^{1/2}\sin[2(\eta+\eta_0)], & k=+1, \quad (64)\\ -\tfrac{1}{2}\Gamma+\tfrac{1}{2}(A^2-\Gamma^2)^{1/2}\sinh[2(\eta+\eta_0)], & A^2>\Gamma^2,\ k=-1. \quad (65)\end{cases}$$

Now, integrating (15) we obtain

$$\int \frac{(2\omega+3)^{1/2}}{\phi}d\phi=\begin{cases}\sqrt{3}\ln\left[\dfrac{2\Gamma\eta+2\Gamma\eta_0-A}{2\Gamma\eta+2\Gamma\eta_0+A}\right], & k=0, \quad (66)\\ \sqrt{3}\ln\left[\dfrac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right], & k=+1, \quad (67)\\ 2\sqrt{3}\operatorname{arctanh}\left[\dfrac{-\Gamma\tanh(\eta+\eta_0)-(A^2-\Gamma^2)^{1/2}}{A}\right], & k=-1\ , \quad (68)\end{cases}$$

where we will only write the $A^2\geq\Gamma^2$ expressions for the $k=-1$ models. To complete the solution by specifying ω, again we consider two specific examples of the coupling function $\omega(\phi)$ for which the integration on the left-hand side of (66)–(68) can be performed.

A. $\omega(\phi)=\omega_0=$const

The radiation Friedmann solutions for the Brans-Dicke theory are given by, for $k=0$,

$$\phi=\phi_0\left[\frac{\eta+\eta_0-\frac{1}{2}A\Gamma^{-1}}{\eta+\eta_0+\frac{1}{2}A\Gamma^{-1}}\right]^{\beta}, \quad (69)$$

$$a^2=y\phi^{-1}=\phi_0^{-1}[\Gamma(\eta+\eta_0)^2-\tfrac{1}{4}A^2\Gamma^{-1}]\times\left[\frac{\eta+\eta_0-\frac{1}{2}A\Gamma^{-1}}{\eta+\eta_0+\frac{1}{2}A\Gamma^{-1}}\right]^{\beta}, \quad (70)$$

where β is given by (28). If we fix the origin of time so that $a(0)=0$ then $2\Gamma\eta_0=A$ and so we have the simple forms

$$a^2(\eta)=\Gamma\phi_0^{-1}\eta^{1-\beta}(\eta+2\eta_0)^{1+\beta}\ , \quad (71)$$

$$\phi(\eta)=\phi_0\eta^{\beta}(\eta+2\eta_0)^{-\beta}\ . \quad (72)$$

These solutions are included in the collection of $k=0$ perfect fluid Friedmann models given by Gurevich, Finkelstein, and Ruban [15]; for $k=\pm1$ solutions also see Refs. [9,16].

For $k=+1$,

$$\phi=\phi_0\ln\left[\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right]^{\beta}, \quad (73)$$

$$a^2=\frac{1}{2}\phi_0^{-1}[\Gamma+(\Gamma^2+A^2)^{1/2}\sin2(\eta+\eta_0)]\ln\left[\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right]^{\beta}. \quad (74)$$

For $k=-1$,

$$\phi=\phi_0\left[\frac{(A^2-\Gamma^2)^{1/2}\exp[2(\eta+\eta_0)]-\Gamma-A}{(A^2-\Gamma^2)^{1/2}\exp[2(\eta+\eta_0)]-\Gamma+A}\right]^{\beta}, \quad (75)$$

$$a^2=\frac{1}{2}\phi_0^{-1}[-\Gamma+(A^2-\Gamma^2)^{1/2}\sinh2(\eta+\eta_0)]\left[\frac{(A^2-\Gamma^2)^{1/2}\exp[2(\eta+\eta_0)]-\Gamma-A}{(A^2-\Gamma^2)^{1/2}\exp[2(\eta+\eta_0)]-\Gamma+A}\right]^{\beta}. \quad (76)$$

B. $2\omega(\phi)+3=\mu^2\phi^{2(n+1)}$, μ const

As before, the radiation solutions are given as follows.

For $k=0$,

$$\phi^{n+1}=\mu^{-1}(n+1)\sqrt{3}\ln\left[\frac{2\Gamma(\eta+\eta_0)-A}{2\Gamma(\eta+\eta_0)+A}\right]+\mu^{-1}(n+1)\phi_0\sqrt{3}\ , \quad (77)$$

$$a^2=\eta(\Gamma\eta+A)[(n+1)\mu^{-1}\sqrt{3}]^{-1/(n+1)}\ln^{-1/(n+1)}[\phi_0\Gamma\eta/(\Gamma\eta+A)]\ . \quad (78)$$

For $k=-1$,

$$\phi^{n+1}=\mu^{-1}(n+1)\sqrt{3}\ln\left[\frac{(A^2-\Gamma^2)^{1/2}\exp(2\eta+2\eta_0)-A-\Gamma}{(A^2-\Gamma^2)^{1/2}\exp(2\eta+2\eta_0)+A-\Gamma}\right]+\mu^{-1}(n+1)\phi_0\sqrt{3}\ , \quad (79)$$

$$a^2=\tfrac{1}{2}[-\Gamma+(A^2-\Gamma^2)^{1/2}\sinh 2(\eta+\eta_0)][(n+1)\mu^{-1}\sqrt{3}]^{-1/(n+1)} \times \ln^{-1/(n+1)}\left[\frac{(A^2-\Gamma^2)^{1/2}\exp(2\eta+2\eta_0)-A-\Gamma}{(A^2-\Gamma^2)^{1/2}\exp(2\eta+2\eta_0)+A-\Gamma}\right]. \tag{80}$$

For $k=+1$,

$$\phi^{n+1}=\mu^{-1}(n+1)\sqrt{3}\ln\left[\frac{(A^2+\Gamma^2)^{1/2}+\Gamma\tan(\eta+\eta_0)-A}{(A^2+\Gamma^2)^{1/2}+\Gamma\tan(\eta+\eta_0)+A}\right]+\mu^{-1}(n+1)\phi_0\sqrt{3}\,, \tag{81}$$

$$a^2=\tfrac{1}{2}[\Gamma+(A^2+\Gamma^2)^{1/2}\sin 2(\eta+\eta_0)][(n+1)\mu^{-1}\sqrt{3}]^{-1/(n+1)}\ln^{-1/(n+1)}\left[\frac{(A^2+\Gamma^2)^{1/2}+\Gamma\tan(\eta+\eta_0)-A}{(A^2+\Gamma^2)^{1/2}+\Gamma\tan(\eta+\eta_0)+A}\right]. \tag{82}$$

C. $\omega(\phi)=\frac{1}{2}(4-3\phi)/(\phi-1)$

Using the forms for $y(\eta)$ in (63)–(65) the solutions for the scale factor are as follows [$\phi(\eta)$ can be obtained from the relation $a^2=y\phi^{-1}$ if required].

For $k=0$,

$$a(\eta)=\left[\Gamma\eta^2+\frac{1}{2}\eta\right]^{1/2}\cos\left[\frac{\sqrt{3}}{2}\ln\left[\frac{c\eta}{\eta+A\Gamma^{-1}}\right]\right]. \tag{83}$$

For $k=+1$,

$$a^2(\eta)=\{[\tfrac{1}{2}\Gamma+\tfrac{1}{2}(\Gamma^2+A^2)^{1/2}]\sin[2(\eta+\eta_0)]\}\cos^2\left[\frac{\sqrt{3}}{2}\ln\left[\frac{c\{(A^2+\Gamma^2)^{1/2}-A+\Gamma\tan(\eta+\eta_0)\}}{(A^2+\Gamma^2)^{1/2}+A+\Gamma\tan(\eta+\eta_0)}\right]\right]. \tag{84}$$

For $k=-1$,

$$a^2(\eta)=\{-\tfrac{1}{2}\Gamma+\tfrac{1}{2}(A^2-\Gamma^2)^{1/2}\sinh[2(\eta+\eta_0]\}\cos^2\left[\frac{\sqrt{3}}{2}\ln\left[\frac{c\{(A^2\Gamma^2)^{1/2}\exp[2(\eta+\eta_0)]-A-\Gamma\}}{(A^2-\Gamma^2)^{1/2}\exp[2(\eta+\eta_0)]+A-\Gamma}\right]\right]. \tag{85}$$

An analogous analysis can be performed for the case of the Bekenstein scalar-tensor theory as was given for the vacuum case in the previous section. The solutions are straightforwardly obtained but are extremely lengthy and so we shall not present them explicitly here.

VI. DISCUSSION

The aim of this paper has been to show how exact cosmological solutions can be found which model the radiation or vacuum-dominated period the evolution of the early Universe for general scalar-tensor gravity theories by generalizing a method used by Lorenz-Petzold to study Brans-Dicke models [17]. We have shown how, by an appropriate choice of variables, the problem can always be reduced to the solution of a single integral over $\omega(\phi)$. We have displayed some particular solutions for interesting classes of scalar-tensor theory but it is straightforward to apply the method to a wide range of other specifications for $\omega(\phi)$. There is no preferred general form at present for the function $\omega(\phi)$ although specific forms will be imposed in particular problems and we have discussed the particular theories derived by Brans and Dicke [7], Barker [11], and Bekenstein [13,14] for illustration. Most past interest has been focused upon the Brans-Dicke theory in which $\omega(\phi)$ is a constant.

One important constraint upon the forms of $\omega(\phi)$ that are appropriate in cosmological problems comes from considering the weak-field limit of the gravity theory defined by $\omega(\phi)$. It is well known that the general relativity limit of the Brans-Dicke theory arises when $\omega\rightarrow\infty$, but the general relativity limit of an $\omega(\phi)$ theory requires $\omega(\phi)\rightarrow\infty$ and

$$(4+3\omega)^{-1}(3+2\omega)^2\omega'(\phi)\rightarrow 0$$

to hold simultaneously [12] [a closely related expression determined the rate of change of G in Eq. (42) above]. Thus in the $\omega\rightarrow\infty$ limit we require that $\omega'/\omega^3\rightarrow 0$ for accord with the predictions of general relativity. This will ensure that the solar system tests of general relativity maintain their agreement between theory and observation. If we apply this consideration to the example theory (B) with $2\omega(\phi)+3=\mu^2\phi^{2(n+1)}$, used above, we then see that, for large ω,

$$\omega'\omega^{-3}\propto(n+1)\phi^{-4n-5} \tag{86}$$

and so the general relativity results are approached in the weak-field limit when $n>-1$ and if $n<-\frac{5}{4}$. However, the weak-field predictions diverge from those of general relativity in the $\omega\rightarrow\infty$ limit if $-\frac{5}{4}<n<-1$. The case with $n=-1$, and hence ω constant, corresponds to Brans-Dicke theory. A detailed study of the cases where $\omega'\omega^{-3}\rightarrow 0$ as $\omega\rightarrow\infty$ allows us to place constraints upon the allowed form of $\omega(\phi)$ using both the solar system tests and the consequences for the evolution of the very early Universe. These questions will be examined in a subsequent paper.

ACKNOWLEDGMENTS

I would like to thank José Mimoso for many helpful discussions.

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