Average rate of separation of trajectories near the singularity in mixmaster models

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The system of equations for a mixmaster cosmological model is reduced to a geodesic flow on a pseudo-Riemannian manifold. This geodesic flow is, on the average, locally unstable in the first and second Belinskij-Khalatnikov-Lifshitz (BKL) approximations. In the geometrized model of dynamics we define an average rate of separation of nearby trajectories with the help of a geodesic deviation equation in a Fermi basis. It turns out that the standard indicator for detecting chaotic behavior, a principal Lyapunov exponent, can be obtained from a normal separation vector. We also show that the principal Lyapunov exponents are always positive in the first and second BKL approximations. If the period of oscillations in the long phase (the second BKL approximation) is infinite, the principal Lyapunov exponent tends to zero.

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I. INTRODUCTION

It is well known that the chaotic regime appears near the initial singularity in mixmaster cosmological models [1]. The evolution of these models can be approximated by a series of Kasner epochs, the duration of which turns out to be stochastic as one moves backwards in time toward the initial singularity. During a given Kasner epoch two of the scale factors oscillate, while the third one decreases monotonically. In minisuperspace models of dynamics the epochs correspond to bounces outward along a corner of the potential, while a new era begins with a change of a corner. In our picture, trajectories can concentrate around three "attractors" (there are three chaotic centers which form the stochastic domains in the neighborhood of unstable Taub solutions [2]). Rugh and Jones [2] have numerically found the qualitative correctness of the Belinskij-Khalatnikov-Lifshitz (BKL) approximation near the singularity if the mixmaster model satisfies the Einstein equations.

Transitions between subsequent Kasner epochs provide a return map with the positive Lyapunov exponent $\lambda=\pi^2/6(\ln 2)^2$ [3], whereas the Lyapunov exponents extracted from continuous dynamics, in accordance with Einstein equations, are zero [2,4] in τ time [$\tau=\int dt/abc$, t is the cosmological time, and a, b, c are the scale factors for Bianchi type-IX (BIX) models].

Barrow used this map to compute the probability that an era has r epochs. The mixmaster cycles are most likely to be very short; over 41% of them will involve a single oscillation. Burd, Buric, and Ellis attribute the vanishing of the "Lyapunov exponents," computed from the numerical trajectories, to the exponentially increasing "time" interval between the successive eras (as suggested by Francisco and Matsas). Both the τ (Hobil, Berstein, Welge, and Simkins and Berger) and α (Burd, Buric, and Tavakol) intervals between eras increase greatly as the trajectories approach the singularity [4].

Lyapunov exponents have never been defined in an invariant manner (with respect to time coordinate transformations), and there is the problem of constructing a gauge-invariant measure of separation of nearby trajectories in time [5]. The problem of a gauge-invariant description of an indicator of chaos is also important in classical dynamics. Since topological properties of the attractor manifold in phase space (particularly, generalized dimensions) do not depend on the choice of parametrization of trajectories, they remain intact under reparametrization of the time coordinate. Thus, the choice of the time parameter in a dynamical equation is rather arbitrary and, generally speaking, a curve in phase space can be parametrized in the most convenient manner by dynamical equations depending on the particular problem. In order to avoid confusion it is worth stressing that the evolution in time, and hence the Lyapunov exponents and generalized entropies, will change under such a transformation, and only the geometric characteristics in phase space will remain invariant. In this paper we show that the "Lyapunov exponents" are positive in the first and second BKL approximations. If the period of oscillations tends to infinity in the second BKL approximation, the Lyapunov exponents go to zero.

Chitre [6] showed that a mixmaster model in superspace could asymptotically be represented by a geodesic flow on a negative curved space, and consequently, that the model was chaotic.

Chitre-Misner variables for the dynamics are one of many possible variables which allow for the reduction to the geodesic problem (see Pullin's discussion concerning the superiority of the Chitre gauge choice over other gauge variables [7]). These variables should not be distinguished in any way since the chaos characteristics have to be defined so as not to depend on this particular choice [8].

II. THE LOCAL INSTABILITY OF THE GEODESIC FLOW FOR BIX COSMOLOGICAL MODELS

Vacuum Einstein equations for the Bianchi IX model are equivalent to the Hamiltonian equations with the Hamiltonian [9]

$$H=T+V\ , \tag{1}$$

where

$$T=\tfrac{1}{2}g^{ij}p_ip_j\ ,\quad g^{ij}=\begin{bmatrix}-q_1^2 & q_1q_2 & q_1q_3\\ q_2q_1 & -q_2^2 & q_2q_3\\ q_3q_1 & q_3q_2 & -q_3^2\end{bmatrix}\ ,$$

$$V=\frac{1}{4}\left[2\sum_{i=1<j}^{3}q_iq_j-\sum_{i=1}^{3}q_i^2\right]\ .$$

T, V, are the kinetic and potential energy, respectively, and $q_i=A_i^2$, $i=1,2,3$) (A_i are scale factors for the anisotropic BIX evolution).

The total energy is conserved and the constraint equation is $H=0$. By virtue of the Maupertuis variational principle, the Hamiltonian system generates a geodesic flow on a space with the Jacobi metric [5,10]

$$ds^2=|V|g_{ij}dq^idq^j\ . \tag{2}$$

Therefore, while investigating the sensitivity of the system with respect to initial conditions, it is suitable to start with the geodesic deviation equation. This equation determines a deviation vector n normal to a velocity vector u along the geodesic. The normal vector n describes a separation of neighboring geodesics. The Jacobi deviation equation has, by analogy with the Newtonian equation of motion, the form

$$\nabla_u\nabla_u n=\frac{D^2n}{ds^2}=-\mathrm{grad}_n V_u(n)\ , \tag{3}$$

where

$$(\mathrm{grad}_n)^i=g^{ij}\frac{\partial}{\partial n^j}$$

and the potential

$$V_u(n)=-\tfrac{1}{2}\langle n,R(u,n)u\rangle=\tfrac{1}{2}R(n,u,n,u)$$
$$=\tfrac{1}{2}K_{u;n}g(n,n)g(u,u)\ ,$$
$$R(n,u,n,u)=K_{u;n}[g(n,n)g(u,u)-g(n,u)g(u,n)]\ ,$$
$$ds=2W\,d\tau\ ,\quad W=|V|\ ,$$

$R(u,n)$ is a Riemannian tensor, τ is a time coordinate in the Hamiltonian equation, and $K_{u;n}$ is the curvature in the two-direction determined by the vectors n and n.

Generally, the problem of solving deviation equation (3) is complex, but there is a simple averaging procedure [10] which gives the sign of the Ricci scalar of the Jacobi metric and informs us about the local instability of the geodesic flow. An average potential $\widehat{V}(n)$ can be created by choosing the tangent vector u and the normal one n at random (any direction $u\wedge n$ is equally probable). The corresponding equation assumes the form

$$\frac{D^2\hat{n}}{ds^2}=-\mathrm{grad}_n\widehat{V}(n)=-\frac{2R}{N(N-1)}\hat{n}\ , \tag{4}$$

where R is the Ricci scalar of a pseudo-Riemannian space with the Jacobi metric and N is a dimension of the configuration space.

If we introduce a Fermi basis $\{E_\alpha,\alpha=1,\ldots,N-1,E_N=u\}$ along the randomly chosen geodesic, formula (4) yields

$$\frac{d^2\hat{n}^\alpha}{ds^2}=-\frac{2R}{N(N-1)}\hat{n}^\alpha\ , \tag{5}$$

and the Ricci scalar R for the space with metric (2) has the form [5]

$$R=-\frac{3}{4}\widehat{W}^{-3}\left[4\widehat{W}\sum_{i=1}^{3}q_i^2+\sum_{i<j}^{3}q_i^2q_j^2-\frac{1}{2}\sum_{i=1}^{3}q_i^4\right]\ ,$$
$$\widehat{W}=|V|\ . \tag{6}$$

According to the idea originally developed by Belinskij, Khalatnikov, and Lifshitz [1], the evolution of the Bianchi IX model can be approximated by a series of Kasner epochs $q_i\propto t^{p_i}$ (the first BKL approximation) with $p_i<0$ dominating in the potential. Taking into account the well-known Kasner constraints, one can suitably parametrize the Kasner exponents [1,9]

$$p_i(u)=\frac{-u}{1+u+u^2}\ ,\quad p_2(u)=\frac{1+u}{1+u+u^2}\ ,$$
$$p_3(u)=\frac{u(1+u)}{1+u+u^2}\ .$$

The rules governing transitions between the subsequent Kasner eras are $u\to u-1$ for $u\geq 2$ and $u\to 1/(u-1)$ for $1\leq u\leq 2$. The state of the system is determined by pairs (u,σ), where

$$\sigma=\begin{bmatrix}1 & 2 & 3\\ i & j & k\end{bmatrix}\ ,\quad \sigma_{12}=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{bmatrix}\ ,$$
$$\sigma_{23}=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{bmatrix}$$

are the corresponding permutations. The mixmaster transformation exchanging the order of Kasner exponents $p_i<p_j<p_k$ is $(u,\sigma)\to(u-1,\sigma\sigma_{12})$ for $u\geq 2$, and $(u,\sigma)\to((u-1)^{-1},\sigma\sigma_{12}\sigma_{23})$ for $1\leq u<2$.

In the second BKL approximation we have $q_i\approx q_j\gg q_k$, which corresponds to small oscillations of two scale factors q_i and q_j, whereas q_k is negligibly small (corner oscillations in the Misner regime).

The Jacobi metric and the Ricci scalar have the forms

$$ds^2=\tfrac{1}{4}q_i^2g_{ab}dq^adq^b\ ,\quad R=-\frac{3}{2W}$$

in the first BKL approximation ,

and

$$ds^2=\tfrac{1}{4}(q_i-q_j)^2g_{ab}dq^adq^b\ ,\quad R=-\frac{6}{W}$$

in the second BKL approximation ,

where

$$g_{ab}=\frac{1}{2q_1q_2q_3}\begin{bmatrix} 0 & q_3 & q_2 \\ q_3 & 0 & q_1 \\ q_2 & q_1 & 0 \end{bmatrix} . \tag{7}$$

Deviation equation (5) in the original time coordinate τ (from the Hamiltonian equations) takes the form

$$\frac{d^2\hat{n}^\alpha}{d\tau^2}+\frac{1}{W}\partial_i V\frac{dq_i}{d\tau}\frac{d\hat{n}^\alpha}{d\tau}=-\frac{8RW^2}{N(N-1)}\hat{n}^\alpha , \tag{8}$$

where $\alpha=1,\ldots,N-1$, and

$$d\tau=\frac{dt}{\sqrt{q_1q_2q_3}} , \quad \partial_i\equiv\frac{\partial}{\partial q_i} .$$

In any era dominated by p_i one has

$$R=-\frac{3}{2W} , \quad W=\tfrac{1}{4}q_i^2=\tfrac{1}{4}t^{4p_i}=\tfrac{1}{4}\exp(4p_i\tau) ,$$

where t is the cosmological time.

By the substitution $\hat{n}^\alpha=W^{1/2}z^\alpha$ the dissipative term in (8) can be eliminated, and in this case we obtain

$$\frac{d^2z^\alpha}{d\tau^2}=[4p_i^2+\tfrac{4}{3}\beta^2\exp(4p_i\tau)]z^\alpha , \tag{9}$$

where $\alpha=1,2,\beta^2=\text{const}$ ($R=-\beta^2/W$, $\beta=\sqrt{3}/2$, and $\beta=\sqrt{6}$ in the first and second BKL approximation, respectively).

In deriving Eq. (9) the first term has been neglected at the limit $\tau\to-\infty$. Equation (9) can be reduced to the form

$$16p_i^2\xi\frac{d^2z^\alpha}{d\xi^2}+16p_i^2\frac{dz^\alpha}{d\xi}-\frac{4}{3}\beta^2z^\alpha=0 , \tag{10}$$

where $\xi=\exp(-4p\bar{\tau})$ and $\bar{\tau}=-\tau$. The solution of this equation is

$$z^\alpha(\xi)=C_1\cosh^2\left[\frac{\beta^2}{12p_1^2}\right]^{1/2}\xi+C_2\sinh^2\left[\frac{\beta^2}{12p_i^2}\right]^{1/2}\xi ; \tag{11}$$

thus,

$$\hat{n}^\alpha=\hat{n}_0^\alpha\exp(2p_i\tau)\exp\left[\frac{\beta}{\sqrt{3}|p_i|}\exp(4p_i\tau)\right] . \tag{12}$$

Equation (8), for z^α rewritten in the cosmological time t $[dt=(q_1q_2q_3)^{1/2}d\tau]$, assumes the form

$$\frac{d^2z^\alpha}{dt^2}+\frac{d\ln V}{dt}\frac{dz^\alpha}{dt}=\left[\frac{1}{4}\left[\frac{d\ln W}{dt}\right]^2+\frac{1}{2}\frac{d^2\ln W}{dt^2}-\frac{dW}{dt}\frac{d\ln V}{dt}-\frac{4}{3}R\frac{W^2}{V^2}\right]z^\alpha . \tag{13}$$

For the Kasner era dominated by p_i the above formula takes the form

$$\frac{d^2z^\alpha}{dt^2}+\frac{1}{t}\frac{dz^\alpha}{dt}=\left[\frac{4p_i^2}{t^2}+\frac{4}{3}\beta^2t^{(4p_i-2)}\right]z^\alpha , \tag{14}$$

which can be recognized as the Bessel equations. Hence, at the limit $t\to0$, the solution reads

$$z^\alpha=z_0^\alpha J_0\left[\frac{i\beta}{\sqrt{3}p_i}t^{-2|p_i|}\right] .$$

Therefore, the solution of the deviation equation near the singularity is

$$\hat{n}^\alpha(t)=\hat{n}_0^\alpha t^{2p_i}J_0\left[\frac{i\beta}{\sqrt{3}p_i}t^{-2|p_i|}\right] . \tag{15}$$

Taking into account that

$$J_0(z)=\left[\frac{2}{\pi z}\right]^{1/2}\left[\cos\left[z-\frac{\pi}{4}\right]+\frac{1}{8z}\sin\left[z-\frac{\pi}{4}\right]\right] ,$$

we arrive at the formula

$$\hat{n}^\alpha(t)=\frac{n_0^\alpha}{\sqrt{2\pi\alpha}}t^{2p_i}\left[\exp(\alpha t^{2p_i})+\frac{t^{-2p_i}}{8\alpha}\exp(\alpha t^{2p_i})\right] \xrightarrow[t\to0]{} \frac{n_0^2}{\sqrt{2\pi\alpha}}t^{2p_i}\exp(\alpha t^{2p_i}) , \tag{16}$$

where $\alpha=\beta/\sqrt{3}|p_i|>0$. One can see that for $t\to0$ the local instability arises and is the superposition of the power law and the exponential instability. This corresponds to a hyperexponential instability in time τ.

The principal Lyapunov exponent can be defined in the following way (let us note that the natural parameter s is measured along the geodesics):

$$\lambda_{\text{Lap}}=\lim_{\|n(0)\|\to0}\ln\left\|\frac{n(s)}{n(0)}\right\| . \tag{17}$$

In the first BKL approximation near the singularity ($\tau\to-\infty$) we have

$$s(\tau)=\frac{1}{8p_i}\exp(4p_i\tau) \text{ and } s(\tau\to-\infty)=-\infty , \quad p_i<0 ,$$

$$n^\alpha(s)=n_0^\alpha\sqrt{8p_is}\exp\left[\frac{\beta}{\sqrt{3}|p_i|}8p_is\right] , \tag{18}$$

and

$$\lambda_{\text{Lap}}=\lim_{s\to-\infty}\frac{\ln\|n(s)\|}{s}=\frac{8\beta}{2\sqrt{3}}=2\sqrt{2}>0 . \tag{19}$$

In the second BKL approximation we obtain

$$W(\xi)=\frac{1}{4}(q_1-q_2)^2=4A^2a_0^4\frac{\xi}{\xi_0^2}\sin^2(\xi-\xi_0) , \tag{20}$$

where

$$q_1=a_0^2\frac{\xi}{\xi_0}\left[1+\frac{A}{\sqrt{\xi}}\sin(\xi-\xi_0)\right]^2 ,$$

$$q_2=a_0^2\frac{\xi}{\xi_0}\left[1-\frac{A}{\sqrt{\xi}}\sin(\xi-\xi_0)\right]^2 ,$$

$$\xi_0=2\pi m, \quad m\in N .$$

ξ is now a new time parameter such that

$$\xi=\xi_0 \exp\left[\frac{2a_0^2}{\xi_0}(\tau-\tau_0)\right],$$

$$\tau\in[\tau_0,-\infty) \text{ and } \xi\in[\xi_0,0].$$

a_0, ξ_0, τ_0, A are constants (for terminology see Belinskij, Khalatnikov, and Lifshitz in [1]).

During the long phase ξ_0 becomes large, and the natural parameter s measured along geodesics is connected with ξ by the relation

$$s=2a_0^2A^2\frac{\xi}{\xi_0},\quad s\in[0,2a_0^2A^2]$$

The conformal factor W expressed in terms of s has the form

$$W(s)=2a_0^2 s\sin^2\left[\frac{\xi_0}{2a_0^2A^2}s\right]\le\overline{W}(s)=2a_0^2 s \tag{21}$$

and the deviation equation in this case takes the form

$$\frac{d^2\hat{n}^\alpha}{ds^2}=\frac{1}{a_0^2 s}\hat{n}^\alpha . \tag{22}$$

The long-phase approximation is valid if $\xi\ge 1$ ($s\ge A^2$), and if $\xi\approx 1$ it can be destroyed. The exact solution of deviation equation (22) can be obtained by Kelli solutions, but we can restrict our attention to

$$\exp\left[\left[\frac{1}{2a_0^4A^2}\right]^{1/2}s\right] \le\frac{\hat{n}^\alpha(s)}{\hat{n}_0^\alpha}\le\exp\left[\left[\frac{\xi_0}{2a_0^4A^2}\right]^{1/2}s\right]. \tag{23}$$

Now we can define the Lyapunov exponent as

$$\lambda_{\text{Lap}}=\left[\frac{1}{2a_0^4A^2}\right]^{1/2}=\left[\frac{m}{\Delta s 0_0^2}\right]^{1/2}. \tag{24}$$

The constant ξ_0 measures the period of oscillations m during the long phase, e.g., $m=\xi_0/2\pi$. If the period of oscillations Δs is infinite Lyapunov exponent (24) goes to zero. The relaxation time should be defined as

$$\tau_{\text{relax}}\propto\lambda_{\text{Lap}}^{-1}=\left[\frac{\Delta s a_0^2}{m}\right]^{1/2}. \tag{25}$$

From the above formula we can see that, if the period of oscillations is finite, the relaxation time is also finite and it goes to infinity as the period of oscillations grows during the long phase (or $a_0\to\infty$). The presence of the exponential rate of separation of neighboring trajectories during the long phase guarantees that the effective "loss of memory" of initial conditions will take place after the relaxation time τ_{relax}.

III. CONCLUSION

In our approach the problem of determining the average separation rate of nearby geodesics has been reduced to determining the normal vector n of the geodesic deviation. The natural parameter s, $ds=2W\,d\tau$, is measured along the geodesics. The deviation vector n measures the actual distance between the nearby geodesics of the congruence (but not between points on them).

In connection with the gauge-invariant character of the Lyapunov exponents one should make the following comments. One often distinguishes the Lyapunov exponents "per time" $\lambda(t)$ and the Lyapunov exponents defined in the standard way as $\lim\sup_{t\to\infty}\lambda(t)$. It is evident from the definition that the Lyapunov exponents "per time" are not invariant with respect to time reparametrization along phase trajectories. The Lyapunov exponents understood in the standard manner are in principle invariant with respect to time reparametrization.

They are defined for linear systems $\dot{x}=A(t)x$, $x\in D^n$ (D^n is compact). Their invariance follows from the fact that the time reparametrization is equivalent to the multiplication of vector field X on a compact space by a positive function $\mu(t)$, $\mu(t)dt=d\tau$, where τ is a new time parameter, and t is the original time such that $dx/dt=\dot{x}$.

The Lyapunov transformation $y=L(t)x$ of the equation of the field variation does not change the Lyapunov exponents of the corresponding solutions [a continuous differentiable functional matrix L defined on $[s,+\infty)$ is called the Lyapunov matrix if matrices L and DL are bounded on $[s,+\infty)$, and $|\det L(t)|\ge m>0, \forall t\ge s]$, i.e., $\omega(y)=\lim\sup_{t\to\infty}(1/t)\ln|y(t)|=\omega(x)$. It can be easily seen that the above property follows from the boundedness of the matrices L and L^{-1}. In general, if the integral of the logarithm of the derivative of the scaling function is finite, the characteristic Lyapunov exponents do not change. If the manifold D^n is not compact (e.g., in the mixmaster models), the characteristic exponents will change with the time reparametrization. The compactness guarantees that all smooth measures are finite. It is essential to be able to use the Oseldec theorem, which almost everywhere assures the finite limit in the definition of the Lyapunov exponent.

In our approach the Lyapunov exponents are defined by the properties of the maximal geodesic. A similar method was elaborated by Pesin and Klingenberg [11]. The group of gauge invariants of the Lyapunov exponents is a group of canonical transformations (the time variable is one of the generalized variables). It means that any change of the form of the Hamiltonian does not modify the characteristic Lyapunov exponents. Unfortunately, in our case the gauge group is not a gauge group for general relativity.

When Lyapunov exponents are investigated numerically, the field X is determined on a noncompact constant energy level $H=0$, and different time variables are interconnected by the exponential function (for details see [11]). Consequently, the integral of the logarithm of the derivative of the scaling function is not finite. This evidently explains the various numerical controversies concerning Lyapunov exponents, which in this case are not invariant with respect to time reparametrization.

In the first and second BKL approximations, the potential in the Misner parametrization is positive, and hence the tangent vector to the geodesic is timelike, which implies that the normal vector is spacelike. The local instability appears if $K_{u;n}g(n,n)g(u,u)<0$, [or equivalently $R(u,n,u,n)<0$]. One can show that in the first and second BKL approximation, the Riemannian curvature in the two-direction containing the vectors u and n is always nonpositive at any point and any two-direction, whereas for the Taub solution it vanishes. This fact means that in such a case, in the neighborhood of the singularity, the system has no noncontinuous additional first integrals.

With the help of the deviation vector one determines the principal Lyapunov exponent and demonstrates that it is positive in the first and second BKL approximation [if the period of oscillations Δs is finite]. In the second approximation the Lyapunov exponent tends to zero if the number of oscillations tends to infinity. It can be shown that, in the corner approximation with the Arnowitt-Deser-Misner (ADM) potential $V(\alpha,\beta_+,\beta_-) = 48e^{4\alpha}e^{4\beta_+}\beta_-^2$ ($|\beta_-| \ll 1, \beta_+ \to +\infty$), the Ricci scalar is positive and goes to zero as $\beta_- \to 0$, i.e., the solution of the deviation equation oscillates, whereas in the second BKL approximation it is negative. Based on this fact, one can interpret the effect of vanishing of the Lyapunov exponents near the singularity, observed numerically [4], as an effect of the trajectory staying long near the boundary $V=0$. This layer is formed around the symmetry axes of the triaxial potential where the axes represent the Taub solution (see Fig. 1).

The chaotic behavior is a qualitative property, and after being averaged it can be destroyed. However, chaos cannot be created by an averaging procedure. A negative value of the Ricci scalar is a sufficient (but not necessary) condition of the local instability of the geodesic flow; evidently the Ricci scalar is a gauge-invariant detector of chaos. In the space with the Jacobi metric (2), s is a natural parameter measured along a geodesic. Since the vector u tangent to the geodesic is normed to unity, $\|u\|^2 = \mathrm{sgn}(E-V)$ (in the sense of the Jacobi metric), s is an invariant quantity (with respect to the canonical and coordinate transformations in the configuration space). Instead of investigating the system in the BKL variables, we can use the Misner variables (α,β_+,β_-) [6], and in such a case the time variable is one of the generalized coordinates. We know that the mixmaster models are very special solutions of general relativity, and one could speak about the fully invariant description only if one would be able to formulate the space-time dynamics in a covariant way. Moreover, in our approach the group of canonical transformations is a gauge group, whereas in general relativity this role is played by the group of all diffeomorphisms.

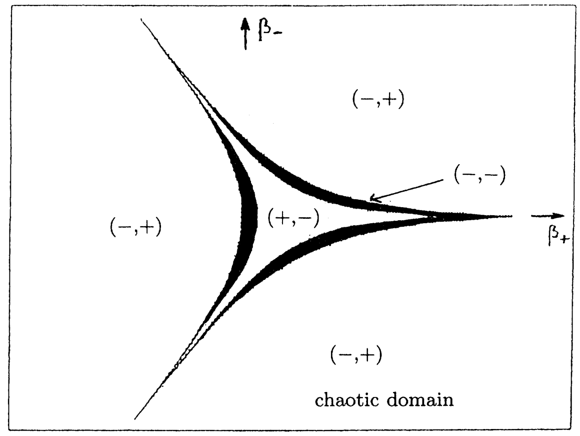

FIG. 1. Signs of the Ricci scalar $R(W)$ [presented as a pair $(\mathrm{sgn}R, \mathrm{sgn}V)$] for the BIX model in Misner's parametrization. In the first and second approximations $V(\alpha,\beta_+,\beta_-)$ and the tangent vectors are timelike, whereas the normal deviation vector is spacelike.

The Chitre-Misner variables for the dynamics should not be distinguished in any way since the chaos characteristics have to be defined so as not to depend on any particular choice of variables. In our approach the Chitre-Misner variables are one of many possible variables which allow for the reduction to the geodesic problem. In such a case the potential is constant in the Jacobi metric at the cost of the shape of the kinetic energy form. In our approach the local instability (in the average) depends only on the Ricci scalar for the space, with any metric which is equivalent to the Jacobi metric (compare [7]). The negative character of the potential energy or the sign of the sectional curvature is the precise criterion of the local instability [12].

As we can see, the problem of chaos in the BIX model is very subtle. But we must stress that there is chaos in that system if the BKL approximation is valid. And all controversies concerning the numerical calculation of the chaos are connected with the lack of accepted tools for detecting the chaos in the context of general relativity.

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