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Stochastic analysis of the initial condition constraints on chaotic inflation

Insu Yi
Astronomy Department, University of Texas, Austin, Texas 78712
and Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, Massachusetts 02138

Ethan T. Vishniac
Astronomy Department, University of Texas, Austin, Texas 78712
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We use stochastic dynamics to examine the probability of successful inflation from various initial conditions at the Planck epoch. This approach allows us to approximate the effect of strong quantum fluctuations at the earliest epoch of inflation, including fluctuations in the local expansion rate. The well-known classical attractors along the slow-roll trajectory exist even when large quantum fluctuations in the early Planck epoch are included. However, our results suggest that a significant fraction of the initial condition phase space which leads to a successful inflationary stage in the classical analysis actually fails to inflate.

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I. INTRODUCTION

The chaotic inflation model [1] (with various models of scalar-field potentials) is one of the simplest models that successfully solves the well-known cosmological problems [2]. Although this model may be realized in various theories of physics of the early universe [3], only certain initial conditions lead to inflation [4]. The issue of whether initial conditions, which lead to inflation, are likely, i.e., whether they occupy a reasonably large volume in phase space, is an important challenge to efforts to construct a self-consistent inflation model. The new inflation model fails to provide a self-consistent solution to the cosmological problems [1,5]. In addition, the phase-space constraint on initial conditions in this model has been shown to be so severe that inflation has a vanishingly small probability [6]. It is naturally of great interest to examine similar constraints on the chaotic inflation model.

Previous studies on the initial condition problem have been carried out almost entirely in the classical limit [6], i.e., without considering quantum fluctuations. These effects will be especially important close to the Planck epoch when the energy density is $\sim m_P^4$. (Throughout this paper we use units such that $G=m_P^{-2}$, $c=\hbar=1$). In these studies, the initial kinetic term, $\dot{\phi}^2/2$, associated with the inflation-driving scalar field (hereafter the inflaton field) ϕ has been shown to have a negligible effect on the probability of successful inflation for a wide range of initial conditions. However, initial spatial inhomogeneities may affect the probability of successful inflation [7]. In any case, a proper investigation of initial condition constraints must include quantum fluctuations, especially since they will initially dominate the inflaton field dynamics. Here we reexamine the initial condition constraints on chaotic inflation including the effects of quantum fluctuations.

For our purposes, we use stochastic dynamics as a first-order, semiclassical approximation to the proper quantum-mechanical scalar-field dynamics [8]. This treatment of quantum fluctuations is only approximate, but fruitful, especially since an exact quantum mechanical solution technique for the Klein-Gordon equation is not available. For simplicity, we do not consider the effects of spatial inhomogeneities [7]. Linde has suggested that it is reasonable to assume that the "tension" energy associated with spatial inhomogeneities will rapidly redshift away as a space-time emerges from the pre-Planckian stage and starts its expansion [3]. However, we note that classical studies of the effects of an initial spatial inhomogeneity show that there is a significant chance that in any particular realization such inhomogeneities may prevent successful inflation. We will study a set of homogeneous universes with varying cosmic kinetic energies, including the effects of quantum fluctuations.

In stochastic dynamics, the scalar field is split into two components [9], a long-wavelength (or coarse-grained) one and a short-wavelength (or high-frequency) one, with respect to the physical horizon $\sim 1/H$, where H is the Hubble expansion parameter. H is assumed to vary slowly during inflation. By decomposing the scalar field in this manner, quantum fluctuations appear in the short-wavelength part [9]. Since we neglect spatial correlations in the fluctuations, the equation of motion for the scalar field becomes a classical stochastic differential equation (in the sense that the random noise term is similar to that encountered in the conventional stochastic theory [10]). In this model, the scalar-field defined in a particular comoving volume of the universe evolves independently. Statistical correlations among different volumes appear only through spatially correlated random (quantum) noise. However, the evolution of the coarse-grained component is not deterministic but stochastic due to the influence of the stochastic noise. The conventional classical homogeneous background of the scalar field [5] is ob-

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tained only after taking the average over the ensemble of independent universes. We can interpret the evolution of the scalar field by taking the average and comparing it to the results of classical deterministic evolution.

By considering the initial condition problem in terms of stochastic dynamics, we aim at obtaining a better understanding of chaotic inflation with quantum fluctuations taken into account. We will show that the stochastic constraints differ substantially from the classical ones and interpret this as a manifestation of quantum effects. We also find that different scalar-field potentials may show somewhat different behavior, suggesting that a purely classical treatment of inflaton dynamics will miss some important quantum-mechanical ingredients.

II. CHAOTIC INFLATION

Chaotic inflation [1,3] is an attempt to invent a simple physical model of the early universe that is free from the various inconsistencies encountered in the previous inflationary models [1–3]. The basic idea of the model is that the early evolution of the universe is driven by a scalar field (the inflaton field), which emerges at the Planck epoch with a strongly nonequilibrium distribution, i.e., with an amplitude much greater than m_P. Once a patch of spacetime foam emerges with an inflaton potential energy density comparable to the kinetic energy $\dot{\phi}^2/2$ and the tension $(\nabla\phi)^2$, it is assumed that the universe evolves until the potential of the field becomes dominant and the universe begins to inflate. In this stage the field evolves very slowly, while the acceleration of the field becomes small [11]. Although various physical models have been proposed as bases for chaotic inflation [2,3], the model is poorly constrained by fundamental physical considerations [4].

In any case, getting to an inflationary stage in the chaotic inflation model is not always possible. Cosmic tension (spatial gradient terms in the energy density) can prevent inflation [7]. (A similar problem can occur in the new inflation model [12]). The effect of kinetic energy $\dot{\phi}^2/2$ is also nontrivial, although the region in the initial condition phase space, which fails to inflate, is rather narrow [4]. In the new inflationary model, the effect of kinetic energy is very serious and only a small fraction of initial conditions leads to inflation [6]. Since the chaotic inflation model invokes "chaotic" initial conditions, i.e., the most general set possible, followed by inflation in some fraction of the independent volumes [1], it is quite important to study the probability of realization of successful inflation from the earliest epoch of inflation (i.e., near the Planck epoch) to make the model self consistent. Originally, Linde [1,3] suggested, based on a crude equipartition argument, that the universe emerges from the preinflationary era with $\sim\frac{1}{3}$ chance for the domination of the potential of the scalar field over kinetic and tension energy. The purpose of the present paper is to include the effects of quantum fluctuations and compare the results with those of the previous classical analyses [4].

In our analysis, we consider a homogeneous and isotropic universe, which is described by the Robertson-Walker line element [13]

$$ds^2=-dt^2+a(t)^2d\mathbf{x}^2=g_{\mu\nu}dx^\mu dx^\nu \ . \tag{2.1}$$

The Lagrangian density associated with the inflaton field ϕ is

$$L=-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi) \tag{2.2}$$

and the stress-energy tensor is

$$T_{\mu\nu}=-\partial_\mu\phi\partial_\nu\phi-Lg_{\mu\nu} \ . \tag{2.3}$$

We will assume that any radiation energy has already been redshifted away. By considering a homogeneous scalar-field cosmology, we have restricted our analysis to the region of phase space in which the tension is already small. The resultant dynamics are described by the Friedmann equation

$$\begin{aligned}H^2&=\left[\frac{1}{a(t)}\frac{\partial a(t)}{\partial t}\right]^2\\&=\frac{8\pi}{3m_P^2}\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2+V(\phi)+\frac{1}{2}\frac{(\nabla\phi)^2}{a(t)^2}\right]-\frac{k}{a(t)^2}\ ,\end{aligned} \tag{2.4}$$

the Raychaudhuri equation

$$\frac{\partial H}{\partial t}+H^2=\left[\frac{8\pi}{3m_P^2}\right]\left[V(\phi)-\left[\frac{\partial\phi}{\partial t}\right]^2+\frac{(\nabla\phi)^2}{a(t)^2}\right]\ , \tag{2.5}$$

and the Klein-Gordon equation

$$\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2}+3H\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\frac{\partial V(\phi)}{\partial\phi}-\frac{\nabla^2\phi}{a(t)^2}=0 \ . \tag{2.6}$$

We will assume a flat universe [$k=0$ in Eq. (2.4)] hereafter. Also, we will denote the time derivative by an overdot whenever convenient.

In the following, for simplicity, we will start the classical evolution of the scalar field ϕ from the epoch at which the space-time foam emerges with the probability for the potential energy distributed evenly in the range, $0\le V_0\le m_P^4$ [14]:

$$\mathcal{P}(V_0)=\frac{1}{m_P^4},\quad 0\le V_0\le m_P^4\ , \tag{2.7a}$$

$$\mathcal{P}(V_0)=0,\quad \text{otherwise}\ . \tag{2.7b}$$

This is meant only as a toy model for the initial conditions of chaotic inflation. We also assume that the total initial energy density is equal to the critical energy density, m_P^4. This is meant as a definition of the emergence of a volume from pre-Planckian conditions. For the massive field case with the potential,

$$V(\phi)=\tfrac{1}{2}m^2\phi^2\ , \tag{2.8a}$$

the probability distribution for the initial value of the scalar field becomes

$$\mathcal{P}(\phi_0)=\frac{m^2}{m_P^4}\phi_0 \tag{2.8b}$$

for the part of phase space where $\phi_0 \geq 0$ and $\dot{\phi}_0 \geq 0$. For the quartic potential,

$$V(\phi)=\tfrac{1}{4}\lambda\phi^4 , \tag{2.9a}$$

$$\mathcal{P}(\phi_0)=\frac{1}{4}\frac{\lambda}{m_P^4}\phi_0^3 \tag{2.9b}$$

in the same region of phase space.

III. CLASSICAL CONSTRAINTS ON CHAOTIC INITIAL CONDITIONS

In the classical analysis (i.e., without allowing for quantum fluctuations), the homogeneous mutually independent evolution equations in a flat space-time (particle generation due to scalar field acceleration will be a small effect and we neglect it here) become [Eqs. (2.4), (2.5), and (2.6)]

$$\dot{\phi}=v , \tag{3.1}$$

$$\dot{v}=-3Hv-\frac{dV}{d\phi} , \tag{3.2}$$

$$H^2=\frac{8\pi}{3m_P^2}(\tfrac{1}{2}v^2+V) , \tag{3.3}$$

which are integrated numerically [4,6].

We can understand the classical behavior of the field qualitatively using a simple analytic solution [6]. If the inflaton field starts its evolution with an initial condition in which the potential energy dominates, $V(\phi) \gg v^2/2$, the field will evolve rapidly towards the slow-roll solution

$$v=\dot{\phi}=-\frac{1}{3H}\frac{dV(\phi)}{d\phi} \tag{3.4}$$

due to the large friction provided by cosmic expansion [11]. Therefore, the initial conditions which need to be examined are those where the kinetic-energy dominates. In this case, $V(\phi_0) \ll v_0^2/2$, the expansion factor is approximately

$$H=\frac{2\sqrt{\pi}}{\sqrt{3}m_P}|v| \tag{3.5}$$

and the evolution equation becomes

$$\frac{d|v|}{dt}=-\frac{2\sqrt{3\pi}}{m_P}|v|^2 . \tag{3.6}$$

The solution is

$$v=\frac{v_0}{1+(2\sqrt{3\pi}/m_P)(t-t_0)|v_0|} , \tag{3.7}$$

where $v_0=v(t=t_0)$. In this limit the magnitude of the roll-down speed is monotonically decreasing from the start. That is, the solution approaches $v=0$ asymptotically until $V(\phi)\sim v_0^2/2$, and it becomes invalid. This is the well-known classical attractor solution in chaotic inflation, which has been previously derived for the massive quadratic potential case [15]. We note that the tendency of the field to evolve toward the attractor is independent of the specific form of the scalar-field potential, as long as it is continuous. The evolution of the inflaton field is given by

$$\phi(t)=\phi_0+\frac{m_P}{2\sqrt{3\pi}}\mathrm{sgn}(v_0)\ln\left[1+\frac{2\sqrt{3\pi}}{m_P}(t-t_0)|v_0|\right] \tag{3.8a}$$

or

$$\phi(t)=\phi_0+\frac{m_P}{2\sqrt{3\pi}}\mathrm{sgn}(v_0)\ln(v_0/v) . \tag{3.8b}$$

Once the field reaches the part of phase space where the potential dominates over the kinetic energy, $V(\phi) \gg v^2/2$, the solution is approximately

$$v(t)=v_1\exp\left[-\frac{\sqrt{24\pi V}}{m_P}(t-t_1)\right] , \tag{3.9a}$$

$$\phi(t)=\phi_1+\frac{m_P}{\sqrt{24\pi V}}(v_1-v) , \tag{3.9b}$$

where the subscript 1 denotes the point at which the potential starts to dominate over the kinetic energy. We see that in this limit the trajectory continues to approach the slow-roll curve. Once the field reaches the slow-roll stage its evolution is given by

$$v(t)\approx 2=\mathrm{const} , \tag{3.10a}$$

$$\phi(t)\approx\phi_2+v_2(t-t_2) , \tag{3.10b}$$

where the subscript 2 denotes the point at which the slow-roll condition [11] is first met. In order for inflation to successfully solve the cosmological problems [2], we require that the number of e foldings >60, which implies that

$$\left|\phi_0+\frac{m_P}{2\sqrt{3\pi}}\mathrm{sgn}(v_0)\ln\left[1+\frac{2\sqrt{3\pi}}{m_P}(t_1-t_0)|v_0|\right] + \frac{v_0}{1+(2\sqrt{3\pi}/m_P)(t-t_0)|v_0|}\frac{m_P}{\sqrt{24\pi V(\phi_1)}}\right|>\phi_s \tag{3.11}$$

where ϕ_s is large enough so that

$$N\equiv\int_{\phi_s}^{\phi_e}H dt>60 \tag{3.12}$$

and ϕ_e is the value of the inflaton field near the end of the slow roll (or beginning of the oscillatory stage) [11]. Equation (3.11) is an initial condition constraint for successful inflation. For the new inflation model with a Coleman-Weinberg–type potential, the allowed initial conditions lie in a very narrow range in phase space [6], whereas the chaotic quadratic potential leads to successful inflation from a wide range of initial conditions [4]. We expect a qualitatively similar conclusion for the chaotic quartic potential, since the two chaotic potentials have essentially the same solution topology in the phase space (see below). The classical behavior described by these asymptotic solutions will now be compared with numerical results and later with the results of our stochastic calculation. In the classical case this approximate solution turns out to be a reasonable guide to the time evolution of the inflaton field.

Before proceeding to the stochastic calculation we will present some numerical results to support our assertion that the approximate analytic argument given above is a reasonable guide to the exact classical evolution of the field. First, we consider the quartic potential, $V(\phi)=\lambda\phi^4/4$. We define the dimensionless variables

$$x \equiv \frac{\phi\lambda^{1/4}}{\sqrt{2}m_P} , \tag{3.13a}$$

$$y \equiv \frac{v}{\sqrt{2}m_P^2} , \tag{3.13b}$$

$$\tau \equiv m_P t , \tag{3.13c}$$

so that the evolution equations [Eqs. (3.1)–(3.3)] become

$$\frac{dx}{d\tau}=\lambda^{1/4}y , \tag{3.14a}$$

$$\frac{dy}{d\tau}=-\sqrt{24\pi}(y^2+x^4)^{1/2}y-2\lambda^{1/4}x^3 . \tag{3.14b}$$

Initial conditions are chosen from the curve, $y_0=\pm\sqrt{1-x_0^4}$ [Eq. (2.8)]. We show the results of numerical integration in Figs. 1 and 2. In Fig. 1, we have chosen a very large self-coupling constant for the purpose of illustration, and in Fig. 2, a realistic (in the sense of producing acceptably small density fluctuations when quantum effects are included) self-coupling constant is used. The classical trajectories (dotted lines) have an obvious symmetry in the phase space. For almost all initial conditions, the trajectories approach the slow-roll curve

$$\dot{\phi}=-\sqrt{\lambda/6\pi}m_P\phi . \tag{3.15}$$

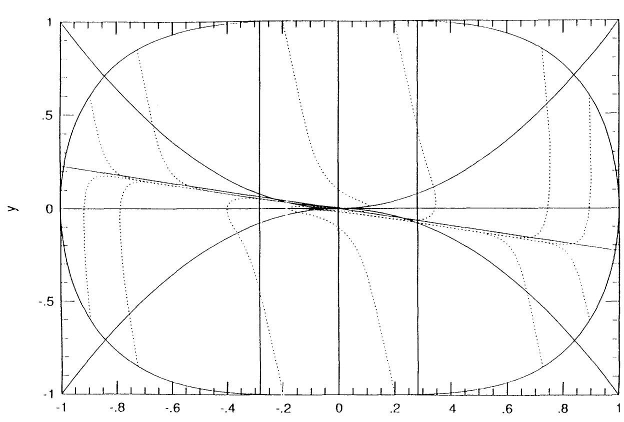

FIG. 1. The classical deterministic trajectories (dotted lines) from 12 different initial conditions for the quartic potential model with an unrealistically large self-coupling constant, $\lambda=1$ for the purpose of illustration. Two solid vertical lines correspond to the slow-roll boundaries, $|x|<0.28\lambda^{1/4}$. The two solid curves, $|y|=x^2$, are boundaries defined by the equal potential-kinetic energies, inside of which the potential energy dominates. The initial conditions are taken on the curve $y_0^2+x_0^4=1$. The slow-roll solution, $y=-\lambda^{1/4}x/\sqrt{6\pi}$, is clearly shown as an attractor in the phase space for all initial condition.

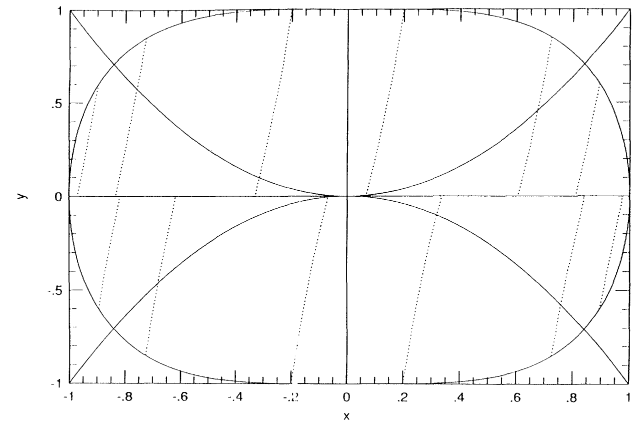

FIG. 2. The classical deterministic trajectories for a realistic self-coupling constant $\lambda=10^{-14}$. In this case, the attractor (slow roll) is apparently very close to the central horizontal line.

We see that successful inflation is easily achieved [4] starting with $|\phi_0|>|\phi_s|$. In the quartic potential case

$$|\phi_s|\geq\sqrt{\phi_e^2+(60/\pi)m_P^2} \tag{3.16a}$$

and

$$\phi_e\approx m_P/\sqrt{2\pi} , \tag{3.16b}$$

as determined by the slow-roll self-consistency condition [11],

$$9H^2\gg\frac{d^2V(\phi)}{d\phi^2} . \tag{3.16c}$$

Although Fig. 2 is not as detailed as Fig. 1 (due to the very small self-coupling constant), the topology of the solution is clearly seen to be independent of the self-coupling constant and the probability of successful inflation is quite large [6].

For the quadratic potential, $V(\phi)=m^2\phi^2/2$, we have the dimensionless evolution equations

$$\frac{dx}{d\tau}=my , \tag{3.17a}$$

$$\frac{dy}{d\tau}=-\sqrt{24\pi}(y^2+x^2)^{1/2}y-\frac{m}{m_P}x , \tag{3.17b}$$

where

$$x\equiv\frac{m\phi}{\sqrt{2}m_P^2} , \tag{3.17c}$$

$$y\equiv\frac{v}{\sqrt{2}m_P^2} . \tag{3.17d}$$

Initial conditions now correspond to points on the curve

$$y_0=\pm\sqrt{1-x_0^2} . \tag{3.18}$$

The results of numerical integration are similar to those for the quartic potential. We summarize the result in Fig. 3. Figure 3 is for a large self-coupling constant (i.e.,

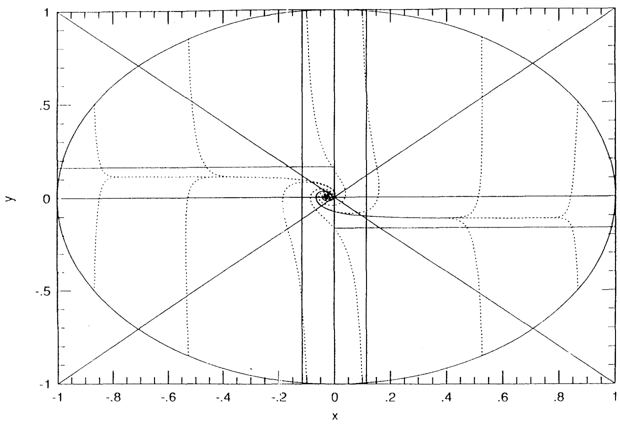

FIG. 3. The classical deterministic trajectories (dotted lines) from 12 different initial conditions for the quadratic potential model with an unrealistically large mass, $m=1.0m_P$, for the purpose of illustration. Two solid vertical lines correspond to the slow-roll boundaries, $|x|<0.43(m/m_P)$. The solid lines $|y|=x$ divide the phase space into the potential-dominated and the kinetic-energy-dominated regions. The initial conditions have been taken from the curve $y_0^2+x_0^2=1$. The slow-roll solution $y=-\text{sign}(x)(m/m_P)/\sqrt{24\pi}$ acts as an attractor for all initial conditions. Some of the trajectories enter the oscillation regime without the inflationary stage.

for illustrative purposes only). We see that the solutions for the quadratic and quartic potentials are almost identical (i.e., show the same solution topology) in the classical analysis. Therefore, any initial condition with

$$|\phi_0|>\phi_s\approx\sqrt{(30/\pi)m_P^2+\phi_e^2} \tag{3.19}$$

where $\phi_e=m_p/\sqrt{12\pi}$ (from the slow-roll condition, $9H^2\gg d^2V(\phi)/d\phi^2$ [11]), will result in successful inflation, in good agreement with the approximate result given in Eq. (3.11).

We conclude that successful inflation is generally expected in chaotic models, and that the exact shape of the potential has very little effect on the topology of the field trajectories. It remains to be seen whether or not this result is still true when the effects of quantum fluctuations are included. In the remainder of this paper we will show that the stochastic solutions for the two potentials are quite different and that the stochastic solution of the quartic potential is closer to its classical solution than the quadratic potential (see below).

IV. STOCHASTIC INFLATION

In this section, we derive stochastic equations which are generally valid, i.e., not only in the slow-roll limit [8] but when $\ddot{\phi}$ is important [16]. In most stochastic analyses, the scalar field is considered only in the slow-roll limit, which results in the Langevin equation and the corresponding Fokker-Planck equation. An exact solution for the quartic potential in this limit was derived by Yi *et al.* [8]. The solution was applied to a very late stage of chaotic inflation in order to analyze the effects of nonlinear stochasticity on generating non-Gaussian inflationary fluctuations. However, in the situations considered here the initial conditions may not be adequately described by the slow-roll equations [8]. In this case, we will show that the equations of stochastic evolution become similar to Kramers equation from classical stochastic theory [10]. The evolution of the Kramers equation can be very different from the Langevin equation, since the former contains the effect of the acceleration of the stochastic system [10]. The stochastic approach is a semiclassical method used to understand the dynamics of a scalar field described by the Klein-Gordon equation, Eq. (2.6). The idea is to take into account the effects of quantum fluctuations through a statistical interpretation of the quantum system.

The derivation of stochastic equations for scalar field dynamics is based on splitting the scalar field into long (or coarse-grained) and short-wavelength parts [9],

$$\phi(\mathbf{x},t)\equiv\phi_c(\mathbf{x},t)+\phi_s(\mathbf{x},t)\ , \tag{4.1}$$

$$\dot{\phi}(\mathbf{x},t)\equiv v_c(\mathbf{x},t)+v_s(\mathbf{x},t)\ , \tag{4.2}$$

where the subscripts c and s denote the coarse-grained (long-wavelength) part and the short-wavelength part, respectively. Eventually we will want to interpret ϕ_c and v_c as classical random variables under the influence of stochastic noise (quantum fluctuations represented by the short-wavelength modes). We will consider the stochastic average of the long-wavelength (coarse-grained) mode as the classical homogeneous background [5]. We now take a specific splitting criterion [9] and define the short components by

$$\phi_s(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\theta(k-\epsilon a(t)H)[a_{\mathbf{k}}\phi_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^\dagger\phi_{\mathbf{k}}^*(t)\exp(-i\mathbf{k}\cdot\mathbf{x})]\ , \tag{4.3}$$

$$v_s(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\theta(k-\epsilon a(t)H)[a_{\mathbf{k}}\dot{\phi}_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^\dagger\dot{\phi}_{\mathbf{k}}^*(t)\exp(-i\mathbf{k}\cdot\mathbf{x})] \tag{4.4}$$

and the long components by

$$\phi_c(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\theta(\epsilon a(t)H-k)[a_{\mathbf{k}}\phi_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^\dagger\phi_{\mathbf{k}}^*(t)\exp(-i\mathbf{k}\cdot\mathbf{x})]\ , \tag{4.5}$$

$$v_c(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\theta(\epsilon a(t)H-k)[a_{\mathbf{k}}\dot{\phi}_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^\dagger\dot{\phi}_{\mathbf{k}}^*(t)\exp(-i\mathbf{k}\cdot\mathbf{x})] \tag{4.6}$$

where $k=|\mathbf{k}|$, $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ are the usual annihilation and the creation operators, respectively and $*$ denotes the complex conjugate of the mode function. The sharp cutoff (or window) function for the splitting is chosen for convenience and is not expected to affect the final results [8,9]. However, it does lead to the Markov property of the noise function [8,10] (see below), which is entirely dependent on the nature of the cutoff [14]. The cutoff function effectively splits the scalar field into the sum of subhorizon modes $[k>\epsilon a(t)H]$ and the superhorizon modes $[k<\epsilon a(t)H]$ with respect to the apparent comoving physical horizon $\sim 1/a(t)H$. ϵ is taken to be much smaller than unity in our analysis, which makes calculations of the short modes quite simple [8].

The mode function, $\phi_{\mathbf{k}}(t)$, is defined in the conventional canonical quantization approach;

$$\phi(\mathbf{x},t)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}[a_{\mathbf{k}}\phi_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^{\dagger}\phi_{\mathbf{k}}^{*}(t)\exp(-i\mathbf{k}\cdot\mathbf{x})] \tag{4.7}$$

with the mode function satisfying

$$\ddot{\phi}_{\mathbf{k}}(t)+3H\dot{\phi}_{\mathbf{k}}(t)+\left[\frac{k^2}{a(t)^2}+\langle V''(\phi)\rangle\right]\phi_{\mathbf{k}}(t)=0\,, \tag{4.8}$$

where a prime means $d/d\phi$ and $\langle\ \rangle$ is the expectation value for a chosen vacuum state [8]. We note that this crude treatment of the self-interaction term is not clearly understood [5,8]. The canonical quantization condition for the field is [8]

$$[\phi(\mathbf{x},t),\dot{\phi}(\mathbf{x},t)^{*}]=\frac{i}{a(t)^3}\,. \tag{4.9}$$

For modes well inside the cutoff scale, $k>\epsilon a(t)H$, we assume

$$\frac{k^2}{a(t)^2}>\epsilon^2H^2\gg\langle V''(\phi)\rangle\,. \tag{4.10}$$

In the slow-roll limit with $H^2\approx 8\pi V(\phi)/3m_P^2$, this limit is easily satisfied. However, at the earliest epochs when we can have $\dot{\phi}^2/2\sim V(\phi)$, this may not be obvious. Nevertheless, we approximate the short modes as those of a massless free field, which has no self-interaction term and automatically satisfies Eq. (4.10). We do this because this assumption makes subsequent calculations simpler [16].

The following definitions will be useful in our derivation:

$$\xi(\mathbf{x},t)=\epsilon a(t)H^2\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta(k-\epsilon a(t)H)[a_{\mathbf{k}}\phi_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^{\dagger}\phi_{\mathbf{k}}(t)^{*}\exp(-i\mathbf{k}\cdot\mathbf{x})]\,, \tag{4.11}$$

$$\zeta(\mathbf{x},t)=\epsilon a(t)H^2\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta(k-\epsilon a(t)H)[a_{\mathbf{k}}\dot{\phi}_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^{\dagger}\dot{\phi}_{\mathbf{k}}(t)^{*}\exp(-i\mathbf{k}\cdot\mathbf{x})]\,. \tag{4.12}$$

Using these definitions we immediately find that

$$\frac{\partial\phi_c(\mathbf{x},t)}{\partial t}=v_c(\mathbf{x},t)+\xi(\mathbf{x},t)\,, \tag{4.13}$$

where we have made use of the relation

$$\frac{\partial}{\partial t}\theta(\epsilon a(t)H-k)=\epsilon\frac{d}{dt}[a(t)H]\delta(k-\epsilon a(t)H)=\epsilon a(t)H^2\delta(k-\epsilon a(t)H)\,. \tag{4.14}$$

In the last relation, we assume that the space-time is in a quasi–de Sitter stage. From Eq. (4.2) and Eq. (2.6), we get

$$\frac{\partial}{\partial t}v_c(\mathbf{x},t)=-\frac{\partial}{\partial t}v_s(\mathbf{x},t)-3H[v_c(\mathbf{x},t)+v_s(\mathbf{x},t)]-\frac{\partial V(\phi)}{\partial\phi}+\frac{\nabla^2\phi(\mathbf{x},t)}{a(t)^2}\,, \tag{4.15}$$

and using

$$\frac{\partial}{\partial t}v_s(\mathbf{x},t)=\zeta(\mathbf{x},t)+\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\theta(k-\epsilon a(t)H)[a_{\mathbf{k}}\ddot{\phi}_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^{\dagger}\ddot{\phi}_{\mathbf{k}}^{*}(t)\exp(-i\mathbf{k}\cdot\mathbf{x})] \tag{4.16}$$

we can rewrite Eq. (4.15) as

$$\begin{aligned}\frac{\partial}{\partial t}v_c(\mathbf{x},t)=&-3Hv_c(\mathbf{x},t)+\zeta(\mathbf{x},t)-\frac{\partial V(\phi)}{\partial\phi}+\frac{\nabla^2\phi}{a(t)^2}-3Hv_s(\mathbf{x},t)\\&-\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\theta(k-\epsilon a(t)H)[a_{\mathbf{k}}\ddot{\phi}_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})+a_{\mathbf{k}}^{\dagger}\ddot{\phi}_{\mathbf{k}}^{*}(t)\exp(-i\mathbf{k}\cdot\mathbf{x})]\,.\end{aligned} \tag{4.17}$$

Using Eq. (4.8), we can combine the last two terms in Eq. (4.17) and get

$$\frac{\partial}{\partial t}v_c(\mathbf{x},t)=-3Hv_c(\mathbf{x},t)+\zeta(\mathbf{x},t)-\frac{\partial V}{\partial\phi}\phi(\mathbf{x},t)+\frac{\nabla^2\phi(\mathbf{x},t)}{a(t)^2}+\langle V''(\phi)\rangle\phi_s(\mathbf{x},t)\ . \tag{4.18}$$

We will now drop the spatial inhomogeneity term. This is partly for simplicity, although it is true that this term will rapidly decrease as the universe expands. We can expand the gradient of the potential as

$$V'(\phi)\approx V'(\phi_c)+V''(\phi_c)\phi_s+O(\phi_s^2)\ , \tag{4.19}$$

so that the equation of motion for $v_c(\mathbf{x},t)$ becomes

$$\frac{\partial}{\partial t}v_c(\mathbf{x},t)=-3Hv_c(\mathbf{x},t)+\zeta(\mathbf{x},t)-\frac{\partial V(\phi_c)}{\partial\phi_c}\ . \tag{4.20}$$

Equations (4.5), (4.6), (4.11), (4.12), (4.13), and (4.20) are the stochastic equations for the inflaton dynamics. We note that the short-wavelength parts of $\phi(\mathbf{x},t)$ and $v(\mathbf{x},t)$ have been removed from the equations and replaced by Eqs. (4.11) and (4.12). We see that ϕ_c and v_c are not identical to their deterministic counterparts [Eqs. (3.4) and (3.5)]. We have dropped spatial inhomogeneity from the equations and the only spatial aspects of these equations enter through ξ and ζ, which still retain their c-number properties. We will eventually remove all spatial correlations by considering the limit ($\epsilon\to0$) in which the stochastic effects occur only through instantaneous time correlations [8].

In order to proceed we now consider the quantum operators ξ and ζ and calculate their expectation values by choosing a specific vacuum state, $|\text{vac}\rangle$. First, we have

$$\langle\text{vac}|\xi(\mathbf{x},t)|\text{vac}\rangle=0\ , \tag{4.21}$$

$$\langle\text{vac}|\zeta(\mathbf{x},t)|\text{vac}\rangle=0\ , \tag{4.22}$$

which are independent of any specific choice of vacuum state since ($a_{\mathbf{k}}|\text{vac}\rangle=0$). To proceed further we make some simplifying assumptions. We assume that for modes with $k\gg\epsilon a(t)H$, H is approximately constant so that we can use the de Sitter result. This is reasonable, since H will not vary over the short mode time scales. The necessary mode functions are solutions of the equation [from Eq. (4.8)]

$$\ddot\phi_{\mathbf{k}}(t)+3H\dot\phi_{\mathbf{k}}(t)+\frac{k^2}{a(t)^2}\phi_{\mathbf{k}}(t)=0\ . \tag{4.23}$$

Without specifying the vacuum state, we can calculate the commutation relations. For ξ and ζ, we easily see from Eqs. (4.11) and (4.12)

$$[\xi(\mathbf{x},t),\xi(\mathbf{x}',t')]=0\ , \tag{4.24}$$

$$[\zeta(\mathbf{x},t),\zeta(\mathbf{x}',t')]=0\ , \tag{4.25}$$

which almost implies a classical behavior for these quantum operators. However, we get a nontrivial cross-commutation relation [16]

$$[\xi(\mathbf{x},t),\zeta(\mathbf{x}',t')]=\epsilon^3\left[\frac{iH^4}{2\pi^2}\right]\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}\delta(t-t')\ , \tag{4.26}$$

where we have made use of the quantization relation, Eq. (4.9). This commutation relation is also independent of the vacuum state. Unless we take a suitable limit (which we will do shortly), the analysis of the scalar field using the stochastic dynamics becomes very complicated at this point.

The stochastic nature of the operators requires evaluation of their vacuum expectation values for a specific choice of the vacuum state. For this purpose we take the Bunch-Davies vacuum state (with positive frequencies) [17],

$$\phi_{\mathbf{k}}(t)=\left[\frac{1}{2k}\right]^{1/2}Hs\exp(-iks)\left[1-\frac{i}{ks}\right]\ , \tag{4.27}$$

where $s\equiv\exp(-Ht)/H$ and the "constant" H is defined only instantaneously. This is a special solution for the massless free field in the perfect de Sitter space-time. A general solution is in fact [16,18]

$$\phi_{\mathbf{k}}(t)=\frac{\sqrt{\pi}}{2}Hs^{3/2}H_\nu^{(2)}(-ks)\ , \tag{4.28a}$$

where $H_\nu^{(2)}$ is the Hankel function of the second kind of the order [19]

$$\nu=\sqrt{(9/4)-(\langle V''(\phi)\rangle/H^2)}\ . \tag{4.28b}$$

Using the mode function, Eq. (4.27), we obtain to lowest order in $s=1/aH$,

$$\dot\phi_{\mathbf{k}}(t)=i\left[\frac{k}{2}\right]^{1/2}H^2s^2\exp(-iks) \tag{4.29}$$

or at $k=\epsilon a(t)H=\epsilon/s$

$$\phi_{\mathbf{k}}(t)\approx-\left[\frac{i}{\epsilon}\right]\left[\frac{s}{2\epsilon}\right]^{1/2}Hs\ , \tag{4.30}$$

$$\dot\phi_{\mathbf{k}}(t)=i\left[\frac{\epsilon}{2s}\right]^{1/2}H^2s^2 \tag{4.31}$$

in the same limit.

We now present the correlation functions for ξ and ζ. We begin with the $\xi-\xi$ correlation function. Using the definition of ξ, Eq. (4.11), we obtain

$$\langle\text{vac}|\xi(\mathbf{x},t)\xi(\mathbf{x}',t')|\text{vac}\rangle=\frac{H^3}{4\pi^2}\delta(t-t')\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}\ , \tag{4.32}$$

where we have used Eqs. (4.30) and (4.31). The self-correlation function for ζ is

$$\langle \text{vac}|\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t')|\text{vac}\rangle = \epsilon^4 \frac{H^5}{4\pi^2}\delta(t-t')\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|} , \quad (4.33)$$

which is not uniquely determined unless ϵ is specified. The cross correlation function has a nontrivial negative sign,

$$\langle \text{vac}|\xi(\mathbf{x},t)\zeta(\mathbf{x}',t')|\text{vac}\rangle = -\epsilon^2 \frac{H^4}{4\pi^2}\delta(t-t')\frac{\sin\epsilon a(t)|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|} \quad (4.34)$$

or

$$\langle \text{vac}|\xi(\mathbf{x},t)\zeta(\mathbf{x}',t')+\zeta(\mathbf{x}',t')\xi(\mathbf{x},t)|\text{vac}\rangle = -\epsilon^2 \frac{H^4}{2\pi^2}\delta(t-t')\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|} . \quad (4.35)$$

If we use the general expression for the mode function, Eq. (4.28a), we obtain for example [20]

$$\langle \text{vac}|\xi(\mathbf{x},t)\xi(\mathbf{x}',t')|\text{vac}\rangle = \epsilon^3 \frac{H^3}{8\pi}|H_\nu^{(2)}(-\epsilon)|^2\delta(t-t')\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|} \quad (4.36)$$

which becomes

$$\langle \text{vac}|\xi(\mathbf{x},t)\xi(\mathbf{x}',t')|\text{vac}\rangle = \frac{H^3}{4\pi^2}\delta(t-t')\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|} \quad (4.37)$$

in the limit $\epsilon \ll 1$ for $\nu = 3/2$ (i.e., massless free), which is in agreement with Eq. (4.32). We have made use of the asymptotic limit of the Hankel function [19]

$$H_\nu^{(2)}(x) \approx i\frac{(\nu-1)!}{\pi}\left[\frac{2}{x}\right]^\nu \quad (4.38)$$

in the limit $x \ll 1$. Other correlation functions for the general mode function, Eq. (4.28a), can be similarly derived with the above results for the massless free-field corresponding to the special case of $\nu = 3/2$ [20].

In $\epsilon \to 0$ limit, the stochastic analysis becomes especially simple. In this limit the physical (comoving) splitting scale is sufficiently far away from the potentially complicated physical (comoving) horizon, which makes the choice of the mode function, Eq. (4.27), a physically meaningful approximation. If ϵ gets close to unity, the assumption, Eq. (4.10), becomes less justifiable and the short modes cannot be adequately approximated by a massless free scalar field. For a finite $a(t)H|\mathbf{x}-\mathbf{x}'|$, the limit $\epsilon \to 0$ gives

$$\frac{\sin\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|}{\epsilon a(t)H|\mathbf{x}-\mathbf{x}'|} \to 1 . \quad (4.39)$$

In this limit the spatial dependence of the noise terms specified by the vacuum expectation values (or correlation functions) disappears [Eqs. (4.32)–(4.35)]. We will use this limit in our subsequent analysis. In the same limit, we also see from Eqs. (4.32)–(4.35) that $\langle \zeta\zeta \rangle$ and $\langle \xi\zeta \rangle$ vanish, while the correlation function $\langle \xi\xi \rangle$ is independent of ϵ. As we mentioned earlier, $[\xi,\zeta]$ does not vanish unless $\epsilon \ll 1$. Due to the quantum nature of the short modes it is almost impossible to analyze the inflaton dynamics using a classical method unless the $\epsilon \to 0$ limit is applied to the system of quantum-mechanical equations. In short, by focusing on very large scales (compared with the comoving apparent physical horizon) and by taking the small ϵ limit, we effectively remove the very complicated quantum mechanical features and the spatial correlations of the noise terms. Under these conditions, we may consider the evolution of the scalar field defined in each of the averaging comoving volumes independent of each other with negligible spatial correlations. Finally, we obtain the evolution equations

$$\dot{\phi}(t) = v(t) + \frac{H^{3/2}}{\sqrt{8\pi^2}}\eta(t) , \quad (4.40)$$

$$\dot{v}(t) = -3Hv(t) - V'(\phi) \quad (4.41)$$

with the correlation function for the newly defined stochastic noise

$$\langle \eta(t)\eta(t') \rangle = 2\delta(t-t') , \quad (4.42)$$

where we drop the subscript c from the equations for convenience. All higher-order correlation functions are negligible in the limit $\epsilon \to 0$, which makes the stochastic noise Gaussian random [10]. We note that the noise function's correlation now has the classical stochastic meaning. That is, the statistical distribution of ϕ and v may be interpreted as the distribution over an ensemble of averaging volumes in the universe. We also note that in the small ϵ limit, the assumption that H is instantaneously constant is a good approximation. However, for the evolution of the long-wavelength modes, the Hubble expansion parameter, H, is determined by the stochastic variables ϕ and v, which means that we take into account the back reaction of the evolution of the scalar field with minimal coupling between gravity and the scalar field [8]. The stochastic equations we have derived are similar to the classical Kramers equation [10],

$$\dot{x}(t) = v(t) , \quad (4.43)$$

$$\dot{v}(t) = -\alpha v(t) - f'(x) + \Gamma(t) , \quad (4.44)$$

where the noise is given by

$$\langle \Gamma(t)\Gamma(t') \rangle = 2D(t-t') . \quad (4.45)$$

In the above equations, x is the position of the Brownian particle (ϕ) and v is its velocity (v in the scalar field equations). α is equivalent to $3H$, f is to V, and D is to $H^{3/2}/\sqrt{8\pi^2}$. However, the classical Kramers equation is different, since the noise term appears in the "velocity" equation [Eq. (4.44)], whereas it appears in Eq. (4.40) for "position" in our case. Nevertheless, in a broad sense, our system of equations are equivalent to the classical Kramers equation and its solutions may be derived using the classical methods [10]. In the classical limit of the stochastic dynamics, it is interesting to see that the slow-roll (nonaccelerating) equation leads to the classical Langevin equation and the non-slow-roll (accelerating) equation of motion leads to the classical Kramers equation.

V. SOME COMMENTS ON INITIAL CONDITIONS

In the exact quantum-mechanical sense, the initial conditions of the universe at the Planck epoch are not clear. The naive classical distribution given in Eq. (2.7) has no particular significance, especially if we consider the quantum aspects of the inflaton field dynamics. There is no exact solution to the problem of calculating the "initial" probability distribution, but we can consider it using stochastic methods.

If the universe emerges from the pre-Planckian epoch with the inflaton field dominating the stress-energy content of the universe (and also in its slow-roll stage), the equation of motion for the field will have the form of the Langevin equation [8],

$$\dot{\phi} = -\frac{V'(\phi)}{3H(\phi)} + \frac{H(\phi)^{3/2}}{\sqrt{8\pi^2}}\eta(t) \tag{5.1}$$

with

$$H(\phi)^2 = \frac{8\pi}{3m_P^2}V(\phi) \tag{5.2}$$

and the noise term given by Eq. (4.42). The corresponding Fokker-Planck equation is not uniquely given due to the noise term's multiplicativity [10,21]. By taking the parameter ω for this uncertainty, the Fokker-Planck equation becomes

$$\frac{\partial}{\partial t}\mathcal{P}(\phi;t) = \frac{\partial}{\partial\phi}\left[\frac{V'(\phi)}{3H(\phi)}\mathcal{P}(\phi;t)\right] + \frac{1}{8\pi^2}\frac{\partial}{\partial\phi}H(\phi)^{3\omega/2}\frac{\partial}{\partial\phi}H(\phi)^{3(2-\omega)/2}\mathcal{P}(\phi;t) \tag{5.3}$$

where $\omega=0$ and $\omega=1$ correspond to Ito's and Stratonovich's rules, respectively [10,21]. The most general stationary solution, $\partial\mathcal{P}(\phi;t)/\partial t = J = \text{const}$, is [22]

$$\mathcal{P}(\phi) = C_0 V(\phi)^{-3(2-\omega)/4}\exp\left[\frac{3m_P^4}{8V(\phi)}\right] - J(8\pi^2)\left[\frac{3m_P^2}{8\pi}\right]^{3/2} V(\phi)^{-3\omega/4}\exp\left[\frac{3m_P^4}{8V(\phi)}\right] \times \int^{\phi} V(s)^{-3\omega/4}\exp\left[-\frac{3m_P^4}{8V(s)}\right]ds\,, \tag{5.4}$$

where C_0 is a constant of integration, in principle, determined by the overall normalization but in practice uncertain due to uncertain boundary conditions. The zero-probability current ($J=0$) solution is

$$\mathcal{P}(\phi) \propto V(\phi)^{-3(2-\omega)/4}\exp\left[\frac{3m_P^4}{8V(\phi)}\right]. \tag{5.5}$$

The exponential term is the dominant factor in this distribution and the form is reminiscent of the Hartle-Hawking wave function [23]. This zero-current solution says that the smallest potential is most probable. If this probability distribution is applied in our case, then the most probable initial condition would be dominated by the kinetic term. However, the distribution is not self-consistently normalized in the stochastic analysis, since the equation is not valid at the boundaries (i.e., slow-roll and Planck density) and since the boundary conditions are not clear [24].

It is not clear if the slow-roll version of the stochastic equation is valid in this case, but a self-consistent stochastic solution for the initial condition is obtained if the scalar-field dynamics is dominated by a large diffusion effect caused by quantum fluctuations. We note, however, that this solution still lacks self-consistent boundary conditions. The equation for the probability distribution is, after neglecting the drift term,

$$\frac{\partial}{\partial t}\mathcal{P}(\phi;t) = \frac{1}{8\pi^2}H(\phi)^{3/2}\frac{\partial}{\partial\phi}H(\phi)^{3/2}\frac{\partial}{\partial\phi}\mathcal{P}(\phi;t) \tag{5.6}$$

with a solution given by

$$\mathcal{P}(\phi;t) \propto \exp\left[-A\frac{m_P^4}{V(\phi)}\right], \tag{5.7}$$

where A is a factor of order unity, and we have assumed that the field diffuses from the initial value ϕ_0 with $V(\phi_0)$ to the value ϕ with $V(\phi) < V(\phi_0)$ [24]. Although, the diffusion equation itself is independent of the slow-roll approximation its validity has not been rigorously proven. However, we point out that this solution is in agreement with the estimate of the probability of quantum creation of an inflationary universe [24]. We may interpret this probability as the probability for the quantum creation of a miniuniverse filled with a field ϕ from the diffusive stage of the earliest epoch.

In either of these two stochastic solutions, the crude classical estimate of the initial probabilities, Eq. (2.7), is not recovered. In general, we do not expect each point in the phase space to have the same accessibility [i.e., $V(\phi) \sim m_P^4$ will be very likely in the case of Eq. (5.7)]. However, since the exact probability is not known, we will consider each possible initial condition in the phase space with equal weight so that we can examine the least favorable initial conditions for successful chaotic inflation. Our main purpose is to consider the case for a significant contribution of kinetic term ($\dot{\phi}^2/2$) in the initial conditions to compare the stochastic results with the classical results and assess the importance of the quantum fluctuations.

VI. STOCHASTIC EVOLUTION FROM PLANCK INITIAL CONDITIONS

We now solve the stochastic equations, Eqs. (4.40) and (4.41), together with Eq. (3.3) with initial conditions set near the Planck epoch. In order to gain some insight into the stochastic behavior of the scalar fields, we first consider an analytic example.

A. A simple analytic example

We consider a model of stochastic evolution for the scalar field in which the scalar potential is assumed to have the form

$$V(\phi)=V_0+\tfrac{1}{2}m^2\phi^2 \tag{6.1}$$

where $V_0 \gg m^2\phi^2/2$ remains valid throughout the evolution so that $H\approx$const becomes a good approximation. The point of this model is to understand the behavior of the scalar field in the exact de Sitter space-time. In this case, the equations become

$$\dot{\phi}=v+\frac{H^{3/2}}{\sqrt{8\pi^2}}\eta(t)\ , \tag{6.2}$$

$$\dot{v}=-3Hv-m^2\phi^2 \tag{6.3}$$

with $H=$const. We immediately see that these equations correspond to the linear Ornstein-Uhlenbeck process [10]. In the matrix form, we get

$$\frac{\partial}{\partial t}\begin{bmatrix}\phi\\ v\end{bmatrix}=-\begin{bmatrix}0 & -1\\ m^2 & 3H\end{bmatrix}\begin{bmatrix}\phi\\ v\end{bmatrix}+\frac{H^{3/2}}{\sqrt{8\pi^2}}\begin{bmatrix}\eta(t)\\ 0\end{bmatrix} \tag{6.4}$$

or the generalized Fokker-Planck equation for the joint probability distribution $\mathcal{P}(\phi;v;t)$

$$\frac{\partial}{\partial t}\mathcal{P}(\phi;v;t)=-\frac{\partial}{\partial\phi}[v\mathcal{P}(\phi;v;t)]+\frac{\partial}{\partial v}[(3Hv+m^2\phi)\mathcal{P}(\phi;v;t)]\ . \tag{6.5}$$

After defining the drift matrix

$$\gamma\equiv\begin{bmatrix}0 & -1\\ m^2 & 3H\end{bmatrix} \tag{6.6}$$

and the diffusion matrix

$$D\equiv\begin{bmatrix}0 & H^3/8\pi^2\\ 0 & 0\end{bmatrix}, \tag{6.7}$$

the standard method leads to the formal solution [10]

$$\mathcal{P}(\phi;v;t\,|\,\phi_0;v_0;t=0)=\frac{1}{2\pi}\frac{1}{\sqrt{\det(\sigma)}}\exp\left[-\frac{[\sigma^{-1}(t)]_{\phi\phi}(\phi-\langle\phi\rangle(t))^2}{2}\right]\times\exp\left[-[\sigma^{-1}(t)]_{\phi v}(\phi-\langle\phi\rangle(t))(v-\langle v\rangle(t))-\frac{[\sigma^{-1}(t)]_{vv}(v-\langle v\rangle(t))^2}{2}\right], \tag{6.8}$$

where

$$\langle\phi\rangle(t)=[\exp(-\gamma t)]_{\phi\phi}\phi_0+[\exp(-\gamma t)]_{\phi v}v^0\ , \tag{6.9}$$

$$\langle v\rangle(t)=[\exp(-\gamma t)]_{v\phi}\phi_0+[\exp(-\gamma t)]_{vv}v_0\ , \tag{6.10}$$

the matrix elements are

$$[\exp(-\gamma t)]_{\phi\phi}=\frac{\lambda_1\exp(-\lambda_2 t)-\lambda_2\exp(-\lambda_1 t)}{\lambda_1-\lambda_2}\ , \tag{6.11}$$

$$[\exp(-\gamma t)]_{\phi v}=\frac{\exp(-\lambda_2 t)-\exp(-\lambda_1 t)}{\lambda_1-\lambda_2}\ , \tag{6.12}$$

$$[\exp(-\gamma t)]_{v\phi}=m^2\frac{\exp(-\lambda_1 t)-\exp(-\lambda_2 t)}{\lambda_1-\lambda_2}\ , \tag{6.13}$$

$$[\exp(-\gamma t)]_{vv}=\frac{\lambda_1\exp(-\lambda_1 t)-\lambda_2\exp(-\lambda_2 t)}{\lambda_1-\lambda_2}\ , \tag{6.14}$$

the eigenvalues are

$$\lambda_1=\frac{3H+\sqrt{9H^2-4m^2}}{2}\ , \tag{6.15}$$

$$\lambda_2=\frac{3H-\sqrt{9H^2-4m^2}}{2} \tag{6.16}$$

(satisfying $\lambda_1+\lambda_2=3H$ and $\lambda_1\lambda_2=m^2$) and the matrix elements of σ are given by

$$\sigma_{\phi\phi}=\frac{H^3}{8\pi^2}\frac{1}{(\lambda_1-\lambda_2)^2}\left[\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}+\frac{4(e^{-(\lambda_1+\lambda_2)t}-1)}{\lambda_1+\lambda_2}-\frac{e^{-2\lambda_1 t}}{\lambda_1}-\frac{e^{-2\lambda_2 t}}{\lambda_2}\right], \tag{6.17}$$

$$\sigma_{\phi v}=\sigma_{v\phi}=\frac{H^3}{\sqrt{8\pi^2}}\frac{1}{(\lambda_1-\lambda_2)^2}(e^{-\lambda_1 t}-e^{-\lambda_2 t})^2\ , \tag{6.18}$$

$$\sigma_{vv}=\frac{H^3}{8\pi^2}\frac{1}{(\lambda_1-\lambda_2)^2}\left[\lambda_1+\lambda_2+\frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}(e^{-(\lambda_1+\lambda_2)t}-1)-\lambda_1 e^{-2\lambda_1 t}-\lambda_2 e^{-2\lambda_2 t}\right] \tag{6.19}$$

with

$$\begin{aligned}\mathrm{Det}(\sigma)=\sigma_{\phi\phi}\sigma_{vv}-(\sigma_{\phi v})^2=&\left[\frac{H^3}{8\pi^2}\right]\frac{1}{(\lambda_1-\lambda_2)^4}\\&\times\left[\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}+\frac{4}{\lambda_1+\lambda_2}\{\exp[-(\lambda_1+\lambda_2)t]-1\}-\frac{1}{\lambda_1}\exp(-2\lambda_1 t)-\frac{1}{\lambda_2}\exp(-2\lambda_2 t)\right]\\&\times\left[\lambda_1+\lambda_2+\frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}\{\exp[-(\lambda_1+\lambda_2)t]-1\}-\lambda_1\exp(-2\lambda_1 t)-\lambda_2\exp(-2\lambda_2 t)\right]\\&-\left[\frac{H^3}{8\pi^2}\right]^2\frac{1}{(\lambda_1-\lambda_2)^4}[\exp(-\lambda_1 t)-\exp(-\lambda_2 t)]^2\ .\end{aligned} \tag{6.20}$$

This exact analytic solution says that the evolution of ϕ and v results in Gaussian statistics. This is simply due to the fact that this system of equations is linear [21].

Using the derived solution, we first consider the means of ϕ and v, $\langle\phi\rangle$ and $\langle v\rangle$. From Eqs. (6.9)–(6.16), we obtain

$$\langle\phi\rangle(t)=\frac{\lambda_1\exp(-\lambda_2 t)-\lambda_2\exp(-\lambda_1 t)}{\lambda_1-\lambda_2}\phi_0+\frac{\exp(-\lambda_2 t)-\exp(-\lambda_1 t)}{\lambda_1-\lambda_2}v_0\ , \tag{6.21}$$

$$\langle v\rangle(t)=m^2\frac{\exp(-\lambda_1 t)-\exp(-\lambda_2 t)}{\lambda_1-\lambda_2}\phi_0+\frac{\lambda_1\exp(-\lambda_1 t)-\lambda_2\exp(-\lambda_2 t)}{\lambda_1-\lambda_2}v_0\ . \tag{6.22}$$

In the limit $H^2\gg 4m^2/9$ (i.e., the effective mass of the scalar field is negligible during inflation), the two eigenvalues become

$$\lambda_1\approx 3H\ , \tag{6.23a}$$

$$\lambda_2\approx\frac{m^2}{3H} \tag{6.23b}$$

and the means are

$$\begin{aligned}\langle\phi\rangle(t)=&\left[\exp\left[-\frac{m^2}{3H}t\right]-\frac{m^2}{3H}\exp(-3Ht)\right]\phi_0\\&-\frac{1}{3H}\left[\exp\left[-\frac{m^2}{3H}t\right]-\exp(-3Ht)\right]v_0\ ,\end{aligned} \tag{6.24}$$

$$\begin{aligned}\langle v\rangle(t)=&\frac{m^2}{3H}\left[\exp(-3Ht)-\exp\left[-\frac{m^2}{3H}t\right]\right]\phi_0\\&-\left[\exp(-3Ht)-\frac{m^2}{9H^2}\exp\left[-\frac{m^2}{3H}t\right]\right]v_0\end{aligned} \tag{6.25}$$

or after a time $t\gg 1/H$,

$$\langle\phi\rangle(t)=\left[\phi_0-\frac{v_0}{3H}\right]\exp\left[-\frac{m^2}{3H}t\right]\ , \tag{6.26}$$

$$\begin{aligned}\langle v\rangle(t)=&\frac{m^2}{9H}[v_0-3H\phi_0]\exp\left[-\frac{m^2}{3H}t\right]\\&\approx-\frac{m^2}{3H}\phi_0\exp\left[-\frac{m^2}{3H}t\right]\ .\end{aligned} \tag{6.27}$$

On the other hand, the classical solution in the slow-roll limit ($\dot\phi=-m^2\phi/3H$) is

$$\phi(t)=\phi_0\exp\left[-\frac{m^2}{3H}t\right]\ , \tag{6.28}$$

$$v(t)=\dot\phi(t)=-\frac{m^2}{3H}\phi_0\exp\left[-\frac{m^2}{3H}t\right]\ . \tag{6.29}$$

The deterministic solutions are in good agreement with the means of the stochastic variables in the assumed limit $m^2\ll 3H$. That is, in this limit, the evolution of the field is very close to the classical evolution regardless of the choice of initial conditions (ϕ_0 and v) after a typical time scale $\gg\sim 3H/m^2$. Although this example is just a toy model, chosen to be analytically tractable, this suggests that the classical attractor solutions (i.e., slow-roll solutions) exist even when we include the effects of quantum fluctuation. We will demonstrate numerically that this is true in more realistic inflation models.

The averaged motion of the field does not reveal the fluctuations of the scalar field and its associated canonical momentum. In the exact analytic solutions, Eq. (6.8), we get the dispersion of ϕ and v,

$$\langle(\phi-\langle\phi\rangle)^2\rangle(t)=\{[\sigma^{-1}(t)]_{\phi\phi}\}^{-1}\ , \tag{6.30}$$

$$\langle(v-\langle v\rangle)^2\rangle(t)=\{[\sigma^{-1}(t)]_{vv}\}^{-1}\ , \tag{6.31}$$

which are completely specified by Eqs. (6.17)–(6.20). The qualitative behavior of the dispersion of v can be understood through a direct integration of the stochastic Eq. (4.20),

$$\dot v\approx-3Hv-\epsilon^2\frac{H^{5/2}}{\sqrt{8\pi^2}}\eta(t)\ , \tag{6.32}$$

where the self-interaction term has been neglected at early epochs during which fluctuations are dominant. However, this estimate is not rigorously correct, since the cross correlation becomes important unless ϵ is negligible. The solution for the linear stochastic equation, Eq. (6.32), is [10,25]

$$v(t)-v_0\exp(-3Ht)=\frac{\epsilon^2H^{5/2}}{\sqrt{8\pi}}\exp(-3Ht)\int_0^t\exp(3Hs)\eta(s)ds \tag{6.33}$$

and we get

$$\langle v\rangle(t)=v_0\exp(-3Ht)\ , \tag{6.34}$$

$$\langle[v(t)-v_0\exp(-3Ht)]^2\rangle=\frac{\epsilon^4H^5}{24\pi^2}[1-\exp(-6Ht)]\rightarrow\frac{\epsilon^4H^5}{24\pi^2}=\text{const} \tag{6.35}$$

for $t\gg1/H$. Although we have neglected the cross correlation $\langle\xi\zeta\rangle$, this behavior is expected to be qualitatively correct in the classical limit. The dispersion of ϕ can be obtained in the same limit (i.e., assuming the fluctuation dominant case) from Eq. (6.2)

$$\langle(\phi-\langle\phi\rangle)^2\rangle\approx\frac{H^3}{4\pi^2}t\ , \tag{6.36}$$

where we have made use of Eq. (4.42). We notice that the fluctuations of ϕ becomes dominant over the fluctuations of v as

$$\frac{\langle\delta v^2\rangle}{\langle\delta\phi^2\rangle}\approx\frac{\epsilon^4H^2}{6t}\propto t^{-1} \tag{6.37}$$

and the mean of v rapidly becomes very small (within a time scale $\sim1/H$). In short, this approximate solution shows that the behavior of the scalar field quickly approaches the slow-roll solution ($v\rightarrow0$), while fluctuations of ϕ dominate the scalar-field dynamics.

$H=$const is a good approximation for the earliest epoch in models such as the new inflation in which the curvature of the potential may be negligible. In this limit the arrival of the slow-roll phase in the course of evolution is expected, since the dynamics of the scalar field will be dominated by the large friction provided by rapid expansion. The analytic solution and its asymptotic behavior demonstrate this expected behavior. However, in more realistic chaotic inflation models, numerical integration of the derived stochastic equations is necessary in order to include the backreaction of fluctuations in H (induced by field fluctuations) on the evolution of the field. This effect, combined with the large quantum fluctuations in the early epoch of inflation, necessitates a numerical treatment. Here we consider only the effects of minimal coupling between gravity and the inflaton field [7].

B. Numerical integration of the stochastic equations

We now integrate the coupled stochastic equations, Eqs. (4.40) and (4.41). In integrating the ϕ equation, we take the form of the difference equation

$$\phi(n+1)=\phi(n)+v(n)\delta t+\frac{H(n)^{3/2}}{\sqrt{8\pi^2}}\sqrt{2\delta t}\,w(n)\ , \tag{6.38}$$

where n is the integration step number and $w(n)$ is the discrete Wiener process [10] of unit variance and zero mean, i.e.,

$$\langle w(n)w(n')\rangle=\delta_{nn'}\ , \tag{6.39a}$$

$$\langle w(n)\rangle=0\ . \tag{6.39b}$$

This interpretation of the stochastic term is that of the Ito calculus [Eq. (5.3)], which satisfies the conventional sense of causality. (We note that this is not necessarily required for a δ-correlated noise process.) We expect that a different interpretation choice will have negligible consequences [21]. Our equations have been obtained in the small ϵ limit in which the cross correlation and the nonvanishing commutator become arbitrarily small. This limit naturally guarantees our choice of simple mode functions, Eq. (4.27). The discrete Wiener process is simulated by random Gaussian deviates generated through the Box-Muller method using uniform deviates generated by a linear congruence generator [26]. The intrinsic skewness of the distribution of random variables is reduced to $\sim10^{-4}$ and is expected to have negligible effects on the final results of integration.

As initial conditions of integrations, we take six points in the half-phase space for each model potential. As we have seen in the classical results, the evolutionary trajectories are expected to be symmetric so we can consider only the half-phase space. Of those six conditions, two points correspond to the potential energy-dominated initial conditions (i.e., the two points most distant from the origin, $x=0$ and $y=0$, on the x axis). The other four points correspond to the "kinetic-energy"-dominated initial conditions. The former initial conditions are not particularly interesting, since they start in the slow-roll regime. The initial conditions are essentially identical to those of the classical evolution. Of course, if the universe emerges from the pre-Planckian epoch with a large probability for potential energy domination [e.g., Eq. (5.7)], the kinetic-energy–dominated initial conditions could be very unlikely. However, in the absence of any firm distribution estimates it seems safer to consider rather extreme initial conditions. The results of our integration of the stochastic equations constitute an estimate of the effects of the nonlinear back reaction as well as the effects of large amplitude quantum fluctuations. As we mentioned before, we have neglected the effects of pre-existing spatial inhomogeneities. Some classical analyses suggest that they may prevent successful inflation [7].

The results of the integration for the quartic potential, $V(\phi)=\lambda\phi^4/4$ are shown in Figs. 4 and 5. In Fig. 4, the dotted lines correspond to the deterministic trajectories, while the dashed lines are a statistical average of over $\sim 10^5$ stochastically independent realizations starting with six different initial conditions. Classically, all of the six initial conditions lead to the slow-roll curve (running near $x=0$ line), suggesting a large probability of successful inflation. Different self-coupling constants do not change this result. During the early stages of stochastic inflation, the mean values of the scalar field do not increase significantly, while the mean velocity of the field decreases sharply, which is exactly what we have seen in the classical calculations. That is, the mean trajectories in the stochastic calculations behave like the classical trajectories. We see that the classical attractor (slow-roll curve) also exists in the stochastic calculations and most of the mean trajectories arrive at the slow-roll curve, although they spend more time to reach it. In Fig. 4, one trajectory has an energy density, which briefly exceeds the Planck density. One significant difference from the classical result is that one of the six mean trajectories does not reach the slow-roll curve (i.e., failure of successful inflation). Although we have not performed an exhaustive search of initial conditions in phase space, this suggests that the classical results are not entirely accurate. In Fig. 5, we plot the evolution of the ratio of the mean potential energy and the mean kinetic energy. The evolutionary track for the noninflationary case approaches the horizontal line almost monotonically after a brief initial epoch. In Fig. 5, the largest spikes are due to the zero crossings or near zero crossings of the mean trajectories, which start from initial conditions in the upper half plane in phase space. The evolutionary tracks show that potential-energy domination is almost always realized except at the earliest epochs. The asymptotic approach to the central horizontal line traces the evolution of the field into the oscillatory stage at the end of inflation [11]. In short, our stochastic calculations in the quartic potential model show that even if quantum fluctuations are large near the Planck time, the classical slow-roll attractor solution still exists. However, the stochasticity of the scalar-field evolution prevents some initial conditions, which classically inflate, from successfully inflating.

The stochastic behavior of the quadratic potential model, $V(\phi)=m^2\phi^2/2$, is similar to that of the quartic potential. The results are shown in Figs. 6 and 7. Different masses do not affect the classical solutions (Sec. III). It is clear that the classical deterministic solutions converge to the slow-roll attractor solutions from almost any initial condition regardless of the choice of the chaotic potential. In the quadratic potential model, we see that one of the six chosen initial conditions fails to enter the inflationary slow-roll stage (Fig. 6), while the corresponding classical trajectory does. We also note that two of the initial conditions briefly exceed the Planck density ($>m_P^4$) before following the slow-roll trajectory. From Figs. 4 and 6, we see that the stochastic mean trajectories are more significantly affected by the initial sign of the

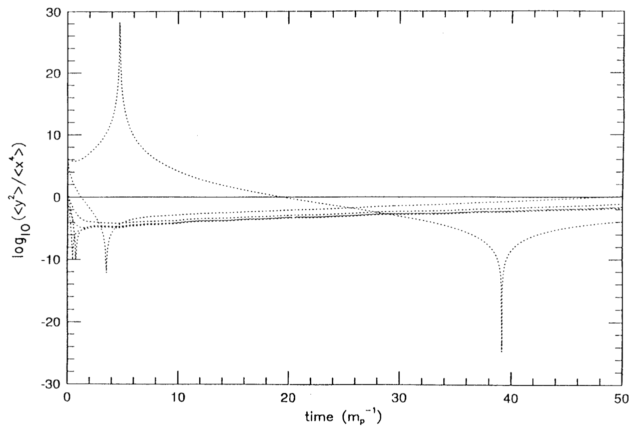

FIG. 5. The time evolution of the mean kinetic-potential energy ratio for stochastic mean trajectories given in Fig. 4.

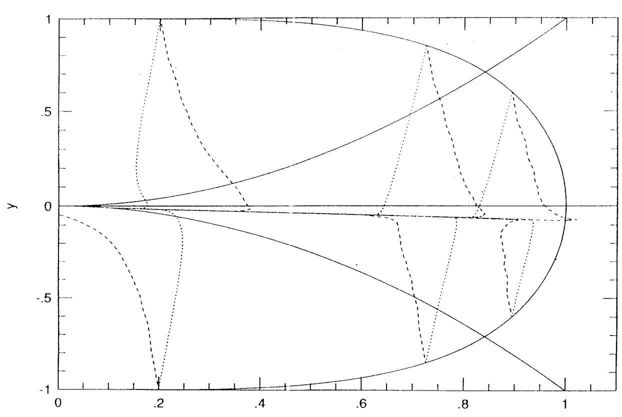

FIG. 4. The stochastic mean trajectories (dashed lines) and the deterministic trajectories (dotted lines) for the same initial conditions in the quartic potential case with $\lambda=10^{-2}$. One of the stochastic trajectories fails to enter the slow-roll stage.

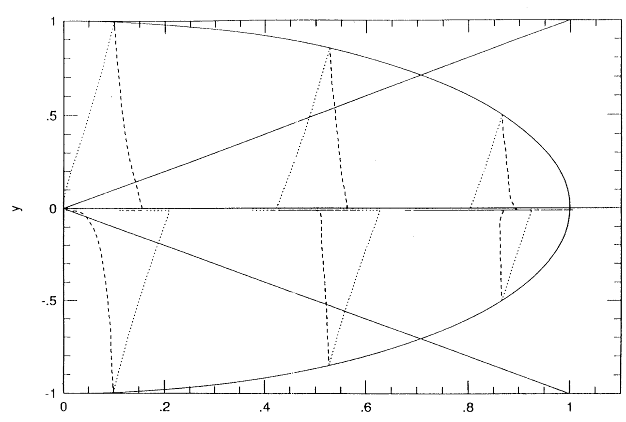

FIG. 6. The stochastic mean trajectories (dashed lines) and the deterministic trajectories (dotted lines) for the same initial conditions in the quadratic potential case with $m=10^{-1}m_P$.

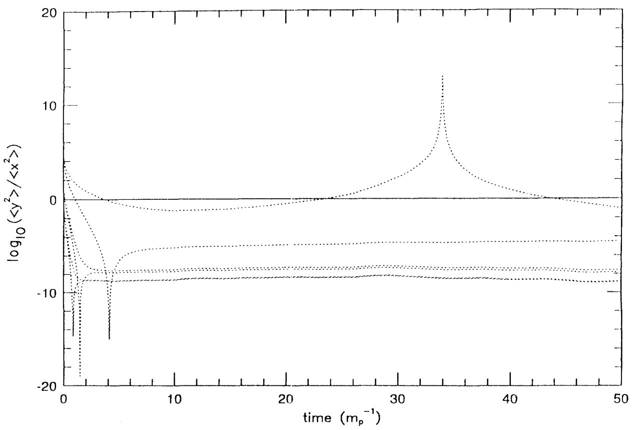

FIG. 7. Similar to Fig. 5 but for the stochastic mean trajectories in Fig. 6.

scalar field velocity. That is, in the initial stages of evolution the mean value of the field shows a substantially larger initial change in the stochastic case. This is the reason the classically successful initial condition fails to enter the inflationary stage. In this sense, the attractor is weaker than in the classical case. Nevertheless, the attractor does exist in the stochastic quadratic potential model. As a related point, after the slow-roll is reached, the evolution of the field in the quartic potential model rapidly approaches the classical results with the field fluctuations being sharply reduced. In the quadratic potential case, the fluctuations of the field always increase and the classical deterministic solution is not prominent in the distribution function. This phenomenon is explained in terms of the nonlinear scaling behavior [27]. It suggests that the transition from the early quantum fluctuation dominated regime to the classical drift dominated one is dependent on the nonlinearity of the self-interaction. Although this might suggest a nontrivial role played by the form of the self-interaction it may also indicate that our approximate classical treatment of the field self-interaction has never taken into account its proper quantum mechanical meaning. In Fig. 7, it is shown that the potential dominated regime (i.e., dotted lines below the central horizontal line) is reached rapidly from most of the initial conditions. We remind the reader that these results depend to some extent on our use of the limit $\epsilon\rightarrow 0$. Allowing for correlated noise (e.g., $\epsilon\sim 1$), may produce significant complications (through nonvanishing cross-correlation functions and the spatial correlations of fluctuations). Nevertheless, we have demonstrated, both by the analytic solution for the simple model and by numerical integration, that the attractor also exists in a model, which allows for some of the effects of quantum noise. Evidently, chaotic inflation is still very likely in the sense that it follows from a wide variety of initial conditions.

VII. CONCLUSIONS

We have examined the initial condition problem in the chaotic inflation model using stochastic techniques and compared the results to the classical calculations. Our stochastic treatment is strictly true only in the limit of $\epsilon\rightarrow 0$ and when the variation of H is instantaneously negligible. We show that inflation is possible for a wide range of initial conditions in spite of the large quantum fluctuations near the Planck epoch. However, the region in the phase space of initial conditions, which will produce inflation is narrower than the classical analysis would indicate. We conclude from this that inflation is still likely when quantum fluctuations are included. However, it seems clear that we need a deeper understanding of the quantum aspects of the inflaton field dynamics to construct a truly self-consistent inflation model. In addition, the fact that the quartic and quadratic potential models exhibit different stochastic behaviors shows that the probability of successful inflation will depend on the details of the inflationary model.

In our analysis we have taken into account only the effect of the kinetic term. However, a complete treatment necessarily requires including spatial inhomogeneities and the possibility of nonvanishing cross-correlation functions. A nonminimal coupling between gravity and the inflaton field may also play a significant role in the dynamic evolution of the quantum field. Our analysis is only a preliminary attempt to include the effects of quantum fluctuation effects and judge their importance for the initial condition constraints. A stochastic study, which includes the effect of spatial inhomogeneity on scales far larger than the averaging scale will be presented elsewhere [28].

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