PHYSICAL REVIEW D VOLUME 47, NUMBER 11 1 JUNE 1993

Improved evaluation of the α^3 vacuum-polarization contribution to the α^4 muon anomalous magnetic moment

Toichiro Kinoshita
Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853

(Received 5 January 1993)

The contribution of the eighth-order vertices containing sixth-order one-electron-loop vacuum-polarization subdiagrams to the muon anomalous magnetic moment, which was evaluated previously using the integration routine RIWIAD on a CDC 7600 computer, is reevaluated using the integration routine VEGAS on an IBM ES/9000 computer. The previous calculation was found to suffer from a severe underestimation of errors. The new result, $-0.241\ 5(19)$, is close to the asymptotic analytic result obtained recently by Broadhurst *et al.* using a renormalization group technique. The difference between the numerical and analytic results may be written as $10.23(39)(m_e/m_\mu)$.

PACS number(s): 12.20.Ds, 12.20.Fv, 13.10+q, 14.60.Ef

Recently, the contribution of the eighth-order diagrams of Fig. 1, which contain sixth-order one-electron-loop vacuum-polarization subdiagrams, to the muon anomalous magnetic moment has been evaluated by Broadhurst *et al.* [1] in the small m_e/m_μ limit using a renormalization group technique, where m_e and m_μ are the electron mass and muon mass, respectively. Their result,

$$a_\mu(\text{Fig. 1}) = \left[-0.290\ 987\cdots + O\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^4, \qquad (1)$$

disagrees with the previous numerical evaluation [2],

$$a_\mu(\text{Fig. 1}) = -0.794\ 5(202)\left(\frac{\alpha}{\pi}\right)^4, \qquad (2)$$

obtained using the integration routine RIWIAD [3] on a CDC 7600 computer.

In order to resolve the discrepancy between (1) and (2), I have reevaluated a_μ(Fig. 1) using the integration routine VEGAS [4] on an IBM ES/9000 computer. The FORTRAN code used in this calculation is identical with that of [2]. The results of the new calculation for individual integrals are listed in Table I. Combining them with the values of auxiliary integrals from Table II, which are needed to implement the two-step on-the-mass-shell renormalization formula [2]

$$\left(\frac{\pi}{\alpha}\right)^4 a_\mu(\text{Fig. 1}) = \sum_{i=A}^{H}\eta_i\Delta M_i - 4\Delta B_2\Delta M^{(\mu,e)}_{2,P_4} + 5(\Delta B_2)^2 M^{(\mu,e)}_{2,P_2} - 2(\Delta L^{(4)} + \Delta B^{(4)})M^{(\mu,e)}_{2,P_2} - 2\Delta\delta m^{(4)} M^{(\mu,e)}_{2,P_2^*}, \qquad (3)$$

one finds

$$a_\mu(\text{Fig. 1}) = -0.241\ 5(19)\left(\frac{\alpha}{\pi}\right)^4. \qquad (4)$$

This is much closer to (1) than to (2).

Since both (2) and (4) are obtained using the same FORTRAN code, one must wonder why they are so different and, in particular, why the uncertainty in (2) is 27 times smaller than their difference. To answer this question, it is necessary to know some details of the calculation leading to (2). Table III lists a summary of the calculation given in Ref. [2] together with some additional information. Comparing Tables I and III, one finds that the values of individual integrals of Table III

TABLE I. The values of integrals $\eta_i\Delta M_i$ corresponding to the diagrams of Fig. 1 evaluated by VEGAS in double precision, where η_i is the multiplicity factor.

Diagram i	$\eta_i\Delta M_i$	Sampling points/iteration	No. of iterations
A	5.674 78(71)	4×10^7	30
B	3.058 95(67)	2×10^7	28
C	1.483 36(45)	2×10^7	30
D	−3.127 30(57)	2×10^7	30
E	−0.083 89(93)	6×10^7	39
F	−4.066 38(68)	2×10^7	27
G	−0.245 31(42)	2×10^7	31
H	2.837 58(33)	2×10^7	30

0556-2821/93/47(11)/5013(5)/$06.00

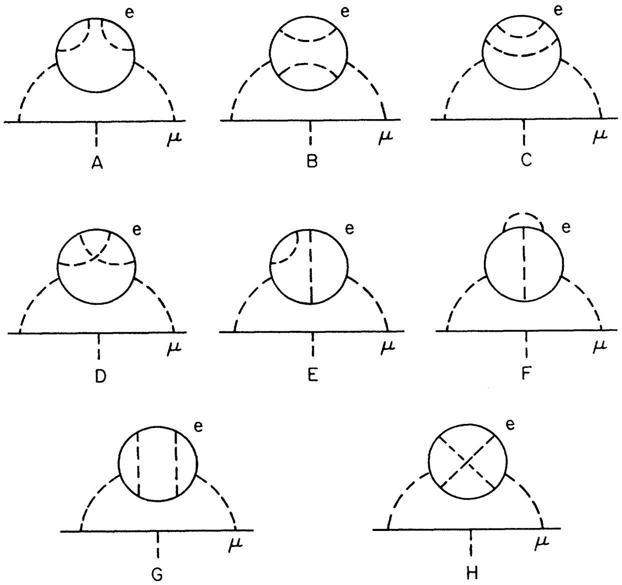

FIG. 1. Eighth-order muon vertices obtained by insertion of sixth-order one-electron-loop vacuum-polarization diagrams in the second-order muon vertex. Diagrams not shown are related to those shown by time reversal and charge conjugation. This is taken into account by appropriate multiplicative factors η_i in (3) and Table I.

(which were evaluated around 1981) agree more or less with those of Table I if some of the stated errors were an order of magnitude larger. Our attention must therefore be focused on the reliability of errors listed in Table III. To understand the nature of these errors it is necessary to know how an adaptive-iterative Monte Carlo integration routine, such as RIWIAD, is used and how the renormalization of the Feynman integral is carried out in this calculation.

RIWIAD starts an iteration procedure by dividing the integration domain (unit hypercube of up to eight dimensions) into a number of subdomains of equal size and evaluating the integrand at two randomly chosen points in each subdomain. The value of the integral is obtained by summing the product of the subdomain volume and the average of the two values of the integrand over all subdomains. The error is estimated collecting the variances of the integrand from all subdomains. The latter information is also used to adjust the sizes and shapes of subdomains for the next iteration. After several iterations the subdomain structure will adapt to a particular integrand and the error estimate from each iteration will stabilize. The most likely value of the integral and its error estimate are obtained by statistically combining the results of many iterations. VEGAS works somewhat differently but gives similar results. See Ref. [4] for details.

The trickiest feature of our treatment of Feynman integral is that the renormalization of ultraviolet divergences and separation of infrared divergences are carried out numerically on the computer. The integral is made finite by point-by-point cancellation of all divergences by carefully tailored ultraviolet and infrared counterterms. This would pose no problem if register could accommodate an arbitrarily large number and if each step of computation were carried out with infinite precision. In reality, registers carry only a finite number of digits and we have to perform calculations in finite precision. The intended cancellation of divergences may occasionally fail because of register overflow which stops the computation. Even when there is no overflow, the difference of canceling terms tends to be dominated near a singularity by round-off errors since these terms have no more than 12 or 13 significant digits (in 64 bits). To the extent that this takes place in very small regions of the integration domain, it does not cause appreciable error in most integrals since the error is proportional to the fractional volume of the integration domain involved. However, it might become noticeable if the *renormalized integrand* itself has steep peaks, as is the case with a_μ(Fig. 1) which diverges logarithmically for $m_e/m_\mu \to 0$.

If the integrand peaks strongly along some axis, the unit interval along that axis will be adaptively subdivided into a large number of subintervals after a few iterations, leaving a relatively small number of subdomains available for adjusting interval sizes along other axes. (The minimum number of subdomains along any axis is chosen to be 2.) Often, the situation is improved and the error reduced significantly by an appropriate remapping of the

TABLE II. The values of auxiliary integrals.

Integral	Value	Sampling points[a]	Reference
ΔB_2	0.75		[5]
$\Delta L^{(4)}$	0.464 85(11)	6 × 30	Improvement of [6]
$\Delta B^{(4)}$	−0.437 65(18)	6 × 30	Improvement of [6]
$\Delta\delta m^{(4)}/m_e$	1.905 24(22)	6 × 35	Improvement of [6]
$M_{2,P_2}^{(\mu,e)}$	1.094 258 28(5)		[7]
$M_{2,P_2^*}^{(\mu,e)} \times m_e$	−0.161 084 05 ... [b]		
$\Delta M_{2,P_4}^{(\mu,e)}$	3.135 059 22(11)		Calculated from [8]

[a] The first and second factors are the number of sampling points per iteration in units of 10^7 and the number of iterations, respectively.

[b] The analytic expression is $M_{2,P_2^*}^{(\mu,e)} \times m_e = -\frac{1}{6} + \frac{1}{8}\pi^2 x + (5 + 4\ln x)x^2 - \frac{15}{8}\pi^2 x^3 + \cdots$, where $x = m_e/m_\mu$.

TABLE III. The values of integrals $\eta_i \Delta M_i$ corresponding to the diagrams of Fig. 1 evaluated by RIWIAD in 60-bit precision, where η_i is the multiplicity factor. They are obtained from Ref. [2]. First few iterations are not included in column 4.

Diagram i	$\eta_i \Delta M_i$	No. of subdomains per iteration	No. of iterations
A	5.411 4(74)	2.5×10^5	11
B	2.900 2(63)	3×10^5	9
C	1.365 5(33)	2.5×10^5	10
D	−3.105 2(59)	3×10^5	11
E	−0.176 5(99)	3×10^5	9[a]
F	−4.037 7(71)	3×10^5	16[b]
G	−0.208 4(86)	1×10^5	10
H	2.834 0(43)	2.5×10^5	10

[a] Available record is incomplete.

[b] Two independent runs, each with eight iterations.

axis i onto itself (which will be called "stretching") such as

$$x_i \to x_i^{\alpha_i} \quad \text{and} \quad 1 - x_i \to (1 - x_i)^{\beta_i}, \quad \alpha_i,\ \beta_i > 1, \tag{5}$$

which helps to reduce the peaking of the integrand. Of course, the integral must be invariant under stretching. As a matter of fact, stretching is performed routinely (for both RIWIAD and VEGAS) to make sure that the value of the integral remains unchanged under various stretchings.

After an appropriate stretching, the distribution of subdomains will become less lopsided. Usually the convergence improves under mild stretching ($\alpha_i, \beta_i \leq 2$). In some cases, however, stronger stretching is desirable. Unfortunately, such a stretching tends to exacerbate the problem of register overflow (which can occur with or without stretching) disrupting the integration process. A simple (but not the only) measure to alleviate this problem is to map the interval (0, 1) onto $(\delta, 1-\delta)$, $0 \leq \delta \ll 1$, for all axes, thereby removing the troublesome regions (as well as some harmless regions) from the integration domain. Let us call this procedure "corner chopping," or simply "chopping." The chopping not only reduces the error estimate generated by an integration routine significantly but also alters the value of the integral by an amount of order $\delta^{1/2}$ (which depends on stretching and can be estimated empirically). Obviously, the result of chopping must be monitored closely by varying δ.

We are now ready to go back to Table III. The best way to examine it is to repeat the calculation under an exactly identical setting. Unfortunately, this task is made practically impossible by the absence of the complete record of calculation. Furthermore, the CDC 7600 computer is no longer available. Under the circumstances, the best one can do is to repeat the RIWIAD calculation on another computer, using the amount of sampling points comparable to that of Ref. [2].

The second column of Table IV lists the best results (namely, with smallest errors) of new RIWIAD calculation (in 64 bits) obtained using various stretchings but no chopping. They agree very well with the VEGAS results of Table I, demonstrating that there is nothing wrong with the RIWIAD itself. One also notices, however, that the errors of the new calculation (in 64 bits) for A, B, E,

TABLE IV. The values of integrals $\eta_i \Delta M_i$ corresponding to the diagrams of Fig. 1 evaluated by RIWIAD in 64-bit precision, where η_i is the multiplicity factor. Column 2 lists results obtained with no or mild stretch. Column 3 lists results of severe stretch with $\delta = 10^{-4}$. Column 4 lists results of severe stretch with $\delta = 10^{-3}$. All calculations are carried out using 2.5×10^5 subdomains per iteration, and iterated 20 times.

	Mild stretch	Severe stretch	Severe stretch
	$\delta = 0$	$\delta = 10^{-4}$	$\delta = 10^{-3}$
Diagram i	$\eta_i \Delta M_i$	$\eta_i \Delta M_i$	$\eta_i \Delta M_i$
A	5.671 3(111)	5.671 0(87)	5.662 1(72)
B	3.061 4(91)	3.055 2(67)	3.074 9(59)
C	1.479 8(28)	1.483 6(26)	1.504 8(26)
D	−3.128 1(61)	−3.108 5(42)	−3.042 9(33)
E	−0.058 0(218)	−0.055 6(176)	−0.003 3(156)
F	−4.057 4(97)	−4.081 7(77)	−4.130 2(60)
G	−0.238 1(55)	−0.248 7(43)	−0.242 6(36)
H	2.834 6(41)	2.835 9(29)	2.819 0(23)

and F are larger than those of the old calculation (in 60 bits) in spite of higher numerical precision. This can be seen more clearly from the results of individual iterations (which are not shown to save space). Based on these observations one can conclude that these errors cannot be made as small as those of Table III by stretching alone. To make them smaller one must rely on chopping or some other similar method. In fact, chopping has been used frequently to avoid register overflow. It was almost a necessity in the early days when securing an adequate amount of computing time was very difficult. Unfortunately the available record of calculation does not show how much chopping was actually made. More seriously, no record has been kept on whether additional calculations with various values of δ were made to correct the distortion of the integral introduced by chopping.

To see how severely an evaluation of integral is affected by chopping, I have carried out sample calculations for our integrals. Results for $\delta = 10^{-4}$ and 10^{-3} are listed in the third and fourth columns of Table IV.

For $\delta = 10^{-4}$, we expect chopping errors of about 1%. For all diagrams except D and F, this δ did not produce significant shifts in the values of integrals. Even for D and F, the shifts are less than 1%: D moved closer to the value of Table III while F moves further away. As expected, all integrals have reduced errors compared with those of column 2 of Table IV. However, the error of E remains large compared with that of Table III.

For $\delta = 10^{-3}$, we expect chopping errors of about 3%. Although the effect of chopping is clearly visible now, especially for D, E, and F, it did not strongly affect A, B, and C, which are the main causes of difference between Tables I and III. The error of E is still far too large compared with that of Table III.

Because of the absence of specific information on stretching and chopping, it is difficult to proceed further in reconstructing the old calculation. In fact, it is unlikely that δ's as large as those chosen above were actually used in Ref. [2], since the register overflow can be easily avoided with a much smaller δ ($\leq 10^{-8}$). It is also possible that a method alternative to chopping was used to avoid register overflow. In any case, our analysis suggests strongly that the severe underestimation of error in (2) has at least two causes. (i) The values and errors listed in Table III are those of chopped integrals, not those of the original integrals; (ii) the value and error of (2) do not include the *required* correction of chopping distortion.

Actually, there is a third possible cause of the discrepancy between (2) and (4): the 60-bit arithmetic of the CDC 7600 computer. The larger round-off errors of 60-bit calculation compared with those of 64-bit calculation could affect the value and error of the integral significantly. It is unfortunate that this possibility cannot be pursued because the CDC 7600 computer is no longer available.

Let me conclude this paper by a discussion of the new result (4). First of all, to establish (4) beyond any doubt, I have examined once again the algebras leading to (3) as well as all FORTRAN programs. No error has been found. The high precision of the new result (4) reflects a 300-fold increase in the total amount of sampling points. Preliminary calculations carried out using various stretchings are all consistent with (4) and indicate strongly that its stated numerical uncertainty is reliable. No chopping was used in all these calculations.

If one assumes that both the numerical result (4) and the asymptotic result (1) are correct, the nature of the difference between them deserves some attention. The question is whether the difference, which corresponds to the uncalculated term of (1), is of order $(m_e/m_\mu)\ln(m_\mu/m_e)$ or m_e/m_μ. Although I have not examined it directly, it is most likely that (1) has no $(m_e/m_\mu)\ln(m_\mu/m_e)$ term. This is because the logarithmic term is absent in the second- and fourth-order vacuum-polarization contributions to a_μ and the same mechanism seems to be at work in all cases. If one accepts this argument, the difference 0.048 5(19) between the coefficients of (1) and (4) is proportional to m_e/m_μ and may be written as $(10.23 \pm 0.39)(m_e/m_\mu)$. This coefficient is rather large but not entirely out of step with the trend set by the coefficients of m_e/m_μ in the second- and fourth-order terms which are about 2.47 and -5.68, respectively. It is hoped that this result is verified analytically some day.

The total α^4 contribution to the difference $a_\mu - a_e$, including the result (4) and an unpublished improvement [1.440 62(10)] of the α^3 two-electron-loop vacuum-polarization contribution to $a_\mu^{(8)}$, which supersedes Eq. (2.14) of Ref. [2], is

$$a_\mu^{(8)} - a_e^{(8)} = 127.55(41)\left(\frac{\alpha}{\pi}\right)^4 . \tag{6}$$

This is larger than the previous result [2] by $0.55(\alpha/\pi)^4$. Note that, of all the terms contributing to Eq. (6), only those given by (2.6), (2.11), (2.15), and (2.16) of Ref. [2] are the ones evaluated by RIWIAD alone. There is no reason to question their accuracy, however, since they are consistent with the renormalization group estimates (A17) and (A18) of Ref. [2].

Collecting (6) and a new evaluation of all sixth-order diagrams in closed analytic form [8, 9] as well as other results of previous calculation [2], one finds the total QED contribution to a_μ to be

$$a_\mu(\text{QED}) = 1\,165\,846\,984(17)(28) \times 10^{-12}, \tag{7}$$

where the first error is a composite of remaining uncertainties in theoretical calculation and the experimental uncertainty in the muon-electron mass ratio [10]

$$\frac{m_\mu}{m_e} = 206.768\,262\ (30), \tag{8}$$

and the second error comes from the uncertainty in the value of α:

$$\alpha^{-1} = 137.035\ 997\ 9(32), \qquad (9)$$

determined from the quantized Hall effect [11].

I should like to thank A. L. Kataev for informing me of their result prior to publication. This research was supported in part by the U.S. National Science Foundation. The numerical computation was carried out on the IBM ES/9000 computer at the Cornell National Supercomputing Facility, which receives major funding from the U.S. National Science Foundation and the IBM Corporation, with additional support from New York State and members of the Corporate Research Institute.

[1] D. J. Broadhurst, A. L. Kataev, and O. V. Tarasov, Phys. Lett. B **298**, 445 (1993).

[2] T. Kinoshita, B. Nizic, and Y. Okamoto, Phys. Rev. D **41**, 593 (1990).

[3] B. E. Lautrup, in *Proceeding of the Second Colloquium on Advanced Computer Methods in Theoretical Physics*, Marseille, France, edited by A. Visconti (University of Marseille, Marseille, 1971).

[4] G. P. Lepage, J. Comput. Phys. **27**, 192 (1978).

[5] P. Cvitanovic and T. Kinoshita, Phys. Rev. D **10**, 4007 (1974).

[6] T. Kinoshita, in *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990), pp. 218–321.

[7] H. H. Elend, Phys. Lett. **20**, 682 (1966); **21**, 720 (1966); M. A. Samuel and G. Li, Phys. Rev. D **44**, 3935 (1991).

[8] S. Laporta, Bologna Report No. DFUB 92-17 (unpublished).

[9] E. Remiddi and S. Laporta, Phys. Lett. B **301**, 440 (1993).

[10] E. R. Cohen and B. N. Taylor, Rev. Mod. Phys. **59**, 1121 (1987).

[11] M. E. Cage *et al.*, IEEE Trans. Instrum. Meas. **38**, 284 (1989).