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Model of random surfaces with long-distance order

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We describe a theory of a rigid string with long-distance order induced by a constrained Wess-Zumino term. The renormalization group analysis shows the existence of a nontrivial fixed point. The corresponding surface has a finite fractal dimension.

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The inclusion of extrinsic curvature-dependent terms into the string action is necessary for the description of flux tubes in gauge theories [1–3] and of certain biological and/or chemical membranes [4,5] (see [6] for a review and references on the membrane line of development). For the membranes it has been demonstrated experimentally that the curvature energy dominates over that of the surface tension. In gauge theories such as QCD[1] or the three-dimensional Ising model, to have a scaling of string tension near the critical point is necessary for obtaining, in the continuum limit, smooth surfaces with finite tension (a proof of this statement was presented in [8]).

The action of the rigid string theory proposed in [2–4] reads (modulo the total derivative)

$$S=\mu_0\sqrt{g}+\frac{1}{2\alpha_0}\sqrt{g}\; g^{\alpha\beta}\nabla_\alpha\partial_\gamma x\,\nabla_\beta\partial^\gamma x +\sqrt{g}\;\lambda^{\alpha\beta}(g_{\alpha\beta}-\partial_\alpha x\;\partial_\beta x) \tag{1}$$

($\alpha,\beta,\gamma=1,2$) and contains, in addition to the Nambu term, also a term equal to the square of the extrinsic curvature tensor of the surface. In the limit $\alpha_0\to\infty$, it reduces to a Nambu action so that the system is governed by "conformal matter + Liouville"-type dynamics [9] and for physical dimensions $1<d<25$ is known to be in a strong-coupling phase [10] with infinite fractal dimension of the corresponding geometrical object. In the limit $\mu_0\to 0$, the action is given purely by a curvature term. However, it was shown in Refs. [2,3,11–13] that the coupling α_0 turns out to be asymptotically free. As discussed by Polyakov [2], this most probably means that in the infrared limit the Lagrange multiplier field develops a nonzero expectation value, thus generating a mass gap and driving the theory to a Nambu phase with the tension completely determined by radiative corrections. Such a scenario is confirmed by subsequent analysis (see [14] for the list of references known to the author).

This picture would change drastically if there were another critical point at a nonzero value of α_0. This point would be infrared stable and would correspond to an ordered phase with smooth surfaces of finite fractal dimension. Indeed, some numerical evidence was found in favor of the existence of such a point, but the data do not exclude other interpretations (see the recent works [15,16] and references therein). These numerical results seem to contradict present analytic knowledge, most of which is, however, either of perturbative or of large-d nature [2,3,6,11–14].

In the present work, a modification of the action (1) leading to the existence of a nontrivial fixed point is considered. The main result is expressed by the formulas for the one-loop β function and fractal dimension [Eqs. (9) and (10)], which show that in the infrared limit the modified theory describes smooth objects with finite fractal dimension. For the simplest case of three-dimensional space, the existence of the fixed point has already been demonstrated in [17], but the method used there made the investigation of the general case very complicated (though a corresponding conjecture was made). In the present work, we prove, using the Monge form for almost-planar surfaces, the general result and calculate one of the scaling exponents. The theory discussed below differs from the only previous construction known to the author (see [18,19]). The hexatic membrane action considered in this work is obtained by adding to the action (1) that of the XY model on the surface. In our construction no new degrees of freedom are necessary.

Before proceeding to the description of the modified action, let us make a remark concerning the possible relevance of the rigid string actions to gauge theories. In a recent work [20], Polchinski and Yang have calculated the high-temperature and high-energy behavior of the partition function of the rigid string (1) and found that it is given by the same power of the temperature as expected for QCD [21], but the sign and, moreover, the phase are different. This raises the problems of unitary on the world sheet [28], expected on general grounds for a theory with a higher-derivative kinetic term. If continued to the space-time amplitudes, this unitary problem will provide a strong argument against the rigid string theories in Minkowski space. This, of course, does not apply to their relevance for the statistical mechanics of membranes [7].

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[1]A recent review of various ideas of the possible equivalence of the large-N QCD to a string theory can be found in [7].

A basic object of our model is a composite matrix field Ω. To construct it, let us denote the tangent vectors of the (closed, orientable) Riemann surfaces Σ embedded in $\mathbb{R}^d$ by $x^\mu_\alpha=(\partial/\partial\xi^\alpha)x^\mu$ ($\alpha=1,2$, $\mu=1,\ldots,d$). Introducing the zweibeins $e_{a\alpha}$ ($a=1,2$) such that $e_{a\alpha}e_{a\beta}=x_\alpha x_\beta=h_{\alpha\beta}$, one can orthonormalize them by writing $x_a=e^\alpha_a x_\alpha$; adding also $d-2$ orthonormal vectors normal to the surface, x_i ($i=3,\ldots,d$), one obtains a set of d orthonormal vectors $x^\mu_m=\{x^\mu_a,x^\mu_i\}$ ($m=1,\ldots,d$). The matrix Ω is a representation of the SO(d) rotation, taking this basis to some arbitrary fixed orthonormal basis $x^{(f)}_m$. It is defined, clearly, only up to an arbitrary rotation belonging to the subgroup SO(2)$\times$SO($d-2$), so that any action depending on Ω must have a corresponding gauge symmetry. In our case this symmetry can be ensured covariantly in a way which we will not describe here; instead, let us accept as a definition that Ω must satisfy the additional constraints

$$\begin{aligned} &e^\alpha_a\nabla_{\bar z}e_{b\alpha}=0\ ,\\ &x_i\partial_{\bar z}x_j=0\ , \end{aligned} \tag{2}$$

with the covariant derivative and the complex structure taken compatible to the metric. A symmetry with respect to the holomorphic SO(2)$\times$SO($2-d$) transformations will remain a symmetry of the action. It will be also independent of the choice of the reference frame $x^{(f)}_m$.

Another comment is that for the surfaces embedded with an open line of self-intersection the field Ω is singular. For $d\geq 4$ these singularities are unstable, but for $d=3$ they are stable and play an important role in the string representation of the three-dimensional Ising model described in [22]. For the present local consideration, they can be ignored.[2]

Let me note also that Ω can be taken in any representation of SO(d). The one relevant for the three-dimensional Ising model is the spinor one. Here we will take Ω to be a $d\times d$ matrix:

$$\Omega_{mn}=x^{(f)}_m x_n\ . \tag{3}$$

With all this in mind, let us write down the simplest possible action depending on Ω, namely, that of the nonlinear σ model:

$$4\alpha_0 S=\sqrt{h}\ h^{\alpha\beta}\mathrm{tr}\partial_\alpha\Omega\,\partial_\beta\Omega^{-1}\ . \tag{4}$$

Substituting for Ω the expansion (3) and using (2), one can show easily that when rewritten in terms of the field x the action (4) coincides with (1).

This fact is just a manifestation of the analogy between (1) and a nonlinear σ model noted by Polyakov in [2]. It suggests generalizing (1) by adding a Wess-Zumino term to it. Namely, consider the action

$$\begin{aligned} S=\mu_0\sqrt{h}+\frac{1}{2\alpha_0}\sqrt{h}\ h^{\alpha\beta}\nabla_\alpha\partial_\gamma x\nabla_\beta\partial^\gamma x\\ +\frac{in}{24\pi}\mathrm{tr}(\Omega^{-1}d\Omega)^3\ , \end{aligned} \tag{5}$$

where in the last term an integration over an arbitrary three-dimensional manifold with a boundary coinciding with the Riemann surface Σ is understood and Ω now denotes an arbitrary extension of the field (3) on this manifold. Let me stress once more that the constraints (2) which have to be added to the action arise as a result of the gauge fixing in the SO(2)$\times$SO($d-2$)-invariant theory.

Once guessed from the simple observation described above, the action (5) is very natural. The matrix Ω as an object describing the embedding was used in Ref. [22], where, following an idea by Polyakov [23] and Dotsenko [23], a possible description of the three-dimensional Ising model in terms of a fermionic string was presented. In the subsequent works [24] and [25], a Wess-Zumino term of the type present in (5) arose as a result of integration over the fermionic variables in the induced Dirac action, describing a gauge-fixed Green-Schwarz superstring theory [26], and has been conjectured to be connected to the density of Hopf's topological invariant [24]. Ideas similar in spirit are developed in detail in Ref. [27].

To determine a scale dependence of the couplings in the one-loop approximation, it is sufficient to keep in the action the terms up to the fourth order in the fields. Choosing a ghost-free gauge $x^\mu=\{\xi^1,\xi^2,x^i\}$ ($i=3,\ldots,d$), one finds (we put below $\mu_0=0$)

$$\begin{aligned} 2\alpha_0 S=&(\partial^2x)^2+\tfrac{1}{2}(\partial^2x)^2(x_\alpha x_\alpha)-2(\partial^2xx_{\alpha\beta})(x_\alpha x_\beta)\\ &-(\partial^2xx_\alpha)(\partial^2xx_\alpha)\\ &+\frac{i\alpha_0 n}{4\pi}\epsilon^{\alpha\beta}[(x^i_\gamma\partial_\alpha x^i_\delta)\partial_\beta\varphi_{\gamma\delta}+(x^i_\gamma\partial_\alpha x^j_\gamma)\partial_\beta\varphi_{ij}]\\ &+O(x^5)\ , \end{aligned} \tag{6}$$

where the fields $\varphi_{\gamma\delta}$ and φ_{ij} parametrize the SO(2)$\times$SO($d-2$) transformation and depend on x via the equations

$$\begin{aligned} &\partial_{\bar z}\varphi_{\alpha\beta}=\tfrac{1}{2}[x^i_\alpha\partial_{\bar z}x^i_\beta-x^i_\beta\partial_{\bar z}x^i_\alpha]\ ,\\ &\partial_{\bar z}\varphi_{ij}=\tfrac{1}{2}[x^i_\alpha\partial_{\bar z}x^j_\alpha-x^j_\alpha\partial_{\bar z}x^i_\alpha]\ . \end{aligned} \tag{7}$$

Now it is quite straightforward to split the field x^i into slow ($|p|<\tilde\Lambda$) and fast ($\tilde\Lambda<|p|<\Lambda$) components and perform a Gaussian integration over the fast ones. The nonlocality present in (7) gets canceled in a final expression, and one obtains, with logarithmic accuracy, the following renormalization of α_0:

$$\frac{1}{\tilde\alpha_0}=\frac{1}{\alpha_0}-\frac{d}{4\pi}\left[1+\frac{n\alpha_0}{8\pi}\right]\ln\frac{\Lambda}{\tilde\Lambda}\ . \tag{8}$$

There is also a generation of the surface energy

$$E_S=-\frac{d}{8\pi^2}\int d^2q\ .$$

From (8) we read off a β function:

$$\beta(\alpha_0)=-\frac{d\alpha_0^2}{4\pi}\left[1+\frac{n\alpha_0}{8\pi}\right]\ . \tag{9}$$

The theory has an infrared-stable fixed point at

[2] These singular configurations can be excluded by allowing for a nonconstant reference frame $x^{(f)}_m$.

$\alpha_0^* = -8\pi/n$ ($n<0$). To get an idea of the phase described by α_0^*, consider an expectation value of an area of the surface of linear length scale R. Calculation gives $\langle A(R) \rangle = R^{d_F}$, with the fractal dimension

$$d_F = 2 - 2\frac{d-2}{n} + O\left[\frac{d-2}{n}\right]. \tag{10}$$

The exact formula for d_F is, most probably,

$$d_F = 2\left[1+\frac{d-2}{n}\right]^{-1}.$$

A more detailed investigation of the phase corresponding to α_0^* will be presented elsewhere.

Let us note finally that, as discussed by Polyakov in [2], the change of the usual coefficient in the β function of the σ model, $d-2$, into $d/2$ in (9) is due to the restriction of the field Ω by the integrability condition stating that Ω must be obtainable from a surface. Formula (9) shows that imposing such a restriction does not spoil the conformal invariance of the Wess-Zumino-Novikov-Witten model, so that the model [Eqs. (5) and (2)] can be regarded as some reduction of it.

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