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Taub numbers at future null infinity

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Taub numbers are studied on asymptotically flat backgrounds with Killing symmetries. When the field equations and the linearized field equations for a metric perturbation are solved, such perturbed space-times admit zeroth-, first-, and second-order Taub numbers. Zeroth-order Taub numbers are Komar constants of the background. For each Killing symmetry of the background, first-order Taub numbers give the contribution of the perturbation to the associated Komar constant, such as the perturbing mass. Second-order Taub numbers give the rate of gravitational radiative loss of the background conserved quantity.

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I. INTRODUCTION

In stationary asymptotically flat space-times, the symmetry generated by the timelike Killing vector k^μ has an associated conserved integral for the mass of the source:

$$\int_\sigma U_0^{\alpha\beta}{}_{;\beta} dS_\alpha \ , \tag{1}$$

where $U_0^{\alpha\beta}=(-g)^{1/2}2\nabla^{[\alpha}k^{\beta]}$ is the Komar [1] superpotential, and dS_α is the three-volume tensor density of the spacelike or null hypersurface σ. For asymptotically flat space-times defined by the properties of null infinity, another type of conserved integral, arising from a generalization of Green's identity, has been shown [2] to yield asymptotic invariants which are the Newman-Penrose constants [3,4]. The generalization of Green's identity is derived from the field equations and the linearized field equations for a metric perturbation. Additionally, for asymptotically flat space-times, a modification of the Komar superpotential, the Winicour-Tamburino [5] superpotential which uses the generators of asymptotic symmetries, yields the Bondi [6] mass when integrated over a cross section of null infinity. The Bondi mass is an asymptotic invariant which is not generally conserved.

The purpose of this work is to explore a method for calculating asymptotic invariants in stationary asymptotically flat space-times: the method of Taub numbers at future null infinity. This method blends elements from the concepts described above. It uses a metric perturbation and Killing symmetries of the background. We will use this method to calculate the background mass and angular momentum, perturbing mass and angular momentum, and mass loss. Taub numbers τ_n are defined in Sec. II. Since we are mainly concerned here with solutions of two sets of equations, the background field equations, and the linearized field equations (i.e., the first functional derivative of the background field equations), only three Taub numbers τ_0, τ_1, and τ_2 are defined. Superpotentials for the Taub numbers are developed and discussed in Sec. III. Results for simple perturbations of well-known solutions (Schwarzschild and Kerr) are given by τ_0 and τ_1 in Sec. IV. τ_0 has the same value as the Komar constant for each Killing vector on the background space-time. τ_1 is the contribution the perturbation makes to each Komar constant. In Sec. V, the Geroch-Xanthopoulos [7] theorem is given wherein any solution $h_{\mu\nu}$ of the linearized field equations preserves the asymptotic simplicity of the vacuum background. τ_2 is evaluated in Sec. VI for nonstationary axial perturbations of Schwarzschild [8]. Working in the conformal manifold which contains future null infinity, Habisohn [9] has proved that τ_2 gives the Bondi mass loss rate due to $h_{\mu\nu}$. Here τ_2 is evaluated in the physical space-time and Habisohn's interpretation verified. The interpretation of τ_1 is confirmed by additional calculations in Sec. VII.

To make the body of this work more readable and reasonably self-contained, we have collected all the lengthy perturbation equations in Appendix A, the details of the Kerr and Schwarzschild metrics in Appendix B, and explicit perturbations of Schwarzschild in Appendix C. The sign conventions used here are $2A_{\nu;[\alpha\beta]}=A_\mu R^\mu{}_{\nu\alpha\beta}$, $R_{\mu\nu}=R^\alpha{}_{\mu\nu\alpha}$, and signature $(+,-,-,-)$.

II. TAUB NUMBERS

Taub numbers [10,11] are defined in the context of a space-time $(M,g_{\mu\nu})$ which admits a Killing vector k^μ and a metric perturbation. Consider a space-time with sources in an interior region which is bounded by vacuum. In the exterior vacuum region, we examine a curve of solutions $\hat{g}_{\mu\nu}$ to the vacuum field equations

$$G_{\mu\nu}(\hat{g})=0 \ , \tag{2}$$

of the form

$$\hat{g}_{\mu\nu}=g_{\mu\nu}+\lambda\dot{g}_{\mu\nu}+\tfrac{1}{2}\lambda^2\ddot{g}_{\mu\nu}+\cdots \ , \tag{3}$$

where $g_{\mu\nu}$ with n overdots denotes $[(d^n/d\lambda^n)\hat{g}_{\mu\nu}(\lambda)]_{\lambda=0}$. One obtains a sequence of perturbation equations from Eq. (2) by expanding $G_{\mu\nu}(\hat{g})$ in powers of λ:

$$\underset{0}{G}{}_{\mu\nu}(g)=0 \ , \tag{4}$$

$$\underset{1}{G}{}_{\mu\nu}(g,\dot{g})=0 \ , \tag{5}$$

$$\underset{2}{G}_{\mu\nu}(g,\dot{g},\dot{g})+\underset{1}{G}_{\mu\nu}(g,\ddot{g})=0\ , \tag{6}$$

$$\underset{3}{G}_{\mu\nu}(g,\dot{g},\dot{g},\dot{g})+3\underset{2}{G}_{\mu\nu}(g,\dot{g},\ddot{g})+\underset{1}{G}_{\mu\nu}(g,\dddot{g})=0\ , \tag{7}$$

etc., where $\underset{n}{G}_{\mu\nu}$ represents the nth functional derivative of $G_{\mu\nu}$ evaluated at $g_{\mu\nu}$. When the first j perturbation equations above are satisfied there are $j+1$ divergence-free terms:

$$\nabla_\nu \underset{n}{G}^{\mu\nu}(g,\dot{g},\ldots,\dot{g})=0\ ,\quad n=0,1,\ldots j\ ,j\leq 2\ . \tag{8}$$

(Here, we will generally consider solutions of $\underset{0}{G}_{\mu\nu}=\underset{1}{G}_{\mu\nu}=0$, which implies that $\nabla_\nu \underset{0}{G}^{\mu\nu}=\nabla_\nu \underset{1}{G}^{\mu\nu}=\nabla_\nu \underset{2}{G}^{\mu\nu}=0$.) $\underset{n}{G}^{\mu\nu}$ is of nth order in $\dot{g}$, and both ∇_ν and semicolons denote covariant derivatives with respect to the unperturbed $g_{\mu\nu}$. Field equations (4) establish the unperturbed vacuum region and (5) are the linearized field equations on a curved background for the metric perturbation $h_{\mu\nu}$ where, henceforth, $h_{\mu\nu}=\dot{g}_{\mu\nu}$ and $h=h_{\mu\nu}g^{\mu\nu}$. When the first two perturbation equations are satisfied, a Killing vector k^μ of the background space-time and (8) yield three conserved vector densities:

$$\partial_\alpha t_n^\alpha=0\ , \tag{9}$$

where

$$t_n^\alpha=(-g)^{1/2}\underset{n}{G}^\alpha{}_\beta k^\beta\ ,\quad n=0,1,2\ . \tag{10}$$

The t_n^α are nonzero because of the presence of sources in the interior region.

An nth-order Taub number is defined as [12]

$$\tau_n=-2\int_\sigma t_n^\alpha dS_\alpha=-2\int_\sigma(-g)^{1/2}\underset{n}{G}^{\alpha\beta}k_\beta dS_\alpha\ , \tag{11}$$

where σ is an initial data surface (null or spacelike). τ_0 is well understood and here we are concerned with τ_1 and τ_2 at future null infinity.

Taub numbers have been studied in the context of linearization stability, where there are global results for two cases. If the background space-time describes an isolated asymptotically flat system, as is done here, then linearization stability holds in general [13]. If the background describes a closed cosmology foliated by compact Cauchy surfaces without boundary then the Einstein field equations are linearization stable about $(M,g_{\alpha\beta})$ if and only if the background has no Killing vector fields. If the closed cosmology has a Killing symmetry then Taub numbers which vanish provide constraints [14] that exclude any spurious solutions of the linearized Einstein equations.

III. SUPERPOTENTIALS

Penrose, in his treatment of conserved quantities in linearized general relativity [15,16], made use of the Killing potential $Q^{\alpha\beta}$, a real antisymmetric tensor field, satisfying

$$\mathcal{P}^{\nu\alpha\beta}=2(\nabla^{(\nu}Q^{\alpha)\beta}-\nabla^{(\nu}Q^{\beta)\alpha}+g^{\nu[\alpha}Q^{\beta]\sigma}{}_{;\sigma})=0\ . \tag{12}$$

Equivalently,

$$Q^{\alpha\beta}{}_{;\nu}=-2\delta_\nu{}^{[\alpha}k^{\beta]}+2(\delta_\nu{}^{[\alpha}\xi^{\beta]})^*\ , \tag{13}$$

where $k^\alpha=\frac{1}{3}Q^{\alpha\beta}{}_{;\beta}$, $\xi^\alpha=\frac{1}{3}{}^*Q^{\alpha\beta}{}_{;\beta}$. When $Q^{\alpha\beta}$ satisfies (12) and (13) then k^α and ξ^α are Killing vector fields in vacuum, $*$ is the dual operator, and $\mathcal{P}^{[\nu\alpha\beta]}=\mathcal{P}_\alpha{}^{\alpha\beta}=0$. Goldberg [17] has generalized Penrose's linearized analysis to curved space. Starting (essentially) with

$$G^{\mu\nu}{}_{\alpha\beta}=-{}^*R^{*\mu\nu}{}_{\alpha\beta}$$
$$=R^{\mu\nu}{}_{\alpha\beta}+2\delta_{[\alpha}{}^\mu R^\nu{}_{\beta]}+2G^\mu{}_{[\alpha}\delta^\nu{}_{\beta]}\ ,$$

where $G^{\mu\nu\alpha\beta}{}_{;\beta}=0$, Goldberg obtains

$$\tfrac{1}{3}\nabla_\nu(G^{\mu\nu}{}_{\alpha\beta}Q^{\alpha\beta})=\tfrac{1}{3}(R^\mu{}_{\nu\alpha\beta}+\delta^\mu{}_\alpha R_{\nu\beta}-\delta^\mu{}_\beta R_{\nu\alpha})\mathcal{P}^{\nu\alpha\beta}$$
$$+2G^\mu{}_\alpha k^\alpha\ . \tag{14}$$

Maintaining k^α and $Q^{\alpha\beta}$ as objects on the background which satisfy $\mathcal{P}^{\nu\alpha\beta}=0$ on the background, Goldberg's equation (14) has perturbation values

$$\tfrac{1}{6}(-g)^{1/2}\nabla_\nu\left[\underset{n}{G}^{\mu\nu}{}_{\alpha\beta}Q^{\alpha\beta}\right]=(-g)^{1/2}\underset{n}{G}^\mu{}_\alpha k^\alpha\ . \tag{15}$$

The left-hand side of (15) contains $\frac{1}{6}(-g)^{1/2}\underset{n}{G}^{\mu\nu}{}_{\alpha\beta}Q^{\alpha\beta}$ as a superpotential for the Taub numbers. It has been shown [14,18] that the Taub numbers are perturbation-gauge invariant and are zero when σ in Eq. (11) is a compact spacelike hypersurface without boundary. The superpotential in Eq. (15) makes the zero result manifest [only heuristically, since the closed three-form $(-g)^{1/2}\underset{n}{G}^\alpha{}_\beta k^\beta dS_\alpha$ is integrated over a three-surface without boundary]. Direct calculation yields

$$(-g)^{1/2}\underset{0}{G}^\alpha{}_\beta k^\beta=(-g)^{1/2}\nabla_\beta k^{[\alpha;\beta]}-(-g)^{1/2}\tfrac{1}{2}k^\alpha\underset{0}{R} \tag{16}$$

and

$$(-g)^{1/2}\underset{1}{G}^\alpha{}_\beta k^\beta=\nabla_\beta U_1^{\alpha\beta}+(-g)^{1/2}\tfrac{1}{2}k^\beta\left[h^{\alpha\nu}\underset{0}{R}_{\nu\beta}-h\underset{0}{R}^\alpha{}_\beta\right]\ . \tag{17}$$

Thus, the zeroth- and first-order numbers have simpler potentials in vacuum:

$$t_0^\alpha=-\tfrac{1}{2}U_0^{\alpha\beta}{}_{;\beta}\ ,\quad t_1^\alpha=U_1^{\alpha\beta}{}_{;\beta}\ , \tag{18}$$

where $U_0^{\alpha\beta}$ is the Komar superpotential and

$$U_1^{\alpha\beta}=(-g)^{1/2}(k^{[\alpha}h^{\beta]}{}_\mu{}^{;\mu}-k^{[\alpha}h^{;\beta]}+\tfrac{1}{2}hk^{[\alpha;\beta]}$$
$$+k^\mu h^{[\alpha}{}_\mu{}^{;\beta]}+k^{\mu;[\alpha}h^{\beta]}{}_\mu)\ . \tag{19}$$

It is clear from Eqs. (4)–(7) why only τ_0 and τ_1 have simple forms for their superpotentials. Suppose one writes $U_2^{\alpha\beta}$ with a finite number of terms, quadratic in h, of the form given in Eq. (19) for $U_1^{\alpha\beta}$. Then Eq. (6) implies $U_2^{\alpha\beta}{}_{;\beta}$ would have to yield

$$\underset{2}{G}^\alpha{}_\beta(h,h)k^\beta=-\underset{1}{G}^\alpha{}_\beta(\ddot{g})k^\beta+f(h)\underset{1}{G}^\alpha{}_\beta(h)k^\beta$$

after vacuum is imposed for the background. h satisfies $\underset{1}{G}^\alpha{}_\beta(h)=0$, but $\ddot{g}$ is known only in principle. Presumably one could obtain it by inverting (6) and expressing $\ddot{g}$ as a Green's function with source $\underset{2}{G}_{\alpha\beta}(h,h)$. Thus, a simple form of the $U_1^{\alpha\beta}$ kind, for $U_n^{\alpha\beta}$ $n>1$, is not possible. The superpotential in Eq. (15) can be useful in calculation if a general solution for the Killing potential $Q^{\alpha\beta}$, independent of symmetry gauge freedom, can be found.

IV. FIRST RESULTS

(A) We evaluate the zeroth Taub number for the vacuum Schwarzschild metric,

$$ds^2=(1-2m/r)du^2+2dudr-r^2d\Omega^2 ,$$

given in a standard outgoing null coordinate system. It is assumed that a source is contained in a sphere of radius $>2m$ interior to the vacuum region. The timelike Killing vector is $k_u{}^\alpha\partial_\alpha=\partial_u$. Integrate $\partial_\alpha t_0^\alpha=0$ over a four-dimensional region bounded by two three-surfaces which meet at future null infinity $\mathscr{I}^+$. The first surface is a $u=$const surface in the vacuum region which smoothly becomes spacelike in the interior. The second surface, lying to the future of the first, is spacelike in the interior and exterior, becoming asymptotically null as it joins the first surface in the same cut of $\mathscr{I}^+$. Gauss's theorem casts the integration onto the bounding three-surfaces. The τ_0 superpotential $U_0^{\alpha\beta}$ (the Komar superpotential) gives rise to an integral [19] over an S^2 cross section of $\mathscr{I}^+$ with result $\tau_0(k_u)=8\pi m$.

(B) The zeroth Taub number for the vacuum Kerr solution, given in Eq. (B3), yields the angular momentum result $\tau_0(k_\varphi)=-16\pi ma$.

(C) The first-order Taub number is evaluated for Schwarzschild viewed as a perturbation of Minkowski space-time with $h_{\mu\nu}=-(2m/r)l_\mu l_\nu$, where $l_\mu dx^\mu=du$. The superpotential $U_1^{\alpha\beta}$ yields $\tau_1(k_u)=-16\pi m$.

V. NULL INFINITY

Since this work studies Taub numbers at $\mathscr{I}^+$, will all perturbation solutions $h_{\mu\nu}$ of the linearized field equations fall off fast enough to maintain $\mathscr{I}^+$? Here we state the Geroch-Xanthopoulos [7] (GX) result wherein $\mathscr{I}^+$ is preserved. Before presenting the GX theorem, asymptotic simplicity is defined.

A space-time $(M,g_{\mu\nu})$ is asymptotically simple [20] if there exists a space $(\tilde{M},\tilde{g}_{\mu\nu})$, $\tilde{M}=M\cup\mathscr{I}$, where M is embedded in $\tilde{M}$ with boundary $\mathscr{I}$ such that (1) there is a smooth scalar field Ω on $\tilde{M}$ with conformal map $\tilde{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$, (2) on boundary $\mathscr{I}$, $\Omega=0$, $\tilde{n}_\mu=\tilde{\nabla}_\mu\Omega\neq 0$, and $\tilde{n}_\mu$ is null, and (3) every maximally extended null geodesic in M has two end points on $\mathscr{I}$. The boundary $\mathscr{I}$ consists of two disjoint parts $\mathscr{I}^+$ (future null infinity) and $\mathscr{I}^-$ (past null infinity).

GX theorem. In an asymptotically simple space-time, when $g_{\mu\nu}$ solves $\underset{0}{G}_{\mu\nu}=0$ and $h_{\mu\nu}$ solves $\underset{1}{G}_{\mu\nu}=0$, asymptotic simplicity is preserved and $h_{\mu\nu}$ satisfies three conditions on $\mathscr{I}$ (here we only need $\mathscr{I}^+$). For $\tilde{h}_{\mu\nu}=\Omega^2 h_{\mu\nu}$ smoothly extending from M to $\tilde{M}$, on $\mathscr{I}^+$: (1) $\tilde{h}_{\mu\nu}|_{\mathscr{I}^+}=0$; (2) $\Omega^{-1}\tilde{h}_{\mu\nu}\tilde{n}^\nu|_{\mathscr{I}^+}=0$; (3) $\Omega^{-2}\tilde{h}_{\mu\nu}\tilde{n}^\mu\tilde{n}^\nu|_{\mathscr{I}^+}=0$.

VI. RADIATIVE RESULTS

The nonstationary axial perturbations of Schwarzschild constructed by Vishveshwara [8] and given in Appendix C will be used to evaluate τ_2 at $\mathscr{I}^+$. We start with

$$\tau_2(k_u)=\int_{\mathscr{N}}\underset{2}{T}(r,\theta)dr\,d\theta\,d\varphi ,\qquad \underset{2}{T}(r,\theta)=(-g)^{1/2}\underset{2}{G}{}^\alpha{}_\beta k^\beta l_\alpha , \tag{20}$$

where $k^\beta=n^\beta+\frac{1}{2}(1-2m/r)l^\beta=\delta_u{}^\beta$. The four-dimensional region containing the Schwarzschild source is bounded by two three-surfaces which meet at $\mathscr{I}^+$. As in the calculation of $\tau_0(k_u)$ above, the first surface is a $u=$const surface $\mathscr{N}$ (with $du=l_\alpha dx^\alpha$) in the vacuum region which smoothly becomes spacelike in the interior. The second surface, lying to the future of the first, is spacelike in the interior and exterior, becoming asymptotically null as it joins the first surface in the same cut of $\mathscr{I}^+$. It is a lengthy exercise to evaluate $\underset{2}{T}(r,\theta)$ since there is no simple superpotential for τ_2. From the relatively simple asymptotic form of $h_{\mu\nu}$,

$$n_{(\mu}y_{\nu)}2\sqrt{2}e^{-ik(u+2r)}\frac{\partial}{\partial\theta}P_l(\cos\theta) ,$$

it is clear that τ_2 at $\mathscr{I}^+$ depends on the spin coefficients of the Schwarzschild tetrad and their derivatives, and coefficients which arise from products of associated Legendre functions such as $P^m{}_l P^m{}_{l'}$. It emerges that $\underset{2}{T}(r,\theta)$ has three kinds of terms: (1) terms of $O(r^{-2})$ and higher which $\to 0$ as $r\to\infty$; (2) terms which integrate to zero on the sphere at any r; (3) terms of the form

$$\frac{\partial f}{\partial r}+\frac{2}{r}f=(fl^\alpha)_{;\alpha}=(2fl^{[\alpha}n^{\beta]})_{;\alpha}l_\beta .$$

It is terms of the third kind which yield τ_2 since they result in [19] $\oint_{\partial\mathscr{N}}(-g)^{1/2}f\,d\theta\,d\varphi$ at $\mathscr{I}^+$. In order to check this calculation and verify the interpretation of $\tau_2(k_u)$ we compute

$$\oint_{\partial\mathscr{N}}\lim_{r\to\infty}\frac{r^2}{4\pi k^2}|\underset{1}{\psi}_4|^2 d\Omega , \tag{21}$$

where

$$\underset{1}{\psi}_4=\frac{1}{r}e^{-ik(u+2r)}{}_{-2}Y_{l,0}a^l(\theta)+O\left[\frac{1}{r^2}\right] .$$

For the perturbation given in (C2) with $l>1$ and associated Legendre functions P_l^m, Eq. (21) has the value

$$\frac{1}{4\pi}\oint P_l^2(\cos\theta)P_{l'}^2(\cos\theta)d\Omega . \tag{22}$$

The results of Eqs. (20) and (21) agree and, since (21) is the Bondi mass loss rate [22] at $\mathscr{I}^+$ due to $h_{\mu\nu}$, Habisohn's interpretation of $\tau_2(k_u)$ is verified. Note that the GX theorem, in which the unphysical $\tilde{h}_{\mu\nu}$ peels on $\mathscr{I}^+$, implies that the perturbed Newman-Penrose Weyl tensor components $\underset{1}{\psi}_n$ $(n=0,\ldots,4)$ peel in the physical space-time.

VII. FURTHER RESULTS

From Eqs. (16) and (17) it is clear that τ_1 is a perturbation of the Komar constants. It follows that τ_1 represents the additional energy and angular momentum contributed by the perturbation. To validate our interpretation of τ_1, we evaluate $\tau_1(k_u)$ and $\tau_1(k_\varphi)$ for a Kerr-type stationary perturbation of Schwarzschild. Use of $h_{\mu\nu}$ given in (C5) and superpotential (19) results in $\tau_1(k_u)=0$ and $\tau_1(k_\varphi)=-16\pi ma$. Since $\tau_0(k_\varphi)$ is zero for Schwarzschild, the perturbation adds precisely the "Kerr" amount of angular momentum.

VIII. CONCLUSION

We have used a well known time-dependent perturbation $h_{\mu\nu}$ of Schwarzschild to evaluate $\tau_2(k_u)$ and verify its value as the Bondi mass loss due to $h_{\mu\nu}$. This was first proved by Habisohn working directly on the conformal boundary $\mathcal{I}^+$. Here we work in the physical space-time. Further understanding of the purely gravitational nature of τ_2 comes from a time-dependent perturbation of Schwarzschild of the form $\hat{g}_{\mu\nu}=g^{\text{Sch}}_{\mu\nu}+2[M(u)/r]l_\mu l_\nu$. We find $\underset{1}{G}_{\mu\nu}=-(dM/du)r^{-2}l_\mu l_\nu$ and, though this is a nonvacuum result, it is true that $\nabla^\nu \underset{1}{G}{}^\mu{}_\nu=0$ [only true for $f(u)r^{-n}$ when $n=2$]. $\tau_1(k_u)$ is defined and has the value $\tau_1=-16\pi M$. The usual interpretation of $\underset{1}{G}_{\mu\nu}$ is of energy loss to $\mathcal{I}^+$ by means of "geometrical optics" photons. Here we find $\underset{2}{G}_{\mu\nu}=-8M^2r^{-4}l_\mu l_\nu$, where $\nabla^\nu \underset{2}{G}{}^\mu{}_\nu\neq 0$. Thus, τ_2 is undefined and, while one can interpret this kind of perturbation as resulting in energy loss to $\mathcal{I}^+$, the mechanism is nongravitational.

$\tau_1(k_u)$ clearly provides a measure of the energy contribution of the perturbation and $\tau_1(k_\varphi)$ a similar measure of the angular momentum contribution. τ_1 can be used to provide an estimate of the energy and angular momentum contributions of any candidate $h_{\mu\nu}$ which satisfies the GX conditions (before solving $\underset{1}{G}_{\mu\nu}=0$). A candidate $h_{\mu\nu}$ cannot cause gravitational radiation if $\underset{2}{G}{}^\mu{}_\nu$ is not divergence free.

A forthcoming work will provide a simpler method of computing τ_2 and will examine other interesting perturbations. The formulation of Taub numbers at $\mathcal{I}^+$ will be extended to the open Friedmann cosmology. (This background has dust rather than vacuum and is not asymptotically simple, but has a well defined $\mathcal{I}^+$.)

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APPENDIX A: PERTURBATION EQUATIONS

For derivative operators $\hat{\nabla}$ and ∇ obeying $\hat{\nabla}_\alpha \hat{g}_{\mu\nu}(\lambda)=\nabla_\alpha g_{\mu\nu}=0$, the connection tensor field $Q^\alpha{}_{\mu\nu}(\lambda)$ is given by $(\nabla_\alpha-\hat{\nabla}_\alpha)A_\beta=Q^\nu{}_{\alpha\beta}(\lambda)A_\nu$ with expansion

$$Q^\nu{}_{\alpha\beta}(\lambda)=\lambda\dot{Q}^\nu{}_{\alpha\beta}+\tfrac{1}{2}\lambda^2\ddot{Q}^\nu{}_{\alpha\beta}+\cdots\ , \tag{A1}$$

where $\dot{Q}^\nu{}_{\alpha\beta}=\frac{1}{2}(h^\nu{}_{\alpha;\beta}+h^\nu{}_{\beta;\alpha}-h_{\alpha\beta}{}^{;\nu})$. The Riemann tensor for $\hat{g}$ is given by

$$R^\mu{}_{\nu\alpha\beta}(\hat{g})=\underset{0}{R}{}^\mu{}_{\nu\alpha\beta}(g)-2Q^\mu{}_{\nu[\alpha;\beta]} -2Q^\sigma{}_{\nu[\alpha}Q^\mu{}_{\beta]\sigma}\ . \tag{A2}$$

The functional derivatives of the connection tensor are given recursively by

$$\underset{n}{Q}{}^\mu{}_{\alpha\beta}=-nh^\mu{}_\nu\,\underset{n-1}{Q}{}^\nu{}_{\alpha\beta},\quad n=2,3,\ldots\ , \tag{A3}$$

where $\underset{1}{Q}{}^\nu{}_{\alpha\beta}=\dot{Q}^\nu{}_{\alpha\beta}$, and functional derivatives of the Ricci tensor can be obtained from the recursion relation [21]

$$\underset{n}{R}_{\mu\nu}=-nh^\alpha{}_\beta\,\underset{n-1}{R}{}^\beta{}_{\mu\nu\alpha}+2n\,\underset{n-1}{Q}{}^\sigma{}_{\mu[\alpha}\underset{1}{Q}{}^\alpha{}_{\nu]\sigma} -2n\,\underset{n-1}{Q}{}^\sigma{}_{\mu[\alpha}\nabla_{\nu]}h^\alpha{}_\sigma\ ,\quad n=2,3,\ldots\ , \tag{A4}$$

where $\underset{1}{R}{}^\beta{}_{\mu\nu\alpha}=-2\underset{1}{Q}{}^\beta{}_{\mu[\nu;\alpha]}$. The functional derivatives of the Ricci and Einstein tensors are

$$\underset{1}{G}_{\mu\nu}=\underset{1}{R}_{\mu\nu}-\tfrac{1}{2}h_{\mu\nu}(g^{\alpha\beta}\underset{0}{R}_{\alpha\beta})-\tfrac{1}{2}g_{\mu\nu}(h^{\alpha\beta}\underset{0}{R}_{\alpha\beta}) -\tfrac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\underset{1}{R}_{\alpha\beta})\ , \tag{A5}$$

$$\underset{2}{G}_{\mu\nu}=\underset{2}{R}_{\mu\nu}-h_{\mu\nu}(h^{\alpha\beta}\underset{0}{R}_{\alpha\beta})-h_{\mu\nu}(g^{\alpha\beta}\underset{1}{R}_{\alpha\beta}) -g_{\mu\nu}(h^{\alpha\beta}\underset{1}{R}_{\alpha\beta})-\tfrac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\underset{2}{R}_{\alpha\beta})\ , \tag{A6}$$

$$\underset{1}{R}_{\mu\nu}=\tfrac{1}{2}(\Box h_{\mu\nu}+h_{;\mu\nu}-h_{\mu\alpha;\nu}{}^\alpha-h_{\nu\alpha;\mu}{}^\alpha)\ , \tag{A7}$$

$$\begin{aligned}\underset{2}{R}_{\mu\nu}=&\,h^{\alpha\beta}(h_{\alpha\mu;\nu\beta}+h_{\alpha\nu;\mu\beta}-h_{\alpha\beta;\mu\nu}-h_{\mu\nu;\alpha\beta})\\ &+h_\nu{}^{\alpha;\beta}(h_{\mu\beta;\alpha}-h_{\mu\alpha;\beta})\\ &+(h^{\alpha\beta}{}_{;\alpha}-\tfrac{1}{2}h^{;\beta})(h_{\beta\mu;\nu}+h_{\beta\mu;\nu}-h_{\mu\nu;\beta})\\ &-\tfrac{1}{2}h_{\alpha\beta;\mu}h^{\alpha\beta}{}_{;\nu}\ .\end{aligned} \tag{A8}$$

When $\underset{0}{R}_{\mu\nu}=0$ one can obtain the compact expression

$$\underset{1}{G}_{\mu\nu}=\tfrac{1}{2}\nabla_\alpha\nabla_\beta[\gamma_{\mu\nu}g^{\alpha\beta}+\gamma^{\alpha\beta}g_{\mu\nu}-2\gamma^\alpha{}_{(\mu}\delta^\beta{}_{\nu)}]\ , \tag{A9}$$

where $\gamma_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}hg_{\mu\nu}$. For $\underset{2}{G}_{\mu\nu}$, with $\underset{0}{R}_{\mu\nu}=\underset{1}{R}_{\mu\nu}=0$, there is the longer expression

$$\begin{aligned}\underset{2}{G}_{\mu\nu}=&\,h^{\alpha\beta}[2h_{\alpha(\mu;\nu)\beta}-h_{\alpha\beta;\mu\nu}-h_{\mu\nu;\alpha\beta}]\\ &+\underset{1}{Q}{}^\beta[2h_{\beta(\mu;\nu)}-h_{\mu\nu;\beta}]-2h_{\mu\alpha;\beta}h_\nu{}^{[\alpha;\beta]}\\ &-\tfrac{1}{2}h_{\alpha\beta;\mu}h^{\alpha\beta}{}_{;\nu}-g_{\mu\nu}\left[\underset{1}{Q}{}^\beta\underset{1}{Q}_\beta-\tfrac{3}{4}h^{\alpha\beta;\sigma}h_{\alpha\beta;\sigma}\right.\\ &\left.+\tfrac{1}{2}h^{\sigma\alpha;\beta}h_{\sigma\beta;\alpha}\right]\ ,\end{aligned} \tag{A10}$$

where $\underset{1}{Q}{}^\beta=\underset{1}{Q}{}^\beta{}_{\mu\nu}g^{\mu\nu}=h^{\alpha\beta}{}_{;\alpha}-\frac{1}{2}h^{;\beta}$.

APPENDIX B: THE KERR AND SCHWARZSCHILD SOLUTIONS

(1) The Kerr metric in Boyer-Lindquist coordinates $(\tilde{t},r,\theta,\tilde{\varphi})$ is given by

$$ds^2=\Psi d\tilde{t}^2+(1-\Psi)2a\sin^2\theta d\tilde{t}\,d\tilde{\varphi}-(\Sigma/\Delta)dr^2-\Sigma d\theta^2 -\sin^2\theta[\Sigma+(2-\Psi)a^2\sin^2\theta]d\tilde{\varphi}^2\ , \quad \text{(B1)}$$

where $\Delta=r^2-2mr+a^2$, $\Sigma=r^2+a^2\cos^2\theta$, $\Psi=1-2mr/\Sigma$. Perturbations of the Kerr metric are discussed by Teukolsky [22] and its stability is treated by Press and Teukolsky [23]. Perturbations viewed at $\mathscr{I}^+$ are described more conveniently in Kerr-Newman coordinates (u,r,θ,φ) where the $u=$const spacelike surfaces are asymptotically null:

$$du=d\tilde{t}-(r^2+a^2)\Delta^{-1}dr\ ,\quad d\varphi=d\tilde{\varphi}-a\Delta^{-1}dr \quad \text{(B2)}$$

relates Boyer-Lindquist and outgoing Kerr-Newman coordinates;

$$ds^2=\Psi du^2+2du\,dr+(1-\Psi)2a\sin^2\theta du\,d\varphi -2a\sin^2\theta dr\,d\varphi-\Sigma d\theta^2 -\sin^2\theta[\Sigma+(2-\Psi)a^2\sin^2\theta]d\varphi^2\ . \quad \text{(B3)}$$

The two principal null vectors are

$$\partial_r$$

and (B4)

$$(r^2+a^2)\partial_u-\tfrac{1}{2}\Delta\partial_r+a\partial_\varphi\ .$$

A Newman-Penrose tetrad $(l^\alpha,n^\alpha,m^\alpha,\bar{m}^\alpha)$ for metric (B3) is given by

$$l^\alpha=\delta^\alpha{}_r\ ,\quad n^\alpha=\delta^\alpha{}_u-\tfrac{1}{2}\Psi\delta^\alpha{}_r\ , \quad \text{(B5)}$$

$$m^\alpha=\frac{ia\sin\theta}{\sqrt{2}\bar{R}}(\delta^\alpha{}_u-\delta^\alpha{}_r)+\frac{1}{\sqrt{2}\bar{R}}\left[\delta^\alpha{}_\vartheta+\frac{i}{\sin\theta}\delta^\alpha{}_\varphi\right]\ ,$$

where $R=r-ia\cos\theta$. The coordinate r is an affine parameter along the outgoing null geodesic l^α. The nonzero spin coefficients for tetrad (B5) are

$$\rho=-1/R\ ,\quad \alpha=-\cot\theta/(2\sqrt{2}R)\ ,\quad \beta=\cot\theta/(2\sqrt{2}\bar{R})\ ,\quad \mu=(-1+2mr/R^2)/(2\bar{R})\ ,\quad \gamma=m/(2R^2)\ ,\quad \nu=ima\sin\theta/(\sqrt{2}\Sigma\bar{R})\ . \quad \text{(B6)}$$

The only nonzero Weyl tensor component is $\psi_2=-m/R^3$. To view the Kerr solution as a perturbation of flat space we give the metric in Kerr-Schild coordinates (t,x,y,z):

$$ds^2=dt^2-dx^2-dy^2-dz^2 -\frac{2mr_a^3}{r_a^4+a^2z^2}\left[\frac{r_a(xdx+ydy)+a(ydx-xdy)}{r_a^2+a^2}+\frac{zdz}{r_a}+dt\right]^2\ , \quad \text{(B7)}$$

where $r_a^4-(x^2+y^2+z^2-a^2)r_a^2-a^2z^2=0$ determines r_a in terms of x,y,z up to a sign,

$$g_{\mu\nu}=\eta_{\mu\nu}-2mN_\mu N_\nu\ , \quad \text{(B8)}$$

where $g^{\mu\nu}N_\mu N_\nu=\eta^{\mu\nu}N_\mu N_\nu=0$.

In the Kerr-Schild coordinate frame the null vector N_μ has components

$$N_\mu=(N_0,N_1,N_2,N_3)\ ,$$

$$N_0^2=r_a^3/(r_a^4+a^2z^2)\ ,\quad N_1=N_0(r_ax+ay)/(r_a^2+a^2)\ , \quad \text{(B9)}$$

$$N_2=N_0(r_ay-ax)/(r_a^2+a^2)\ ,\quad N_3=N_0z/r_a\ .$$

(2) The Schwarzschild metric in outgoing null coordinates (u,r,θ,φ) is given by

$$ds^2=(1-2m/r)du^2+2du\,dr-r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\ . \quad \text{(B10)}$$

A Newman-Penrose tetrad for (B10) is

$$l^\alpha=\delta^\alpha{}_r\ ,\quad l_\alpha=\delta_\alpha{}^u\ ,\quad n^\alpha=\delta^\alpha{}_u-\tfrac{1}{2}(1-2m/r)\delta^\alpha{}_r\ ,\quad n_\alpha=\delta_\alpha{}^r+\tfrac{1}{2}(1-2m/r)\delta_\alpha{}^u\ , \quad \text{(B11)}$$

$$m^\alpha=\frac{1}{\sqrt{2}r}\left[\delta^\alpha{}_\vartheta+\frac{i}{\sin\theta}\delta^\alpha{}_\varphi\right]\ ,\quad m_\alpha=-\frac{r}{\sqrt{2}}(\delta_\alpha{}^\vartheta+i\sin\theta\delta_\alpha{}^\varphi)\ ,$$

with nonzero spin coefficients

$$\rho=-1/r\ ,\quad \alpha=-\cot\theta/(2\sqrt{2}r)\ ,\quad \beta=\cot\theta/(2\sqrt{2}r)\ ,\quad \mu=-(1-2m/r)/(2r)\ ,\quad \gamma=m/(2r^2)\ . \quad \text{(B12)}$$

APPENDIX C: SCHWARZSCHILD PERTURBATIONS

We follow Vishveshwara's [8] treatment of axial (odd-parity) Schwarzschild perturbations:

$$\hat{g}_{\mu\nu}=g^{\text{Sch}}_{\mu\nu}+\lambda h_{\mu\nu},$$

where, using the coordinates and tetrad of Eqs. (B10) and (B11),

$$h_{\mu\nu}=(l_\mu y_\nu+l_\nu y_\mu)[H_0-H_1(1-2m/r)]/(2\sqrt{2}r\sin\theta)+(n_\mu y_\nu+n_\nu y_\mu)[H_0(1-2m/r)^{-1}+H_1]/(\sqrt{2}r\sin\theta)\ . \quad \text{(C1)}$$

Here $y_\mu = i(m_\mu - \overline{m}_\mu)$ is a real spacelike vector with $d\varphi = y_\mu dx^\mu/(\sqrt{2}r \sin\theta)$. The perturbation functions H_0 and H_1 are given by

$$H_a = h_a(r)e^{-ik(u+r^*)}\sin\theta\frac{\partial}{\partial\theta}P_l(\cos\theta)\ ,\quad a=0,1\ , \tag{C2}$$

where stability requires the frequency k to be real, and P_l is the Legendre polynomial with angular momentum l:

$$\begin{aligned} &h_0(r) = (i/k)d(rQ)/dr^*\ ,\\ &h_1(r) = (1-2m/r)^{-1}(rQ)\ ,\\ &r^* = r + 2m\ \ln(r/2m - 1)\ . \end{aligned} \tag{C3}$$

For $l > 1$, Q satisfies the Sturm-Liouville-type equation

$$d^2Q/dr^{*2} + (k^2 - V_{\text{eff}})Q = 0\ ,$$

$$V_{\text{eff}} = (1-2m/r)[l(l+1)r^{-2} - 6mr^{-3}]\ .$$

As $r \to \infty$, and with $l > 1$, both H_0 and H_1 approach

$$re^{-ik(u+2r)}\sin\theta\frac{\partial}{\partial\theta}P_l(\cos\theta)\ . \tag{C4}$$

A Kerr-type stationary Schwarzschild perturbation can be obtained from Vishveshwara's perturbation (C1) with $k=0$ and $l=1$, where $h_0 = c_0/r$ and $h_1 = 0$. We choose $c_0 = 2ma$ to obtain the Kerr angular momentum. The Schwarzschild metric (B10) is perturbed by [24]

$$\begin{aligned} h_{\mu\nu} = &\sqrt{2}mar^{-2}\sin\theta l_{(\mu}y_{\nu)}\\ &+ 2\sqrt{2}mar^{-2}(1-2m/r)^{-1}\sin\theta n_{(\mu}y_{\nu)}\ , \end{aligned} \tag{C5}$$

where y_ν is given in (C1).

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