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Renormalization of overlapping transverse divergences in a model light-front Hamiltonian

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We construct a simplified model of a light-front Hamiltonian and describe its renormalization. Our construction starts from a Hamiltonian which acts in a space spanned by free states of two fermions and states of two fermions and one scalar boson. For the purpose of this paper the starting Hamiltonian is regularized by chopping factors in the interaction vertices. We derive an effective Hamiltonian acting in the space of two fermions. Then we make *ad hoc* simplifications in the effective Hamiltonian to produce a model that we can analyze. We drop fermion self-interactions and replace the eigenvalue in the effective one-boson-exchange term by a constant. Our model Hamiltonian acts in the space of two fermions only. The model involves logarithmic ultraviolet transverse divergences, analogous to overlapping divergences in perturbative Lagrangian S-matrix calculations. We describe the construction of a Hamiltonian counterterm that removes the divergences to all orders in Hamiltonian perturbation theory. The counterterm is local in the transverse direction and contains an arbitrary function of the longitudinal momenta of fermions. We suggest that a suitable choice of the arbitrary function may partly remove the violation of rotational invariance in the spectrum of the model Hamiltonian. The renormalization group transformation for the renormalized two-fermion interaction, V_λ, is determined by a nonlinear integro-differential equation of the form $dV_\lambda/d\lambda = -V_\lambda K_\lambda V_\lambda$, where K_λ is a known kernel and λ is the scale of the relative transverse momenta of fermions at which the interactions are chopped off.

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I. INTRODUCTION

Construction of a renormalized light-front Hamiltonian for quantum chromodynamics (QCD) provides an unexplored alternative to existing programs for the description of hadronic structure and interactions. If we knew such a Hamiltonian at hadronic scales, we could use known methods of Hamiltonian quantum mechanics to find solutions for the hadronic spectrum and scattering amplitudes. However, the light-front QCD Hamiltonian poses complicated renormalization problems. In order to understand the required renormalization procedure, one may consider simple model light-front Hamiltonians which involve typical divergences. In this paper we build such an elementary model and study elements of its renormalization.

Renormalization of light-front Hamiltonians is interesting because of the exceptional properties of the front form of dynamics [1]. Light-front dynamics greatly differs from the commonly used equal-time dynamics. For example, in the equal-time Hamiltonian approach to QCD there is a problem of finding the ground state of the theory. Construction of an effective Hamiltonian for the hadronic excitations of the ground state can hardly be based on first principles without knowing the ground state. The new feature of the light-front Hamiltonian approach to QCD is that the ground-state problem can be transcribed into the renormalization problem for the light-front QCD Hamiltonian [2]. The reason for that is as follows. The canonical light-front QCD Hamiltonian is singular. One has to introduce cutoffs in order to make it finite. Introducing a lower bound on positive longitudinal momenta of hadronic constituents implies that a translationally invariant light-front Hamiltonian cannot produce constituents from the bare vacuum. The longitudinal cutoffs also limit the number of constituents allowed to appear in the Hamiltonian eigenstates, despite the fact that the constituents are allowed to be created and destroyed in the interactions. Thus, introducing cutoffs in order to regularize a light-front Hamiltonian, one cuts off the vacuum problem. The price for cutting off the vacuum is that light-front Hamiltonians depend on infrared cutoffs in a singular way and produce divergences due to these cutoffs in practical calculations, in addition to divergences due to ultraviolet cutoffs in the transverse directions. One may hope that finding suitable counterterms to these divergences by using renormalization group ideas may be, at least partly, equivalent to solving the vacuum problem. The advantage of the renormalization group idea for light-front Hamiltonians is that one works with relatively simple Hamiltonians. One gradually corrects their structure when cutoffs are being removed, and comparison with experimental data is made. Therefore the study of the singularities and defining the renormalization group transformation for light-front Hamiltonians is important. It offers a possibility of investigating the nonperturbative nature of strong interactions without directly relying on guesses about properties of the vacuum state [3].

One of the authors proposed a set of perturbative power-counting rules to classify operators which may appear in a light-front QCD Hamiltonian [2]. The power

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counting determines what counterterms need to be present in the Hamiltonian to all orders in perturbation theory. Transverse and longitudinal dimensions count differently. We adopt the notation $x^+=x^0+x^3$ for the light-front time variable and $x^-=x^0-x^3$ for the light-front longitudinal space variable. The transverse coordinates x^1 and x^2 are denoted by $x^\perp$. A contraction of two four-vectors p and x is written as $px=\frac{1}{2}p^-x^+ +\frac{1}{2}p^+x^- -p^\perp x^\perp$. The light-front fermion fields have dimension $(x^\perp\sqrt{x^-})^{-1}$, while the light-front Hamiltonian density has dimension $x^{\perp -4}$, since the light-front Hamiltonian has dimension $x^-/x^{\perp 2}$ and the density is integrated over the front $x^+=0$ to obtain the Hamiltonian. Divergences in perturbation theory arise in both longitudinal and transverse dimensions. Of particular importance to this paper, power counting tells us that there can be a light-front Hamiltonian counterterm which removes transverse ultraviolet divergences and which involves four-fermion fields. This term must be local in the transverse dimension, does not involve either derivatives in the transverse direction or mass parameters, and must involve a function of longitudinal fermion-field coordinates which has dimension $(x^-)^2$. In this article we discuss a simple model Hamiltonian which exhibits this particular feature. We do not expand on nonperturbative issues arising beyond power-counting analysis to all orders in perturbation theory. We stress that the ultraviolet transverse-momentum cutoffs are not responsible for cutting off the vacuum problem. However, understanding the renormalization procedure in the model is a prerequisite to tackling the renormalization problem in QCD where severe longitudinal singularities begin to play an essential role.

In order to consider models that resemble what we expect the renormalized light-front QCD Hamiltonian to be at hadronic scales, we choose to keep close to the phenomenological picture of hadrons developed in the constituent quark model. We also wish to include relativistic effects of creation and destruction of particles by interaction terms in the Hamiltonian. Creation and destruction of particles are missing in the constituent quark model, which is nonrelativistic in its nature. Creation and destruction of particles by a relativistic equal-time Hamiltonian leads to the unsolved vacuum problems and spoils the constituent quark model. The general postulate is that the intrinsically relativistic light-front Hamiltonians may avoid such discrepancies in renormalization theory. We need to study simple models to learn how we can verify this postulate.

For example, when approaching the physics of mesons, one may start with a Hamiltonian which acts in the space of two sectors: the constituent quark-antiquark pair sector with a constituent quark-antiquark pair and a constituent gluon. Still, working with the constituent picture on the light-front, we need to introduce Hamiltonian counterterms which remove the cutoff dependence. Unfortunately, even the simplest light-front Hamiltonians which describe interactions of constituent quarks with constituent gluons depend on longitudinal- and transverse-momentum cutoffs in a way that is not easy to understand. We need to consider a model where singularities are simple enough that we can understand them. We consider the interaction of fermions with scalar bosons, instead of vector bosons. Vector bosons introduce a complicated longitudinal cutoff dependence which does not appear in the case of scalar bosons.

In order to build our model Hamiltonian, we start from a Hamiltonian which describes dynamics in two Fock sectors, one containing two fermions and another two fermions and a scalar boson. The number of constituents is restricted. The momenta of the constituents are limited by cutoffs. We derive an effective Hamiltonian in the space of two fermions, make *ad hoc* simplifications and consider the resulting model Hamiltonian for two fermions. We study divergences in our model that appear when the cutoffs become extreme. This work follows an earlier study of bound-state equations in a Yukawa model [4]. The earlier study identified kernels in the integral equations which removed divergences of fourth order in a coupling constant. Here we study overlapping transverse divergences and construct counterterms which remove the transverse cutoff dependence from the model Hamiltonian spectrum in a renormalization procedure to all orders of the coupling constant.

In parallel with this paper, Van de Sande and Pinsky [5] have proposed a novel approach to renormalizing model Hamiltonians such as ours. They also arrive at an equation which is identical to a key equation in our paper except that our equation defines an effective Hamiltonian, whereas the "potential" in the equation of Van de Sande and Pinsky has only a nonlinear relation to their effective Hamiltonian. Because of the technical limitations that will be explained later, the paper of Van de Sande and Pinsky does not address the overlapping divergence problem that is the focus of this paper.

Setting up a Hamiltonian in the space of a limited number of light-front constituents makes the Hamiltonian eigenvalue problem look similar to the well-known Tamm-Dancoff approximation to the Hamiltonian eigenvalue problem in quantum field theory. Our ultimate goal is to define the analogous approach to the light-front QCD Hamiltonian and provide a definition of hadronic constituents which is based on first principles. However, this goal is a long way off. The renormalized QCD Hamiltonian which is supposed to act in the space of constituent quarks and gluons is not simply related to a canonical one, and here we are not concerned with this issue. In this paper we merely study a model light-front Hamiltonian that contains a typical divergence. We do not treat our model as a Hamiltonian for some prescribed theory. We should also stress that our analysis is not complete. We discuss only the removal of logarithms of the transverse-momentum cutoff from the model Hamiltonian spectrum. The model is not designed to discuss removal of the longitudinal cutoff dependence which is essential in QCD. Nevertheless, the model illustrates elements of the same renormalization procedure which can be used for QCD.

The Hamiltonian formalism is not explicitly covariant, and cutting off momenta or limiting the numbers of particles violates many symmetries, including Poincaré symmetry, which is not allowed to be broken by formulas to

be compared with experimental data. We have to explain how it may be possible that a renormalized Hamiltonian acting in a limited Hilbert space may lead to relativistic answers for physical quantities. In equal-time dynamics, such Hamiltonians have not been found. In lowest orders of the old fashioned equal-time perturbation theory, relativity is connected with the necessity to produce particles from the vacuum. In cutoff light-front dynamics, the situation is different. Creation of particles from the vacuum is absent. In perturbation theory covariant results can be obtained thanks to the presence of special new terms (seagull terms) in the light-front Hamiltonian.

We expect our Hamiltonians to contain counterterms that lead do relativistic results. The counterterms contain arbitrary functions that must be adjusted so that solutions for physical amplitudes obey the rules of special relativity. In other words, the dictum of renormalization group is not able to fix the finite parts of the counterterms. The finite parts contain whole functions of the constituents' momenta. Those finite parts can be chosen to obtain required symmetries for the physical amplitudes. Thus the marriage of Hamiltonian quantum mechanics and relativity is made by fitting unknown parameters in the Hamiltonian. It is one of the major advantages of light-front renormalized Hamiltonian dynamics that it may lead to relativistic answers through quantum-mechanical calculations in a space of a limited number of constituents with their momenta limited by cutoffs. A special example of such a model Hamiltonian calculation is given by Ref. [6].

So far, we have discussed symmetry violations due to the momentum cutoffs. These violations can be cured, at least at the current level of analysis, by adjusting counterterms in the Hamiltonian. In addition, there are symmetry violations when arbitrary restrictions on the number of constituents are imposed. Such restrictions may be more restrictive than imposing the longitudinal-momentum cutoff itself. The major example is provided by the Tamm-Dancoff procedure of limiting a Hamiltonian domain in quantum field theory by arbitrarily restricting the number of constituents. Symmetry violations due to restrictions on the constituent number cannot be studied in our simple model. In order to clarify this issue, one needs to investigate renormalization-group transformation for Hamiltonians which act in the space with a variable number of constituents. Such considerations go beyond the scope of this paper. Note, however, that once we learn how to remove the momentum cutoff dependence from Hamiltonian calculations, we are free to consider effects of varying the number of constituents and legitimately investigate what kind of finite terms have to be introduced into the Hamiltonian to remove effects of the artificial limiting of the constituent number. Thus renormalization of model light-front Hamiltonians opens a way to make studies which are not possible in the equal-time dynamics where restricting the number of particles so far excludes obtaining relativistic results. For example, violations of rotational symmetry in light-front Hamiltonians seem to be curable by counterterms, while violations of boost invariance in equal-time Hamiltonian dynamics seem to require knowledge about the ground-state properties.

The paper is organized as follows. In Sec. II we construct our model Hamiltonian. Section III describes the transverse divergence. Although it is not necessary for the renormalization procedure, we introduce a partial-wave analysis of the model Hamiltonian in order to make the connection with the closely related work in Ref. [4] more transparent. Section IV describes construction of the counterterms. The structure of the counterterms agrees with the predictions of power counting. Section V concludes the paper. Appendix A contains a sketch of a proof to all orders in the coupling constant that the counterterm has the structure predicted by power counting. Appendix B provides analytic examples of solutions to the renormalization-group equation for renormalized interactions in a few cases which are similar to our model.

II. MODEL HAMILTONIAN

Our initial Hamiltonian acts in a space spanned by Fock states containing two fermions and states containing two fermions and a boson. The Hamiltonian has the structure

$$H = H_0^{ff} + H_0^{ffb} + H_I + V_\Lambda \ , \tag{2.1}$$

where H_0^{ff} and H_0^{ffb} denote free-energy terms. The free-energy terms do not need to be the same in different Fock sectors. Masses squared in the free energies of fermions in the three-body sector may differ from the masses squared in the two-fermion sector [7]. Although this issue will not concern us here, we assume that

$$H_0^{ff} = \sum_{\lambda_1,\lambda_2} \int [dp_1][dp_2] \left[\frac{p_1^{\perp 2} + m^2 + \delta m^2_{\Lambda\epsilon}}{p_1^+} + \frac{p_2^{\perp 2} + m^2 + \delta m^2_{\Lambda\epsilon}}{p_2^+} \right] b^\dagger_{p_1\lambda_1} b^\dagger_{p_2\lambda_2} |0\rangle\langle 0| b_{p_2\lambda_2} b_{p_1\lambda_1} \ , \tag{2.2}$$

and

$$H_0^{ffb} = \sum_{\lambda_1,\lambda_2} \int [dp_1][dp_2][dq] \left[\frac{p_1^{\perp 2} + m^2}{p_1^+} + \frac{p_2^{\perp 2} + m^2}{p_2^+} + \frac{q^{\perp 2} + \mu^2}{q^+} \right] b^\dagger_{p_1\lambda_1} b^\dagger_{p_2\lambda_2} a^\dagger_q |0\rangle\langle 0| a_q b_{p_2\lambda_2} b_{p_1\lambda_1} \ . \tag{2.3}$$

We use the abbreviated notation

$$\int [dp]=\frac{1}{(2\pi)^3}\int \frac{dp^+}{2p^+}d^2p^\perp \ , \tag{2.4}$$

where $p^+=p^0+p^3$ and $p^\perp=(p^1,p^2)$. We formally treat the fermions as if they were different, but do not exhibit the quantum numbers by which they differ. Such details are not important in this paper, and we want to avoid complicated expressions. Proper antisymmetrization of all our formulas in the case of identical fermions is understood. The creation and annihilation operators satisfy commutation or anticommutation relations of the generic form

$$[a_{q_1},a^\dagger_{q_2}]=(2\pi)^3 2q_1^+\delta^3(q_1-q_2) \ , \tag{2.5}$$

where boson operators can be replaced by fermion operators when the commutator is replaced by the anticommutator.

The interaction Hamiltonian H_I is

$$\begin{aligned} H_I=g\sum_{\lambda_{2'},\lambda_2,\lambda_1}\int [dp_2][dp_{2'}][dk][dp_1]2(2\pi)^3\delta^3(p_{2'}+k-p_2) \\ \times\Theta(4\Lambda^2-\kappa^{12}_{fb})\Theta\left[\frac{k^+}{p_2^+}-\varepsilon\right]\Theta\left[\frac{p_{2'}^+}{p_2^+}-\varepsilon\right]\Theta(\Lambda^2-\kappa^{12}_{ff})\Theta\left[\frac{p_2^+}{p_2^++p_1^+}-\varepsilon\right]\Theta\left[\frac{p_1^+}{p_2^++p_1^+}-\varepsilon\right] \\ \times[\bar u_{p_{2'}\lambda_{2'}}u_{p_2\lambda_2}b^\dagger_{p_{2'}\lambda_{2'}}a^\dagger_k b^\dagger_{p_1\lambda_1}|0\rangle\langle 0|b_{p_1\lambda_1}b_{p_2\lambda_2} \\ +\bar u_{p_2\lambda_2}u_{p_{2'}\lambda_{2'}}b^\dagger_{p_2\lambda_2}b^\dagger_{p_1\lambda_1}|0\rangle\langle 0|b_{p_1\lambda_1}a_k b_{p_{2'}\lambda_{2'}}]+[1\leftrightarrow 2] \ . \end{aligned} \tag{2.6}$$

In interactions induced by this term, a fermion of momentum p_2 emits a scalar boson of momentum k to become a fermion of momentum $p_{2'}$ or the boson is absorbed by the fermion. The emission and absorption rates are given by the coupling constant g and a scalar product of the fermion spinors, as in a canonical light-front Hamiltonian for Yukawa theory.

In order to complete the definition of the interaction Hamiltonian, one has to introduce cutoffs on transverse and longitudinal momenta of the fermions and the boson. Since our Hamiltonian induces transitions between only two- and three-body states, we may introduce cutoffs using the momentum of a spectator fermion. The interaction Hamiltonian is written in the form which explicitly indicates the spectator fermion of momentum p_1. Two fermions in the two-fermion state are labeled by momenta p_1 and p_2. Fermions in the three-body states are labeled by momenta p_1 and $p_{2'}$. The relative transverse momentum of the active fermion and the boson, $\kappa^\perp_{fb}$, is defined as

$$\kappa^\perp_{fb}=\frac{p_{2'}^+k^\perp-k^+p^\perp_{2'}}{p_2^+} \ , \tag{2.7}$$

and the relative transverse momentum of the two fermions is

$$\kappa^\perp_{ff}=\frac{p_1^+p_2^\perp-p_2^+p_1^\perp}{p_1^++p_2^+} \ . \tag{2.8}$$

In Eq. (2.6), the subscripts 1 and 2 are understood to indicate also the two kinds of fermions and the symmetrization implies that bosons equally interact with both kinds of fermion.

The fermion spinors are defined as

$$u_{p\lambda}=\frac{1}{(mp^+)^{1/2}}[\Lambda_+p^++\Lambda_-(m+\alpha^\perp p^\perp)]u_\lambda \ , \tag{2.9}$$

where u_λ denotes the spinor for a fermion of mass m at rest, normalized to $\bar u u=2m$. $\Lambda_\pm=\frac{1}{2}(1\pm\alpha^3)$.

The chopping factors in the interaction vertices are introduced here in the form of Θ functions which contain two scales Λ and ε. The first role of the chopping factors is that they chop off interactions which create or annihilate pairs of a fermion and a boson of relative transverse momenta larger than 2Λ. The factor 2 is useful later. Some larger, finite factors in place of 2 are equally possible. The cutoff ε chops out interactions that create or annihilate pairs in which the longitudinal (i.e., "plus") momentum of any particle in a pair is smaller than the ε fraction of the total longitudinal momentum of the pair. The second role of the chopping factors is the following. The interaction Hamiltonian causes transitions between the two-fermion sector and the two-fermion–one-boson sector. The chopping factors ensure that the transitions occur when the relative transverse momentum of fermions in the two-fermion sector is smaller than Λ and each of the fermions carries at least a ε fraction of the sum of their longitudinal momenta. All the chopping factors maintain kinematical symmetries of the front form of dynamics.

The last term in Eq. (2.1) will be necessary for renormalization. A similar term will appear in our simplified model Hamiltonian. V_Λ is introduced in Eq. (2.1) to alert the reader where she may expect counterterms to appear in the initial Hamiltonian of the basis of our simplified analysis. The reader may also find it helpful to consult Refs. [6,8] and particularly the closely related Ref. [4] for more discussion of light-front Hamiltonians for fermions interacting with scalar bosons.

Having set up the initial Hamiltonian, we proceed to the construction of our simplified model.

We consider bound states of two fermions. A bound state with momentum P is defined as

$$|P\rangle=\sum_{\lambda_1\lambda_2}\int[dp_2][dp_1]\Psi^P_{\lambda_1\lambda_2}(p_2,p_1)b^\dagger_{p_2\lambda_2}b^\dagger_{p_1\lambda_1}|0\rangle + \sum_{\lambda_1\lambda_2}\int[dp_2][dp_1][dq]\Psi^P_{\lambda_2\lambda_1}(p_2,p_1,q)b^\dagger_{p_2\lambda_2}b^\dagger_{p_1\lambda_1}a^\dagger_q|0\rangle\ . \tag{2.10}$$

The wave functions are of the form

$$\Psi^P_{\lambda_2\lambda_1}(p_2,p_1)=2(2\pi)^3\delta^3(P-p_2-p_1)(p_2^+p_1^+)^{1/2}\phi_{\lambda_2\lambda_1}(x_2\kappa^\perp) \tag{2.11}$$

and

$$\Psi^P_{\lambda_2\lambda_1}(p_2,p_1)=2(2\pi)^3\delta^3(P-p_2-p_1-q)(p_2^+p_1^+x_0)^{1/2}\phi_{\lambda_2\lambda_1}(x_2x_1\kappa_2^\perp\kappa_1^\perp). \tag{2.12}$$

Such forms are applied by the kinematical symmetries of the front form of dynamics. Factors involving square roots are introduced for later convenience.

The arguments of the wave functions are defined as

$$x_2=\frac{p_2^+}{P^+},\quad x_1=\frac{p_1^+}{P^+},\quad x_0=\frac{q^+}{P^+}\ . \tag{2.13}$$

In the two-fermion sector, we define

$$p_2^\perp=x_2P^\perp+\kappa^\perp\ , \tag{2.14}$$

$$p_1^\perp=x_1P^\perp-\kappa^\perp\ , \tag{2.15}$$

and

$$x_2+x_1=1\ . \tag{2.16}$$

In the two-fermion–one-boson sector, we define

$$p_2^\perp=x_2P^\perp+\kappa_2^\perp\ , \tag{2.17}$$

$$p_1^\perp=x_1P^\perp-\kappa_1^\perp\ , \tag{2.18}$$

$$q^\perp=x_0P^\perp+\kappa_1^\perp-\kappa_2^\perp\ , \tag{2.19}$$

and

$$x_0=1-x_1-x_2\ . \tag{2.20}$$

The bound-state equations are obtained by expressing the initial Hamiltonian eigenvalue equation

$$H|P\rangle=\frac{P^{\perp2}+M^2}{P^+}|P\rangle \tag{2.21}$$

in terms of coupled integral equations for the two wave functions from Eqs. (2.11) and (2.12). The integral equations can be written in an abbreviated fashion as

$$\begin{bmatrix} p_1^-+p_2^-+\dfrac{\delta m^2_{\Lambda\varepsilon}}{p_1^+}+\dfrac{\delta m^2_{\Lambda\varepsilon}}{p_2^+}+\Sigma\int\langle V_\Lambda\rangle & \Sigma\int\langle H_I\rangle \\ \Sigma\int\langle H_I\rangle & p_1^-+p_2^-+q^- \end{bmatrix}\begin{bmatrix}\Psi^P_{\lambda_1\lambda_2}(ff)\\ \Psi^P_{\lambda_2\lambda_1}(ffb)\end{bmatrix}=\frac{P^{\perp2}+M^2}{P^+}\begin{bmatrix}\Psi^P_{\lambda_2\lambda_1}(ff)\\ \Psi^P_{\lambda_2\lambda_1}(ffb)\end{bmatrix}, \tag{2.22}$$

where we have marked symbolically integrations over arguments of the wave functions and sums over spin indices, and indicated operator matrix elements between the corresponding Fock states by angular brackets.

We have eight functions, four of six variables and four of nine variables.

The spectrum of the model Hamiltonian is continuous. The eigenvalues are parametrized by the total momenta P^+ and $P^\perp$ of the eigenstates. The total momentum of the bound state can be eliminated from Eq. (2.22), thanks to our choice of the arguments of the chopping factors. Then one obtains equations for the relative motion wave functions, which appear multiplied by three-dimensional δ functions in Eqs. (2.11) and (2.12). These are functions of only three and six arguments, respectively. The eigenvalue becomes equal to the mass of the eigenstates. The mass make take discrete values for bound states and become continuous for scattering states.

One may observe that the wave function of the three-particle sector can be algebraically expressed in terms of the wave function of the two-particle sector. This is not usually possible when the Hamiltonian contains nontrivial interactions in the three-particle sector. For example, canonical light-front Hamiltonians contain seagull terms which induce nonlocal interactions in the three-body sector. One expects that similar interactions are present in the physically relevant Hamiltonians. Those interactions destroy the possibility of expressing the three-body wave function by the two-body wave function in a direct algebraic way. One also desires to consider confining theories where free constituents do not exist. Nevertheless, here we consider a simple model in which there is no interaction in the sector containing two fermions and one boson. We are motivated to consider such a simplistic model because our aim is to learn about the renormalization problems involved in finding spectra of the simplest possible Hamiltonians which allow the creation and destruction of particles. Including interactions in the three-body sector immediately leads to technical complications, while we need some qualitative understanding of the renormaliza-

tion procedure for light-front Hamiltonians in order to be able to consider more complicated models.

We can express the three-body wave function by the two-body wave function and write an equation for the two-body wave function itself. This equation has the familiar structure

$$\left[\frac{m^2+\kappa^2+\delta m^2_{\Lambda\varepsilon}+\Sigma_2}{x_2 M}+\frac{m^2+\kappa^2+\delta m^2_{\Lambda\varepsilon}+\Sigma_1}{x_1 M}-M\right]\phi + \sum\int [\langle V_{\rm OBE}\rangle+\langle V_\Lambda\rangle]\phi=0 . \quad (2.23)$$

The new features of this equation are the chopping factors hidden in the integration limits and fermion self-energies, and the presence of the kernel $\langle V_\Lambda\rangle$ induced by the Hamiltonian term V_Λ. The wave function ϕ represents a column of four wave functions corresponding to four possible spin configurations of two fermions. $\langle V_{\rm OBE}\rangle$ represents the one-boson-exchange potential, which depends on the eigenvalue M.

At this point we are ready to make two *ad hoc* simplifications which produce the desired simplified model.

First, we completely neglect $\delta m^2_{\Lambda\varepsilon}$, Σ_2, and Σ_1.

Second, the one-boson-exchange term $\langle V_{\rm OBE}\rangle$ contains the eigenvalue M in the energy denominator. We replace the eigenvalue in the denominator by a constant M_0, which is smaller than $2m$. The simplified one-boson-exchange term is denoted by the same symbol $\langle V_{\rm OBE}\rangle$ in further equations.

The resulting bound-state equation can be viewed as an eigenvalue equation for a simplified Hamiltonian which acts in the space of two fermions only. The latter is our simplified model Hamiltonian which we analyze. There is no need for the model Hamiltonian to have any connection to the Hamiltonian we started from. The derivation is used to provide a language to describe our model.

III. TRANSVERSE DIVERGENCE

In order to describe the origin of the transverse divergence, we need to consider details of $\langle V_{\rm OBE}\rangle$. $\langle V_{\rm OBE}\rangle$ has the structure

$$M\langle V_{\rm OBE}\rangle = N(x\kappa^\perp,x'\kappa'^\perp)\left[\frac{\Theta(x-x')C(x\kappa^\perp,x'\kappa'^\perp)}{D(x\kappa^\perp,x'\kappa'^\perp)}+(x\kappa^\perp\leftrightarrow x'\kappa'^\perp)\right], \quad (3.1)$$

where N denotes the numerator factors resulting from the fermion spin and momentum-dependent interaction vertices and D denotes light-front energy denominators multiplied by the exchanged boson longitudinal momentum. We denote x_2 by x and x'_2 by x'. The chopping factor C is

$$\begin{aligned} C(x\kappa^\perp,x'\kappa'^\perp)=&\Theta\left[\frac{x'}{x}-\varepsilon\right]\Theta\left[1-\varepsilon-\frac{x'}{x}\right]\Theta\left[\frac{1-x}{1-x'}-\varepsilon\right]\Theta\left[1-\varepsilon-\frac{1-x}{1-x'}\right]\\ &\times\Theta\left[4\Lambda^2-\left[\kappa'^\perp-\frac{x'}{x}\kappa^\perp\right]^2\right]\Theta\left[4\Lambda^2-\left[\kappa^\perp-\frac{1-x}{1-x'}\kappa'^\perp\right]^2\right]\\ &\times\Theta(x-\varepsilon)\Theta(1-x-\varepsilon)\Theta(\Lambda^2-\kappa^{\perp 2})\Theta(x'-\varepsilon)\Theta(1-x'-\varepsilon)\Theta(\Lambda^2-\kappa'^{\perp 2}) . \end{aligned} \quad (3.2)$$

Equation (3.2) can be simplified because the first, third, fifth, and sixth Θ functions are automatically 1 given all the other Θ functions (the second, fourth, and seventh through 12th Θ functions):

$$\begin{aligned} &C(x\kappa^\perp,x'\kappa'^\perp)\\ &\quad=\Theta(x-x'-\varepsilon x)\Theta[x-x'-\varepsilon(1-x)]\\ &\qquad\times\Theta(x-\varepsilon)\Theta(1-x-\varepsilon)\Theta(\Lambda^2-\kappa^{\perp 2})\\ &\qquad\times\Theta(x'-\varepsilon)\Theta(1-x'-\varepsilon)\Theta(\Lambda^2-\kappa'^{\perp 2}) . \end{aligned} \quad (3.3)$$

This is seen by noting that if both $\kappa=|\kappa^\perp|$ and $\kappa'=|\kappa'^\perp|$ are each smaller than Λ, then their sum is smaller than 2Λ. It is understood that the chopping factors in the counterterm V_Λ are the same as in the one-boson-exchange term. Since the resulting chopping factor is independent of the azimuthal angles of the transverse momenta, we will use a simpler notation in our further discussion, $C(x\kappa,x'\kappa')=C(x\kappa^\perp,x'\kappa'^\perp)+(x\leftrightarrow x')$.

Longitudinal momentum fractions carried by fermions are limited from below by the longitudinal cutoff ε. From Eq. (3.3) we see also that the longitudinal-momentum integrals are chopped off in the region where $x\simeq x'$; i.e., the momentum fraction carried by the boson in the intermediate states is not allowed to vanish. In chromodynamics, $x\simeq x'$ is a region where a strong longitudinal singularity appears and a renormalization-group analysis of the singularity is essential. Therefore we assume here that ε is finite. In the present model, we study divergences that appear in the transverse direction.

Limits on transverse integrals are particularly simple thanks to our choice of the chopping factors in the initial Hamiltonian. Namely, the relative transverse momenta of fermions are limited only in modulus, which is cut off by the transverse cutoff Λ. This feature enables us to carry out simple partial-wave analysis of the model Hamiltonian eigenvalue problem. It is not necessary to make the partial-wave analysis in order to carry out the renormalization procedure, but it should help the reader to understand the connection between the present paper and Ref. [4].

We observe that our Hamiltonian commutes with the

kinematical operator J_z, generator of rotations about the z axis. The total z component of the angular momentum can be decomposed into the orbital angular momentum of the bound state relative to an arbitrary axis parallel to the z axis, L_z, and the total internal angular momentum of the bound state, $j_z=l_z+s_z$. Our model Hamiltonian conserves j_z. Divergences appear when $|j_z|\leq 1$. We begin our discussion with the case $j_z=0$. This allows us to remove two angles from consideration, and we can focus on kernels depending on the magnitudes of the relative transverse momenta and longitudinal-momentum fractions only. We will comment on cases with $j_z\neq 0$ later.

We introduce the notation $\phi^{l_z}_{\lambda_2\lambda_1}$ and obtain the following $j_z=0$ part of the model Hamiltonian eigenvalue equation:

$$(\mathcal{M}_{12}^2-M^2)\begin{bmatrix}\phi^0_{\uparrow\downarrow}\\ \phi^0_{\downarrow\uparrow}\\ \phi^{-1}_{\uparrow\uparrow}\\ \phi^{+1}_{\downarrow\downarrow}\end{bmatrix}-\frac{\alpha_g}{4\pi}\int dx'\int d\kappa' C(x\kappa,x'\kappa')\left\{\begin{bmatrix}\alpha & \beta & \gamma & \delta\\ \beta & \alpha & -\delta & -\gamma\\ \varepsilon & -\varphi & \omega & \rho\\ \varphi & -\varepsilon & \rho & \omega\end{bmatrix}+M\langle V^0_\Lambda\rangle\right\}\begin{bmatrix}\phi^0_{\uparrow\downarrow}\\ \phi^0_{\downarrow\uparrow}\\ \phi^{-1}_{\uparrow\uparrow}\\ \phi^{+1}_{\downarrow\downarrow}\end{bmatrix}=0\ . \tag{3.4}$$

V^0_Λ is proportional to the $j_z=0$ projection of the counterterm. The counterterm commutes with j_z because the initial Hamiltonian does. V^0_Λ is present to make the solutions to Eq. (3.4) independent of the cutoff Λ. The coupling constant $\alpha_g=g^2/4\pi$. $\mathcal{M}_{12}$ denotes the invariant mass of two free fermions of mass m:

$$\mathcal{M}_{12}^2=(p_1+p_2)^2=\frac{m^2+p_1^{\perp 2}}{x_1}+\frac{m^2+p_2^{\perp 2}}{x_2}-(p_1^\perp+p_2^\perp)^2=\frac{m^2+\kappa^2}{x_1x_2}\ . \tag{3.5}$$

The various functions denoted by Greek letters are

$$\alpha=m^2\left[\frac{1}{x}+\frac{1}{x'}\right]\left[\frac{1}{1-x}+\frac{1}{1-x'}\right]A^0\ , \tag{3.6}$$

$$\beta=\left[\frac{\kappa'^2}{x'(1-x')}+\frac{\kappa^2}{x(1-x)}\right]A^0-\kappa\kappa'\left[\frac{1}{x'(1-x)}-\frac{1}{x(1-x')}\right]A^1\ , \tag{3.7}$$

$$\gamma=m\left[\frac{1}{x}+\frac{1}{x'}\right]\left[\frac{\kappa}{1-x}A^1-\frac{\kappa'}{1-x'}A^0\right]\ , \tag{3.8}$$

$$\delta=m\left[\frac{1}{1-x}+\frac{1}{1-x'}\right]\left[\frac{\kappa}{x}A^1-\frac{\kappa'}{x'}A^0\right]\ , \tag{3.9}$$

$$\varepsilon=m\left[\frac{1}{x}+\frac{1}{x'}\right]\left[\frac{\kappa'}{1-x'}A^1-\frac{\kappa}{1-x}A^0\right]\ , \tag{3.10}$$

$$\varphi=m\left[\frac{1}{1-x}+\frac{1}{1-x'}\right]\left[\frac{\kappa'}{x'}A^1-\frac{\kappa}{x}A^0\right]\ , \tag{3.11}$$

$$\omega=m^2\left[\frac{1}{x}+\frac{1}{x'}\right]\left[\frac{1}{1-x}+\frac{1}{1-x'}\right]A^1\ , \tag{3.12}$$

$$\rho=-\left[\frac{\kappa'^2}{x'(1-x')}+\frac{\kappa^2}{x(1-x)}\right]A^1+\kappa\kappa'\left[\frac{1}{x'(1-x)}-\frac{1}{x(1-x')}\right]A^0\ . \tag{3.13}$$

We use the notation

$$A^l=\frac{1}{(a^2-b^2)^{1/2}}\left[\frac{(a^2-b^2)^{1/2}-a}{b}\right]^{|l|}\ , \tag{3.14}$$

$$a=\mu^2+\kappa^2+\kappa'^2+\frac{1}{2}(x'-x)\left[\frac{\kappa^2+m^2}{x}-\frac{\kappa^2+m^2}{1-x}+\frac{\kappa'^2+m^2}{1-x'}-\frac{\kappa'^2+m^2}{x'}\right]+\frac{1}{2}|x'-x|\left[\frac{\kappa^2+m^2}{x(1-x)}+\frac{\kappa'^2+m^2}{x'(1-x')}-2M_0^2\right] \tag{3.15}$$

and

$$b=-2\kappa\kappa'\ . \tag{3.16}$$

The asymptotic behavior of the wave functions $\phi^{l_z}_{\lambda_2\lambda_1}$ when the transverse momentum becomes large is given by the inverse of the invariant mass squared of the two fermions, $\mathcal{M}_{12}^{-2}$, multiplying the potential term.

By inspection of Eqs. (3.6)–(3.13), we see that almost all entries in the one-boson-exchange potential in Eq. (3.4) behave for large transverse momentum as at least one inverse power of the transverse momentum. These terms lead to asymptotic behavior of the wave functions which is as least as convergent as κ^{-3} and produce no large cutoff dependence in the integral equations. However, the first term of β in Eq. (3.7) behaves like a constant (in fact, like a function of x and x') for large κ. Therefore the wave functions with $l_z=0$ obtain contributions which fall off at large κ like κ^{-2}. Substituting such a function under the integral with a constant potential, we see that asymptotic behavior of the wave function in

the transverse direction leads to a logarithmic divergence in the one-boson-exchange potential. This is a common feature of all one-boson-exchange interactions between relativistic light-front fermions. The logarithmically divergent integral in the potential term generates dependence of the eigenvalue problem on the cutoff Λ. We discover that the spectrum is cutoff dependent. Numerical estimates of the cutoff dependence are summarized in Ref. [4]. The counterterm $\langle V_\Lambda^0 \rangle$ is designed in renormalization theory so that the cutoff dependence is removed.

We proceed to the description of how the counterterm is constructed.

IV. COUNTERTERMS

Counterterms that remove the transverse cutoff dependence from the spectrum of the model Hamiltonian are constructed by the renormalization procedure [9,10]. Thanks to the simplicity of the model, we can use particularly simple formulas here.

Let us denote the model Hamiltonian by H and its free part by H_0. Denote the eigenvalue by E and the potential term by gV, where g symbolizes a coupling constant which is equal to α_g in our model. Thus $H=H_0+gV$.

Let us introduce projection operators P_λ and $Q_\lambda=1-P_\lambda$. P_λ projects on the space of states of two fermions with relative transverse momentum squared smaller or equal to $\lambda \leq \Lambda^2$. The corresponding space of two-fermion states is called λ space. The free Hamiltonian H_0 commutes with the projection operators. Note that the dimension of the cutoff λ is transverse momentum squared.

The effective Hamiltonian in λ space, denoted by H_λ, contains the effective interaction V_λ, which is given by the standard formula [11]

$$V_\lambda=P_\lambda gVP_\lambda+P_\lambda gVQ_\lambda \frac{1}{E-H_0-Q_\lambda gVQ_\lambda}Q_\lambda gVP_\lambda \ . \tag{4.1}$$

If the limit of V_λ when $\Lambda\to\infty$ exists, then H_λ has a cutoff-independent spectrum and the renormalization counterterms are not necessary. If the limit does not exist, one can attempt to add counterterms to H so that the dependence of V_λ on Λ disappears for $\Lambda^2 \gg \lambda$. The new finite limit of H_λ when Λ formally tends to infinity is called the renormalized Hamiltonian $H_{R\lambda}$:

$$H_{R\lambda}=\lim_{\Lambda\to\infty} H_\lambda \ . \tag{4.2}$$

Since the cutoff λ is chosen arbitrarily, the spectrum of common eigenstates of the renormalized Hamiltonians $H_{R\lambda}$ must be independent of the cutoff λ. Renormalized Hamiltonians at various cutoffs λ are related by a renormalization group transformation. The objective of our renormalization procedure is to find the necessary counterterms to add to the Hamiltonian H so that the formal limit of the new H when Λ is sent to infinity exists. It is not obvious how to construct the counterterms. Analysis of divergences in the effective Hamiltonians H_λ when $\Lambda\to\infty$ indicates what counterterms should be added. In our model one can also study the renormalization group transformation and the structure of the renormalized Hamiltonians.

There are two steps to take. The first step is to find the general structure of the counterterms. One can use power counting in perturbation theory to isolate divergences in the effective Hamiltonian and discover the structure of the necessary counterterms.

The second step involves identifying special conditions, such as Poincaré invariance of the spectrum, and using these to constrain finite parts of the counterterms. Finite parts of the counterterms can be freely chosen, but should possess the same structure as the divergent parts. The second step may require a numerical procedure involving diagonalization of the full Hamiltonian with counterterms containing the adjustable finite parts. In the remaining part of this paper we take the first step for our model Hamiltonian.

In order to proceed to the discussion of details of the renormalization procedure, we develop useful differential equations for the effective interactions as functions of the cutoff λ. By definition,

$$dV_\lambda=P_\lambda V_{\lambda+d\lambda}P_\lambda-V_\lambda \ . \tag{4.3}$$

If we represent multiplication of matrices in spin space and integration over longitudinal momentum fractions as multiplication, while integration over transverse momenta is denoted by integration over the variable z for κ^2, z' for κ'^2, etc., then the kernels of the projection operators are represented by

$$P_\lambda(z,z')=\Theta(\lambda-z)\Theta(z)\delta(z-z') \ , \tag{4.4}$$

$$Q_\lambda(z,z')=\Theta(\Lambda^2-z)\Theta(z-\lambda)\delta(z-z') \ , \tag{4.5}$$

and we can define

$$dQ_\lambda(z,z')=-d\lambda\,\delta(z-\lambda)\delta(z-z') \ . \tag{4.6}$$

The corresponding space of states is called $d\lambda$ space. Using the identity

$$\frac{1}{E-H_0-Q_\lambda gVQ_\lambda}=\frac{1}{1-[Q_\lambda/(E-H_0)]gVQ_\lambda}\frac{1}{E-H_0} \ , \tag{4.7}$$

we can formally rewrite Eq. (4.1) as

$$V_\lambda=P_\lambda gVP_\lambda+P_\lambda gV\left[\frac{Q_\lambda}{E-H_0}+\frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}+\frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}+\cdots\right]gVP_\lambda \ . \tag{4.8}$$

Then

$$dV_\lambda = P_\lambda gV\left[\frac{dQ_\lambda}{E-H_0}+\frac{dQ_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}+\frac{Q_\lambda}{E-H_0}gV\frac{dQ_\lambda}{E-H_0}+\frac{dQ_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}\right.$$
$$\left.+\frac{Q_\lambda}{E-H_0}gV\frac{dQ_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}+\frac{Q_\lambda}{E-H_0}gV\frac{Q_\lambda}{E-H_0}gV\frac{dQ_\lambda}{E-H_0}+\cdots\right]gVP_\lambda, \tag{4.9}$$

so that

$$dV_\lambda = P_\lambda V_\lambda \frac{dQ_\lambda}{E-H_0}V_\lambda P_\lambda \ . \tag{4.10}$$

Because of the simplicity of our model, the projection operator dQ_λ is infinitesimally small and action of the potential term takes states out of the $d\lambda$ space. Therefore dQ_λ appears at most once in each term of the sum in Eq. (4.9) and the resulting Eq. (4.10) takes a simple form. One is on the lookout for terms in gV that stay within $d\lambda$ space when self-interactions of fermions are included.

In terms of potential kernels, Eq. (4.10) reads

$$\frac{d}{d\lambda}V_\lambda(z,z') = V_\lambda(z,\lambda)\frac{1}{H_0(\lambda)-E}V_\lambda(\lambda,z') \ . \tag{4.11}$$

The same equation can be obtained without using a series expansion of Eq. (4.8). One may derive an expression for the infinitesimal change of the effective kernel when the cutoff $\lambda+d\lambda$ is reduced to λ directly in the integral eigenvalue equation. The derivation given above is provided to show the equivalence of the differential equation with the high-low analysis from Ref. [4].

Divergences in the effective potential V_λ when $\Lambda\to\infty$ and no counterterms are included are only logarithmic, and we may consider such cutoffs λ that eigenvalues E are much smaller than $H_0(\lambda)$. Therefore we may neglect the eigenvalue in the denominator on the right-hand side of Eq. (4.11). Denoting the inverse of the product $P^+H_0(\lambda)$ by K_λ, incorporating another factor of P^+ in $V_\lambda(z,z')$ in subsequent equations, and changing the sign of the potential kernels for convenience, we obtain the required differential equation:

$$\frac{d}{d\lambda}V_\lambda(z,z') = -V_\lambda(z,\lambda)K_\lambda V_\lambda(\lambda,z') \ . \tag{4.12}$$

The integral form of this equation is

$$V_\lambda(z,z') = V_{\Lambda^2}(z,z') + \int_\lambda^{\Lambda^2} ds\ V_s(z,s)K_s V_s(s,z') \ . \tag{4.13}$$

The initial condition at $s=\Lambda^2\to\infty$ is

$$V_{\Lambda^2}(z,z') = gV(z,z') + c_\Lambda(z,z') \ , \tag{4.14}$$

where $c_\Lambda(z,z')$ denotes a counterterm. The problem is to find the structure of the counterterm in the limit $\Lambda\to\infty$. The difficulty is that we have to solve a first-order differential equation with two boundary conditions. Namely, we need to specify the potential at $s=\Lambda^2$ and request that the effective potential be independent of Λ at $s=\lambda\ll\Lambda^2$.

The generally valid structure of the counterterm can be found in perturbation theory. We write the counterterm as a series in the coupling constant. The series for the counterterm starts with a term proportional to g^2. The family of kernels $V_s(z,z')$ for $\lambda_0\le s\le\Lambda^2$ is also written as a series in the coupling constant g. The family is viewed as a single function of three arguments z, z', and s. The first term in the series expansion of the kernel is proportional to g. It is equal to the original kernel $gV(z,z')$ (including the factor $P+$). The term of order g is independent of s. λ_0 is much larger than M^2, but otherwise it is arbitrary.

Order by order in the series expansion, one can show that the divergences due to $\Lambda\to\infty$ appear in the effective potential at small cutoffs only in $V_{\lambda_0}(0,0)$. Therefore, it is sufficient to subtract the divergent part of $V_{\lambda_0}(0,0)$ to obtain cutoff independence of the effective Hamiltonian H_λ. A sketch of the proof to all orders in g is given in Appendix A.

Once it is established that divergences appear only in $V_{\lambda_0}(0,0)$, the renormalized Hamiltonians can be found in the following iterative procedure. We assume that the counterterm is independent of z and z', i.e., $c_\Lambda(z,z')=c_\Lambda$. Equation (4.13) takes the form

$$V_\lambda(z,z') = gV(z,z') + c_\Lambda + \int_\lambda^{\Lambda^2} ds\ V_s(z,s)K_s V_s(s,z') \ . \tag{4.15}$$

The first approximation to the effective kernel is the potential itself,

$$V_{1\lambda}(z,z') = gV(z,z') \tag{4.16}$$

and

$$c_{1\Lambda} = 0 \ . \tag{4.17}$$

The second approximation is given by

$$V_{2\lambda}(z,z') = gV(z,z') + c_{2\Lambda} + \int_\lambda^{\Lambda^2} ds\ gV(z,s)K_s gV(s,z') \tag{4.18}$$

and

$$c_{2\Lambda} = f_2 - \int_{\lambda_0}^{\Lambda^2} ds\ gV(0,s)K_s gV(s,0) \ , \tag{4.19}$$

where f_2 is allowed to be a finite spin matrix which is a function of the longitudinal momenta of the fermions. The procedure can be iterated according to the recursion

$$V_{(n+1)\lambda}(z,z') = gV(z,z') + f_{n+1} - \int_{\lambda_0}^{\Lambda^2} ds\ V_{ns}(0,s)K_s V_{ns}(s,0) + \int_\lambda^{\Lambda^2} ds\ V_{ns}(z,s)K_s V_{ns}(s,z') . \tag{4.20}$$

At the $n+1$ iteration, one has to choose the arbitrary

function f_{n+1} so that some symmetry requirements are satisfied. It is a matter of case-by-case study to find how can one implement such constraints. If the limit of the iteration procedure for $n \to \infty$ exists, the solution to the renormalization problem is given by the potential which satisfies the equation

$$V_\lambda(z,z')=gV(z,z')+f-\int_{\lambda_0}^{\Lambda^2} ds\ V_s(0,s)K_s V_s(s,0) + \int_{\lambda}^{\Lambda^2} ds\ V_s(z,s)K_s V_s(s,z')\ , \tag{4.21}$$

when $\Lambda^2 \to \infty$. $f=O(g^2)$ is the finite part of the counterterm that is defined using the cutoff λ_0 as the lower limit of integration. Therefore f and λ_0 are related.

In practice, it should be sufficient to know V_{Λ^2}, which is a sum of $gV(z,z')$ and a certain unknown function of the longitudinal momentum fractions of the fermions. Equation (4.21) can be solved numerically. One may also study analytic examples that illustrate what kind of solutions may be expected. The renormalized interactions V_λ may develop singularities for some values of the cutoff λ. Elementary examples of solutions for renormalized Hamiltonians for simple choices of bare potential kernels $gV(z,z')$ are given in Appendix B.

At this point we have completed the discussion of finding counterterms that remove divergences in the spectrum of the partial-wave Hamiltonian for $j_z=0$. The construction proceeds the same way for $|j_z|=1$. Higher partial waves are free from divergences.

If neglect of the eigenvalue E in the denominator introduces an untolerable error because the ratio M^2/λ_0 is not small enough, one can retreat to the more accurate procedure of calculating effective Hamiltonians, for example, using the operator R introduced in the renormalization theory in Ref. [10].

An equation essentially identical to Eq. (4.11) appears in Ref. [5]. However, the potential V_λ that appears in Ref. [5] has an auxiliary role unique to that reference. Namely, Van de Sande and Pinsky introduce a nonlinear equation, which gives a finite effective Hamiltonian with a small cutoff μ, in terms of V_0 and μ, as well as the initial cutoff Hamiltonian with a large cutoff Λ in terms of V_0 and Λ. Van de Sande and Pinsky then examine some model Hamiltonians in which V_Λ is local and raise the possibility, but do not prove, that their cutoff Hamiltonian is essentially local also. However, if the formalism of Van de Sande and Pinsky were applied to the problem of this paper, their auxiliary potential, at least in the lowest perturbative order, would be a nonlocal one-boson-exchange potential, and in this case it is highly implausible that their initial cutoff Hamiltonian could be local. Hence their paper does not address the basic issue of this paper—the overlapping divergence problem for our model Hamiltonian, with a nonlocal potential but only local counterterms.

V. CONCLUSION

Although the problem of renormalization of overlapping divergences was solved in Lagrangian perturbation theory, it had to be investigated again in the light-front Hamiltonian formalism. The light-front Hamiltonian framework differs from the Lagrangian framework. The power-counting rules for light-front Hamiltonians are based on the observation that transverse and longitudinal dimensions on the light-front count differently. Our explicit analysis of a model Hamiltonian confirms the prediction of power counting that counterterms involving four-fermion fields are local in the transverse direction, no transverse derivative is needed, and an arbitrary finite function of longitudinal coordinates with dimension $(x^-)^2$ is present. This finite function is explained in Appendix A following Eq. (A6). It can be determined by imposing some of the conditions of rotational invariance around the x and y axes which a fully relativistic theory must satisfy but normally would not be satisfied in Tamm-Dancoff truncation of light-front theory. Initial numerical studies in that direction in lowest-order perturbation theory are described in Ref. [4].

In order to cut off transverse momenta in our model Hamiltonian, we introduce a suitable set of chopping factors in the interaction terms. We describe the removal of the transverse cutoff dependence from the spectrum of the Hamiltonian. The chopping factors also introduce longitudinal momentum cutoffs. We did not discuss the removal of the longitudinal cutoff dependence from the spectrum. The model was designed for investigation of the transverse divergences. It provides some understanding of the elements of the renormalization procedure needed for the complete removal of various cutoff dependences from spectra of light-front Hamiltonians. Our ultimate goal is to define the renormalization program for light-front QCD.

Simple analytic examples of solutions for renormalized Hamiltonians are described in Appendix B. They illustrate potential difficulties that may arise when a light-front Hamiltonian does not have asymptotic freedom. Solutions to the renormalization-group transformation for renormalized light-front Hamiltonians may blow up to infinity. In such cases nonperturbative features prevent analysis order by order in the coupling constants, no matter how small the constants are. We hope to avoid such problems in QCD.

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APPENDIX A

The effective potential can be written as

$$V_\lambda(z,z')=gV(z,z')+c_\lambda(z,z')\ , \tag{A1}$$

where $c_\lambda(z,z')$ satisfies the equation

$$c_\lambda(z,z')=c_\Lambda+\int_\lambda^{\Lambda^2} ds[gV(z,s)+c_s(z,s)] \times K_s[gV(s,z')+c_s(s,z')] \ . \quad \text{(A2)}$$

K_s behaves for large s as s^{-1} (it also contains a factor depending on the fermions' longitudinal momenta). If the potentials under the integral tend to constants (functions of the fermions' longitudinal momenta) at large s, the integral diverges like $\ln\Lambda$.

At some particular value of $\lambda_0 \gg M^2$, we request that

$$c_{\lambda_0}(z,z')=c_\Lambda+\int_{\lambda_0}^{\Lambda^2} ds[gV(z,s)+c_s(z,s)] \times K_s[gV(s,z')+c_s(s,z')] \quad \text{(A3)}$$

be independent of Λ in the limit $\Lambda\to\infty$. When z and z' are smaller than $\lambda \ll \Lambda^2$, the coefficient of the logarithmic divergence is given by the structure of

$$\lim_{s\to\infty} V_s(z,s) \quad \text{(A4)}$$

and

$$\lim_{s\to\infty} V_s(s,z') \ . \quad \text{(A5)}$$

In second order in the coupling constant $g=\alpha_g$, the limits are given by the bare potential itself. One may analyze the behavior of the potential terms in Eqs. (3.6)–(3.13) for the case $j_z=0$ or consider the full one-boson-exchange potential without partial-wave projection. In the latter case, multiplication of operators in this appendix should be understood to contain integrals over azimuthal angles of the relative transverse momenta of fermions. Order by order in perturbation theory, one can check that the limits (A4) and (A5) are independent of z and z'. Therefore one can subtract the term containing $\ln\Lambda$ by subtracting from Eq. (A3) the same integral with $z=z'=0$. Following this observation, we construct counterterms in the following way.

From Eq. (A3) we define

$$c_\Lambda=f-\int_{\lambda_0}^{\Lambda^2} ds[gV(0,s)+c_s(0,s)] \times K_s[gV(s,0)+c_s(s,0)] \ , \quad \text{(A6)}$$

where f is an arbitrary finite kernel acting on the longitudinal momenta and spins of the fermions; i.e., it is independent of the fermions' transverse momenta and represents an interaction which is local in the transverse direction. Then

$$c_\lambda(z,z')=f-\int_{\lambda_0}^{\Lambda^2} ds[gV(0,s)+c_s(0,s)] \times K_s[gV(s,0)+c_s(s,0)] + \int_\lambda^{\Lambda^2} ds[gV(z,s)+c_s(z,s)] \times K_s[gV(s,z')+c_s(s,z')] \ . \quad \text{(A7)}$$

We shall outline the proof that

$$\lim_{\Lambda\to\infty} c_\lambda(z,z') \quad \text{(A8)}$$

exists to all orders in perturbation theory in the coupling constant g.

The limit of Eq. (A7) can be rewritten as

$$\lim_{\Lambda\to\infty} c_\lambda(z,z')=f-\int_{\lambda_0}^{\lambda} ds[gV(0,s)+c_s(0,s)]K_s[gV(s,0)+c_s(s,0)] + \lim_{\Lambda\to\infty}\int_\lambda^{\Lambda^2} ds[gV(z,s)+c_s(z,s)]K_s[gV(s,z')-gV(s,0)+c_s(s,z')-c_s(s,0)] + \lim_{\Lambda\to\infty}\int_\lambda^{\Lambda^2} ds[gV(z,s)-gV(0,s)+c_s(z,s)-c_s(0,s)]K_s[gV(s,0)+c_s(s,0)] \ . \quad \text{(A9)}$$

We define

$$c_s(z,z')=\sum_{k=2}^{\infty} c_{sk}(z,z')g^k \quad \text{(A10)}$$

and

$$f=\sum_{k=2}^{\infty} f_k g^k \ . \quad \text{(A11)}$$

For $k=2$, Eq. (A9) gives

$$\lim_{\Lambda\to\infty} c_{\lambda 2}(z,z')=f_2-\int_{\lambda_0}^{\lambda} ds\ V(0,s)K_s V(s,0)+\lim_{\Lambda\to\infty}\int_\lambda^{\Lambda^2} ds\ V(z,s)K_s[V(s,z')-V(s,0)] + \lim_{\Lambda\to\infty}\int_\lambda^{\Lambda^2} ds[V(z,s)-V(0,s)]K_s V(s,0) \ . \quad \text{(A12)}$$

The limits of the integrals in Eq. (A12) exist if the limits of integrals of norms of the operators under the integrals exist. Direct inspection of the potential kernel $V(z,z')$ gives the estimates

$$|V(z,z')|<D_1 \ , \quad \text{(A13)}$$

$$|V(z,s)-V(0,s)|<E_1\left[\frac{z}{s}\right]^\alpha \ , \quad \text{(A14)}$$

$$|V(s,z')-V(s,0)|<E_1\left[\frac{z'}{s}\right]^{\alpha}, \tag{A15}$$

where D_1 and E_1 are independent of the transverse momenta. For the one-boson-exchange potential, $\alpha=\frac{1}{2}$, but the proof is valid for arbitrary positive α. Direct integration in Eq. (A12) with operators replaced by majorants of their norms from Eqs. (A13)–(A15) produces integrals that are insensitive to the upper limit of integration when $\Lambda\to\infty$. Therefore the limit of $c_{\lambda 2}(z,z')$ when $\Lambda\to\infty$ exists. The same integrals allow us to make estimates of $c_{\lambda 2}(z,z')$, which are necessary to verify the existence and estimate $c_{\lambda 3}(z,z')$. We obtain

$$|c_{s2}(z,z')|<D_2\ln(es/\lambda_0), \tag{A16}$$

$$|c_{s2}(s,z')-c_{s2}(s,0)|<E_2\left[\frac{z'}{s}\right]^{\alpha}, \tag{A17}$$

$$|c_{s2}(z,s)-c_{s2}(0,s)|<E_2\left[\frac{z}{s}\right]^{\alpha}. \tag{A18}$$

Under the above conditions satisfied by $c_{s2}(z,z')$, one can see by explicit calculation that $\lim_{\Lambda\to\infty}c_{\lambda 3}(z,z')$ exists. One can also make estimates for $c_{\lambda 3}(z,z')$. The proof to all orders is made by mathematical induction.

We assume that the following estimates hold for $k=2,3,\ldots,N$:

$$|c_{sk}(z,z')|<D_k\ln^{k-1}(es/\lambda_0), \tag{A19}$$

$$|c_{sk}(s,z')-c_{sk}(s,0)|<E_k\left[\frac{z'}{s}\right]^{\alpha}\ln^{k-2}(es/\lambda_0), \tag{A20}$$

$$|c_{sk}(z,s)-c_{sk}(0,s)|<E_k\left[\frac{z}{s}\right]^{\alpha}\ln^{k-2}(es/\lambda_0). \tag{A21}$$

Direct calculation of terms of order g^{N+1} with counterterms defined by subtraction at $z=z'=0$ establishes the existence of $\lim_{\Lambda\to\infty}c_{N+1}(z,z')$, which satisfies the conditions

$$|c_{sN+1}(z,z')|<D_{N+1}\ln^{N}(es/\lambda_0), \tag{A22}$$

$$|c_{sN+1}(s,z')-c_{sN+1}(s,0)|<E_{N+1}\left[\frac{z'}{s}\right]^{\alpha}\ln^{N-1}(es/\lambda_0), \tag{A23}$$

$$|c_{sN+1}(z,s)-c_{sN+1}(0,s)|<E_{N+1}\left[\frac{z}{s}\right]^{\alpha}\ln^{N-1}(es/\lambda_0). \tag{A24}$$

We conclude that a counterterm which is a function of only the longitudinal momenta of fermions and is local in the transverse direction removes cutoff dependence from the eigenvalue problem of our model Hamiltonian to all orders in perturbation theory.

We should remark at this point that when the longitudinal momentum carried by the scalar boson in the intermediate states is limited from below by a vanishingly small cutoff, the constants E_k and D_k for $k=1,2,\ldots$ obtain contributions from integrating potential kernels over the longitudinal momentum region where the kernels grow inversely with the vanishing longitudinal momentum carried by the boson. The analysis presented above has to be slightly modified. Integration of the kernels over the region $x\simeq x'$ leads to new constants E_k and D_k for $k=1,2,\ldots$, which are multiplied by higher powers of the same logarithms. In the region where the fermions' longitudinal momenta may approach zero, it is important to include fermion seagull terms, which we arbitrarily neglected in order to simplify our model

APPENDIX B

1. Solution for $gV(z,z')=g$

If the kernel is independent of transverse momenta, then the renormalized interaction is too:

$$c_\lambda(z,z')=c_\lambda. \tag{B1}$$

Equation (4.21) simplifies to

$$c_\lambda=f-\int_{\lambda_0}^{\Lambda^2}ds[g+c_s]K_s[g+c_s]+\int_{\lambda}^{\Lambda^2}ds[g+c_s]K_s[g+c_s], \tag{B2}$$

which is equivalent to

$$c_\lambda=f-\int_{\lambda_0}^{\lambda}ds[g+c_s]K_s[g+c_s]. \tag{B3}$$

The solution is

$$c_\lambda=\left[1+(f+g)\int_{\lambda_0}^{\lambda}ds\,K_s\right]^{-1}\times\left[f-(f+g)\int_{\lambda_0}^{\lambda}ds\,K_s g\right], \tag{B4}$$

so that the renormalized interaction is

$$V_\lambda=\left[1+(f+g)\int_{\lambda_0}^{\lambda}ds\,K_s\right]^{-1}(f+g). \tag{B5}$$

The renormalized interaction vanishes when the cutoff λ tends to infinity, which is analogous to asymptotic freedom in QCD. When the fermion spins and longitudinal momenta are absent, the resulting eigenvalue problem for the constant kernel reduces to the two-dimensional Schrödinger equation with a singular δ-function potential which is discussed in Ref. [12].

2. Solution for $gV(z,z')=ig\Theta(z-z')-ig\Theta(z'-z)$

This example illustrates how integration of the differential Eq. (4.12) may produce a singular behavior of the renormalized interaction at certain values of the cutoff λ. The example is motivated by analysis of the partial-wave potential of $|j_z|=1$ in our model, where antisymmetric imaginary kernels appear. We consider

$$V_s(z,z')=ig\Theta(z-z')-ig\Theta(z'-z)+c_s, \tag{B6}$$

where c_s is independent of the transverse momentum as in the case of Appendix B 1 above. Here we assume that g and c_s are numbers independent of the longitudinal momenta and spins of the fermions. Only then can a simple

analytic form for c_s be easily derived. K_s is put equal to s^{-1} for simplicity. With the above simplifications, direct integration of Eq. (4.21) leads to the solution

$$c_\lambda = g\frac{f - g\tan(g\ln\lambda/\lambda_0)}{g + f\tan(g\ln\lambda/\lambda_0)} . \tag{B7}$$

We note that the renormalized coupling tends to infinity for certain values of the cutoff λ. Near such values of the cutoff, perturbation theory becomes invalid. In order to avoid such features, it is desirable to incorporate asymptotic freedom in the model Hamiltonians for quarks and gluons. The advantage of asymptotic freedom is that it forces the renormalized couplings to approach zero so that they cannot go to infinity instead.

3. Factorized kernels and basis functions

If the renormalized kernel V_λ can be approximated by a separable form

$$V_\lambda(z,z') = \sum_{ij} h_i^*(z) c_{ij}(\lambda) h_j(z') \tag{B8}$$

or, equivalently, in abbreviated notation

$$V_\lambda(z,z') = h^\dagger(z) c_\lambda h(z'), \tag{B9}$$

and satisfies Eq. (4.12), then the matrix c_λ may be written explicitly as

$$c_\lambda = \left[1 + c_{\lambda_0}\int_{\lambda_0}^{\lambda} ds\, h(s) K_s h^\dagger(s)\right]^{-1} c_{\lambda_0} . \tag{B10}$$

This result suggests that using basis functions instead of grids may facilitate speedy renormalization procedures in numerical calculations.

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