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Evolution of the density parameter in multidimensional cosmology

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Inflation as a dynamical effect of extra dimensions is discussed from the viewpoint of the evolution of the parameters Ω and q in the physical space. By reducing the dynamics of the models FLRW $(k=\pm1)\times T^D(T^D=S^1\times\cdots\times S^1)$ and FLRW $(k=0)\times S^D$, where FLRW denotes Friedmann, Lemaître, Robertson, and Walker, to two dimensions it is shown that inflation need not imply $\Omega\cong1$ for the present time, i.e., that it does not solve the flatness problem. The energy-momentum tensor of the form $T^\mu_\nu=(\rho,m\rho,m\rho,m\rho,n\rho,\ldots,n\rho)$ is assumed. The results are comparable to those obtained by Madsen and Ellis for four-dimensional models $N=1$, $d=11$ of supergravity theory.

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I. INTRODUCTION

In the standard cosmological models, it is a scalar field ϕ with a potential $V(\phi)$ that is identified as a mechanism responsible for inflation. Within the Kaluza-Klein cosmological scheme, space-time dimensionality becomes a dynamic variable, and it is natural to regard inflation as a result of the dimensional reduction process; the effects of such a process could be equivalent to those of a scalar field with a suitable potential. (See Refs. [1,2]. Some critical remarks with respect to this program are made in [3].) The original motivation for introducing inflation was a desire to avoid paradoxes of the classical cosmology, one of them being the so-called flatness problem. This problem was carefully rediscussed by Ellis and Madsen and Ellis [4] who, by presenting inflationary models (with exponential and power-law inflation) as dynamical systems, showed that the density parameter Ω is nearly 1 only during restricted periods of the world's evolution, that the probability that $\Omega\approx1$ depends on time of observation, and that the set of initial conditions leading to Ω not being close to 1 at a given time is, in general, not of zero measure. We address the questions of how this problem looks in the multidimensional world models FRW $(k=\pm1)\times T^D$ and FLRW $(k=0)\times S^D$ (where FLRW stands for the Friedmann-Lemaître-Robertson-Walker space and $T^D=S^1\times\cdots\times S^1$) with the hydrodynamic energy-momentum tensor $T^\mu_\nu=(\rho,m\rho,m\rho,m\rho,n\rho,\ldots,n\rho)$, where m and n are constants. In these two cases the model equations can be reduced to a two-dimensional autonomical dynamical system, and the assumed for m of the energy-momentum tensor is general enough to contain a physically interesting situation (see [2]).

Following Barrow [5] we shall assume that the inflation takes place in those regions of space for which acceleration of the scale factor of the physical space is larger than zero, $(\ddot R>0)$; if this condition is satisfied long enough, all horizons will grow to sizes bigger than those of the visible Universe. Our results remain in agreement with the above-quoted conclusions obtained by Madsen and Ellis (see Sec. V below). Although there can exist regions of the phase plane in which inflation takes place, and these regions can be of nonzero measure, phase trajectories leaving such regions need not lead to $\Omega\approx1$. It turns out that the probability that Ω will be measured to be nearly 1 depends on the observation time, and $\Omega\approx1$ is valid only for a restricted time interval. However, it can happen that there are trajectories leading to $\Omega\approx1$ which never passed through the inflation region. In this sense the state with $\Omega\approx1$ is not an attractor for the inflation phase.

This failure to explain the flatness problem with the help of multidimensional inflation will be illustrated by the behavior of the FLRW$\times T^D$ world model filled with radiation. The problem can be similarly discussed for the FLRW$\times S^D$ model and for other types of the energy-momentum tensor listed in Table I.

II. GENERAL FORMULAS

In [6] has been shown that the world models FLRW $(k=\pm1)\times T^D$ and FLRW $(k=0)\times S^D$ with the energy-momentum tensor $T^\mu_\nu=(\rho,m\rho,m\rho,m\rho,n\rho,\ldots,n\rho)$ can be reduced to the following two-dimensional dynamical system.

For FLRW $(k=\pm1)\times T^D$,

$$\begin{aligned}x'=\frac{dx}{d\tau}&=-\frac{2D+4-3\alpha}{D+2}x^2-\frac{D(D+2-3\alpha)}{D+2}xy\\&\quad+\frac{\alpha D(D-1)}{2(D+2)}y^2-\frac{2d+4-3\alpha}{D+2}K\,,\\y'=\frac{dy}{d\tau}&=\frac{3\beta}{D+2}x^2-\frac{2(D+2)-3\beta D}{D+2}xy\\&\quad-\frac{D[2D+4-\beta(D-1)]}{2(D+2)}y^2+\frac{3\beta}{D+2}\,.\end{aligned}\tag{1}$$

TABLE I. Energy-momentum tensor for particular cases.

	Energy density ρ	Values of parameters m	n	Cases
1	$\rho/(R^3 r^D)$	0	0	Dust matter
2	$\rho_0/(R^6 r^{2D})$	-1	-1	Massless scalar field
3	Λ	Λ	Λ	Cosmological constant in $D+3$ dimensions
4	$\rho_0/(R^{3k} r^{Dk})$	$-1/(D+3)$	$-1/(D+3)$	Radiation in $(D+3)$ dimensions
5	ρ_0/r^{2D}	1	-1	Field with the Freund-Rubin ansatz [7]
6	ρ_0/r^{D+4} and $\rho_0<0$	1	$-4/D$	Quantum effects of massless scalar field in low-temperature approximation [8]

In the physical region determined by the (0,0) component of Einstein's equations one has

$$\rho R^2=3x^2+3Dxy+3K+\frac{D(D-1)}{2}y^2 \begin{cases} \geq 0 & \text{for cases (1)–(5) in Table I,} \\ \leq 0 & \text{for case (6) in Table I ,} \end{cases} \tag{2}$$

where

$$x=R\frac{d}{dt}(\ln R), \quad y=R\frac{d}{dt}(\ln r) ,$$

R and r being the scale factors of the physical and internal spaces, respectively, $\alpha=1+(1-D)m$ and $\beta=-2n+3m+1$ are constants, K is the curvature constant of the physical space, and D is the dimension of the internal space. Along phase trajectories a new time parameter τ is determined, $dt=R\,d\tau$, which is a monotonic function of t.

For FLRW$(k=0)\times S^D$,

$$\begin{aligned} x'&=\frac{dx}{d\tau}=-\frac{3(D+4-\alpha)}{D+2}x^2-\frac{D^2-2-3\alpha D}{D+2}xy+\frac{\alpha D(D-1)}{2(D+2)}y^2-\frac{\alpha D(D-1)}{2(D+2)}K , \\ y'&=\frac{dy}{d\tau}=\frac{3\beta}{D+2}x^2-\frac{3(D+2)-3\beta D}{D+2}xy-\frac{2D(D+1)-4-\beta D(D-1)}{D+2}y^2+\frac{(D-1)(\beta D-D-2)}{D+2} , \end{aligned} \tag{3}$$

and, analogously to the previous case,

$$\rho r^2=3x^2+3Dxy+\frac{D(D-1)}{2}\left[y^2+\frac{k}{3}\right] \begin{cases} \geq 0, & \text{for cases (1)–(5) in Table I,} \\ \leq 0 & \text{for case (6) in Table I ,} \end{cases} \tag{4}$$

where

$$x=r\frac{d}{dt}(\ln R), \quad y=r\frac{d}{dt}(\ln r) ,$$

and k is the curvature constant of the D-dimensional maximally symmetric space.

Particular cases of the assumed energy-momentum tensor are shown in Table I, where $k=(D+4)/(D+3)$.

Vanishing of the energy-momentum tensor leads to the following relation determining the density parameter:

$$\rho=\frac{\rho_0}{R^{3(1-m)}r^{(1-n)D}} .$$

III. MULTIDIMENSIONAL INFLATION ON A PHASE PLANE

It will be convenient to construct phase portraits in projectile variables. For the right-hand sides of Eqs. (1) and (3), let us introduce the projective variables

$$z=\frac{1}{x} , \quad u=\frac{y}{x} \quad \text{and} \quad v=\frac{1}{y}, \quad w=\frac{x}{y} .$$

The dynamical system in (z,u), and (v,w) variables is equivalent to the original one provided $z\neq 0$ and $v\neq 0$. To infinitely distant points of the (x,y) plane correspond a one-sphere S^1 which can be covered by two lines $z=0$, for $-\infty<u<\infty$, and $v=0$, for $-\infty<w<\infty$. After the time transformation $\tau\to\tau_1$, $d\tau_1=x\,d\tau$, dynamical systems (1) and (3), in the projective variables (z,u), assume

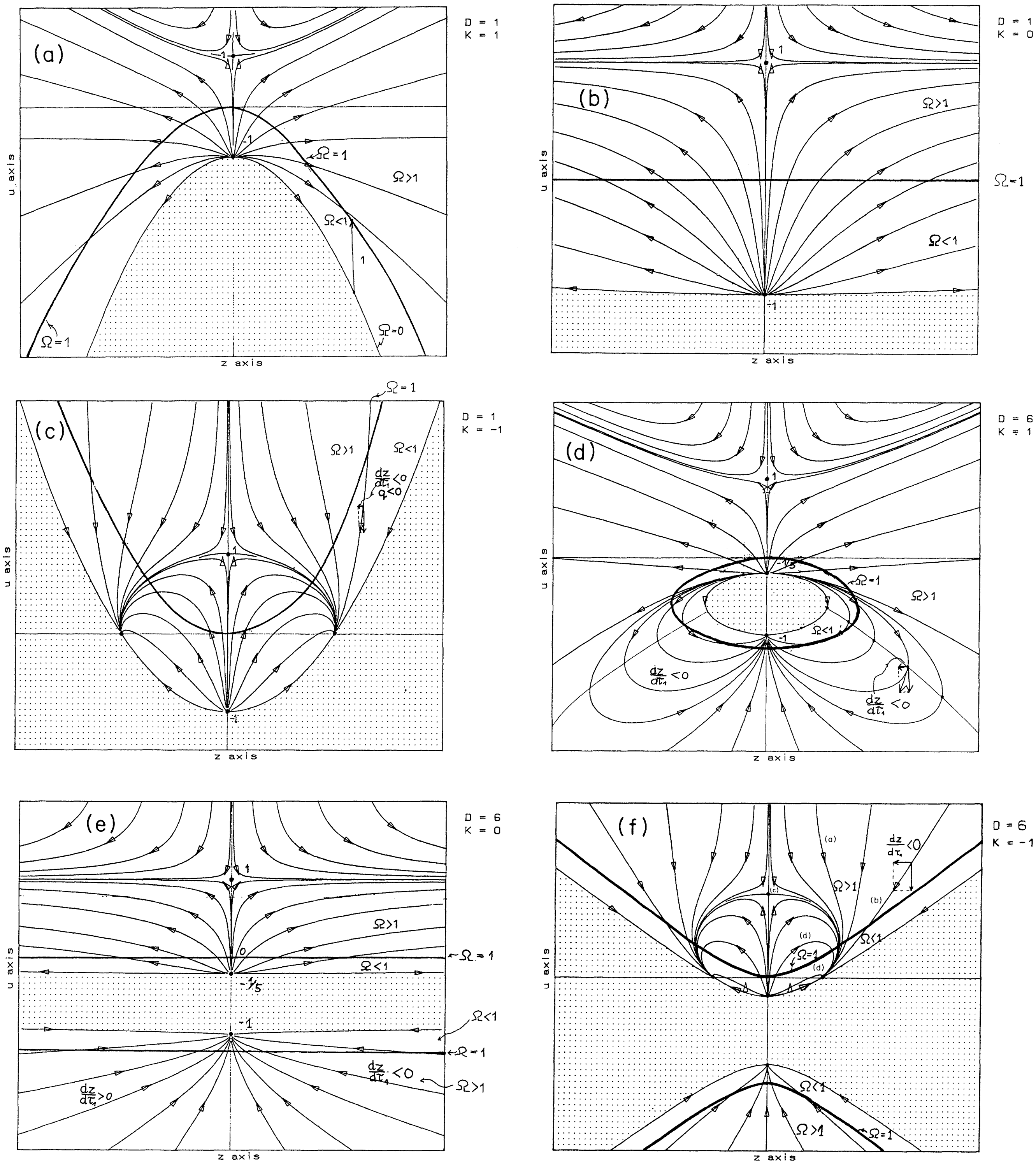

FIG. 1. Phase portraits of multidimensional cosmological models in projective variables (z,u) for different values of D and K. This choice of variables is useful for discussing properties of the system at infinity (on the Poincaré sphere). The enhanced line denotes the curve $\Omega=1$. Note that the phase trajectories cross this curve; in the case of $(1+3)$-dimensional FLRW world models the curve $\Omega=1$ is a phase trajectory on the plane (H,ρ), where H is the Hubble function and ρ is the energy density. The curve $\Omega=1$ is a boundary, shifted up by 1, of the nonphysical domain (excluded by the "constraint conditions") in which $\Omega=0$. The domain $\Omega>1$ is above this curve, and the domain $\Omega<1$ beneath it [(a)–(c)]. In (a)–(c) the curve $\Omega=$const is obtained by stretching the boundary of the nonphysical domain along the z axis by the value $\Omega=$const.

the form

$$\dot{z}=-zP^*(z,u)\ ,$$
$$\dot{u}=Q^*(z,u)-uP^*(z,u)\ , \tag{5}$$

where

$$P^*(z,u)=z^2P^*\left[\frac{1}{z},\frac{u}{z}\right]\ ,$$

$$Q^*(z,u)=z^2Q\left[\frac{1}{z},\frac{u}{z}\right]\ ,$$

and an overdot now denotes differentiation with respect to τ_1.

In the (v,w) variables, dynamical systems (1) and (3) assume the form analogical to that of (6), with the role of z now played by v and that of u by w. In Fig. 1 six phase portraits of system (1) are shown in the (z,u) variables. The topological structure of the phase plane, for a given K, is independent of the dimension $D>1$ of the internal space. The case $D=1$ is distinguished, which can be seen on the phase portraits. Let us notice that dynamical systems (1) and (3), in the (z,u) variables, exhibit the symmetry $z\longrightarrow -z$. We shall be interested in the region $z\geq 0$ (where physical space is noncontracting). Inflation occurs in these regions of the phase plane for which the following conditions are satisfied[1]: for (1),

$$\ddot{R}>0\Longleftrightarrow 0\frac{dx}{d\tau}>0\Longleftrightarrow -\frac{1}{3}\frac{dz}{d\tau_1}>0\ , \tag{6}$$

for (3),

$$\ddot{R}>0\Longleftrightarrow \frac{dx}{d\tau}>(xy-x^2)\Longleftrightarrow -\frac{1}{z^3}\frac{dz}{d\tau_1}>(u-1)\ . \tag{7}$$

From the phase portraits [Figs. 1(c)–1(f)] it can be seen that the regions for which condition (7) is satisfied are of zero measure, whereas in the remaining cases [Figs. 1(a) and 1(b)] they have a finite measure. For physical spaces with a zero or positive curvature, inflation takes place if the internal spaces expand to a constant size, provided $D\neq 1$; for the case $D=1$ such a relationship does not exist.

For the physical space of negative curvature with $D>1$, a double inflation [Fig. 1(f), trajectory a], or single inflation, for $-\infty<\tau<\infty$ [Fig. 1(f), trajectory b], or a preinflation [Fig. 1(f), trajectory c] can take place.

IV. DECELERATION AND DENSITY PARAMETERS

The deceleration parameter and density parameter can be expressed in terms of the variables (z,u).

For FLRW$\times T^D$,

$$q=-\frac{1}{H^2}\frac{\ddot{R}}{R}=-\left[\frac{dx}{d\tau}\right]x^{-2}=-\frac{dz}{d\tau}\quad \text{(see Fig. 3)}\ , \tag{8}$$

$$\frac{\rho}{3H^2}=\Omega=1+Du+Kz^2+\frac{D(D-1)}{6}u^2\quad \text{(see Fig. 2)}\ . \tag{9}$$

For FLRW$\times S^D$,

$$q=-\frac{dx/d\tau-xy+x^2}{x^2}=-\frac{dz}{d\tau}+u-1\ , \tag{10}$$

$$\frac{\rho}{3H^2}=\Omega=1+Du+Kz^2+\frac{D(D-1)}{6}(kz^2+u^2)\ , \tag{11}$$

where $H=(d/dt)(\ln R)$ is the Hubble function.

From (9) and (11) (Figs. 2 and 3) it can be seen that conditions (7) and (8) are equivalent to $q<0$. Formulas (10) and (12) are independent of the form of the energy-momentum tensor. Dynamical system (1), with (10) or (12), determines the evolution of the parameter Ω in multidimensional cosmology. One can also eliminate u from (10) and (11), and determine Ω from the dynamical system:

$$\frac{d\Omega}{d\tau_1}=P(\Omega,z)\ ,\qquad \frac{dz}{d\tau_1}=Q(\Omega,z)\ . \tag{12}$$

Such a dynamical system was used in Ref. [2] to investigate the evolution of the classical FLRW models. From the fact that it also can be done for multidimensional counterparts of these models, the full classification follows from multidimensional world models on the phase plane $(\Omega,\dot{R})$ or (q,R).

V. DISCUSSION

First, let us discuss the relationship of our results to those known for $(1+3)$-dimensional world models. In our case the physical space and the internal space are dynamically conjugated, and consequently the parameters Ω and q, as measured in the physical space, are modified by the dynamics of the internal space. If we put $u=0$ (the internal space is static) we *formally* obtain the transition to the $(1+3)$-dimensional case, in the sense that the corresponding dynamical systems are equivalent. The problem of the configuration FLRW$\times${static internal space} is known as the problem of dimensional dynamical reduction. For instance, such a reduction occurs in the class of models FLRW$(k=-1)\times T^D$ with the hydrodynamic energy-momentum tensor; however, the multidimensional dynamics does not always admit the solution FLRW$\times${static internal space, $u=0$}. In other words, there is no limiting transition, on the level of dynamics, from multidimensional models to classical ones. In the classical models, at the critical point $z_0=0$, $\Omega=1$, whereas in our case at $z_0=0$, $u=u_0$, and

[1]Whether or not conditions (6) and (7) are satisfied depends on the tangent vector $(dz/d\tau,du/d\tau_1)$ to the trajectory. In the case of condition (6) this fact depends only on the sign of $dz/d\tau_1$, which allows one to easily identify domains on the phase portrait in which this condition is not satisfied [in the case of condition (7) this is slightly more difficult].

$\Omega = \Omega(u_0)$. If $u_0 \to 0$ (internal space changes slowly), the dependence of u_0 upon Ω is weak, but $\Omega = 1$ is not a phase trajectory (had it been a trajectory, this would have meant a total dynamical disconnection of the physical and internal spaces).

Let us notice that FLRW world models with the hydrodynamic energy-momentum tensor have the first integral independent of the state equation: namely,

$$\rho - 3H^2 = \frac{3K}{R^2} ,$$

i.e.,

$$\Omega - 1 = \frac{K}{\dot{R}^2} .$$

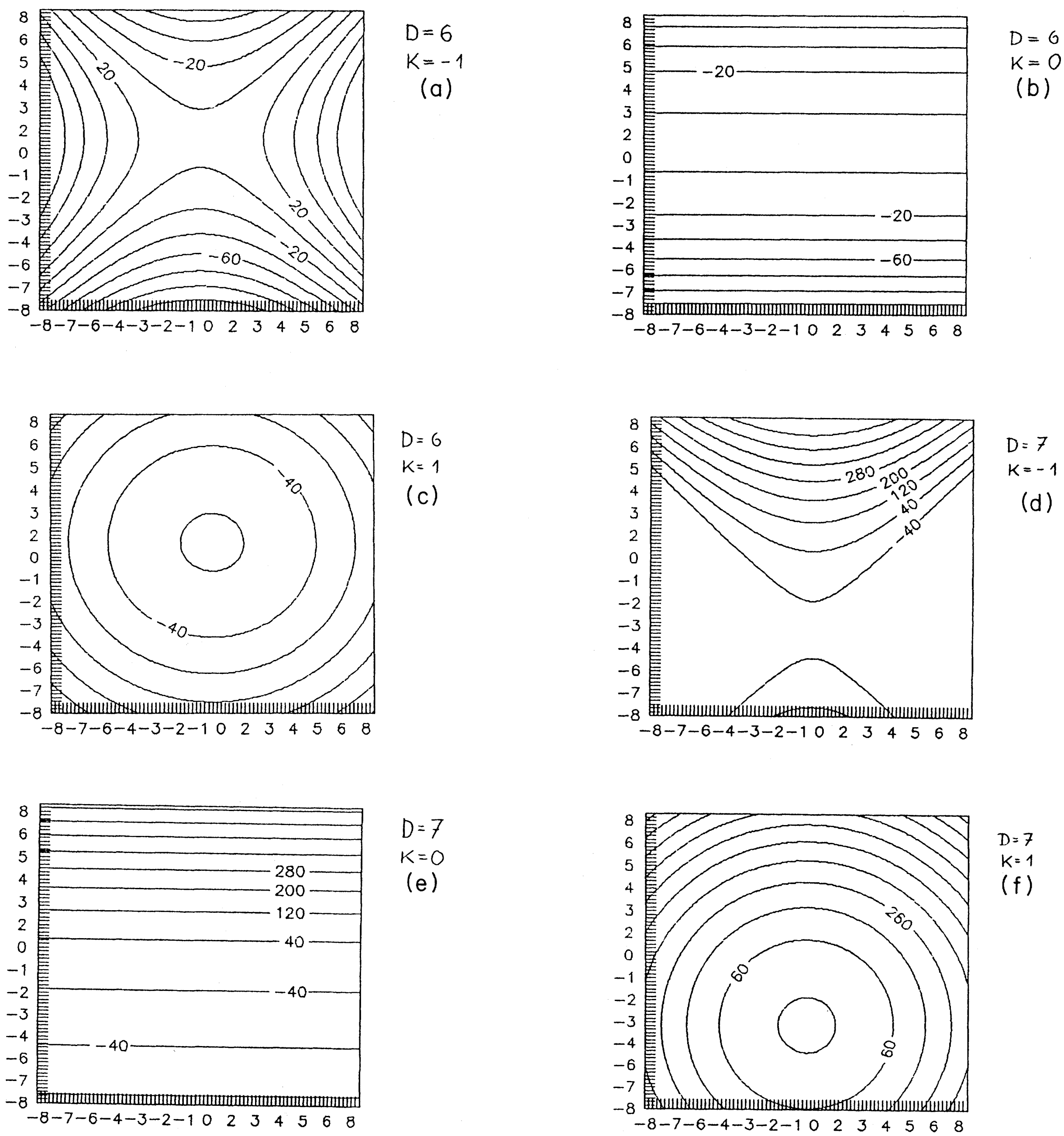

FIG. 2. Level surfaces of the parameter Ω [formula (9)] for different dimensions D of the internal space and curvatures of the physical space for multidimensional world models with topology $\mathrm{FLRW} \times T^D$.

At the critical point $z_0=0$, $\Omega=1$, and this critical point can be an attractor (this depends on the energy conditions). In our case, $\Omega=1$ is not a phase trajectory since the effects of extra dimensions cannot be modeled in terms of the hydrodynamical energy-momentum tensor.

For the reasons stated above, in the present work we discuss the behavior of phase trajectories in a neighborhood of the curve $\Omega=1$ (which is not a trajectory). However, this does not mean that the evolution of the parameter Ω and q in the physical space cannot be compared with that in the classical case. One can investigate to which extent the extra dimensions modify classical results. Therefore, let us compare our results with those obtained by Ellis and Madsen and Ellis [4].

(1) In both cases, there exists a nonzero measure set of initial conditions leading to inflation (understood exactly in the same sense). However, in our case this effect is due to the existence of extra dimensions, whereas in the case

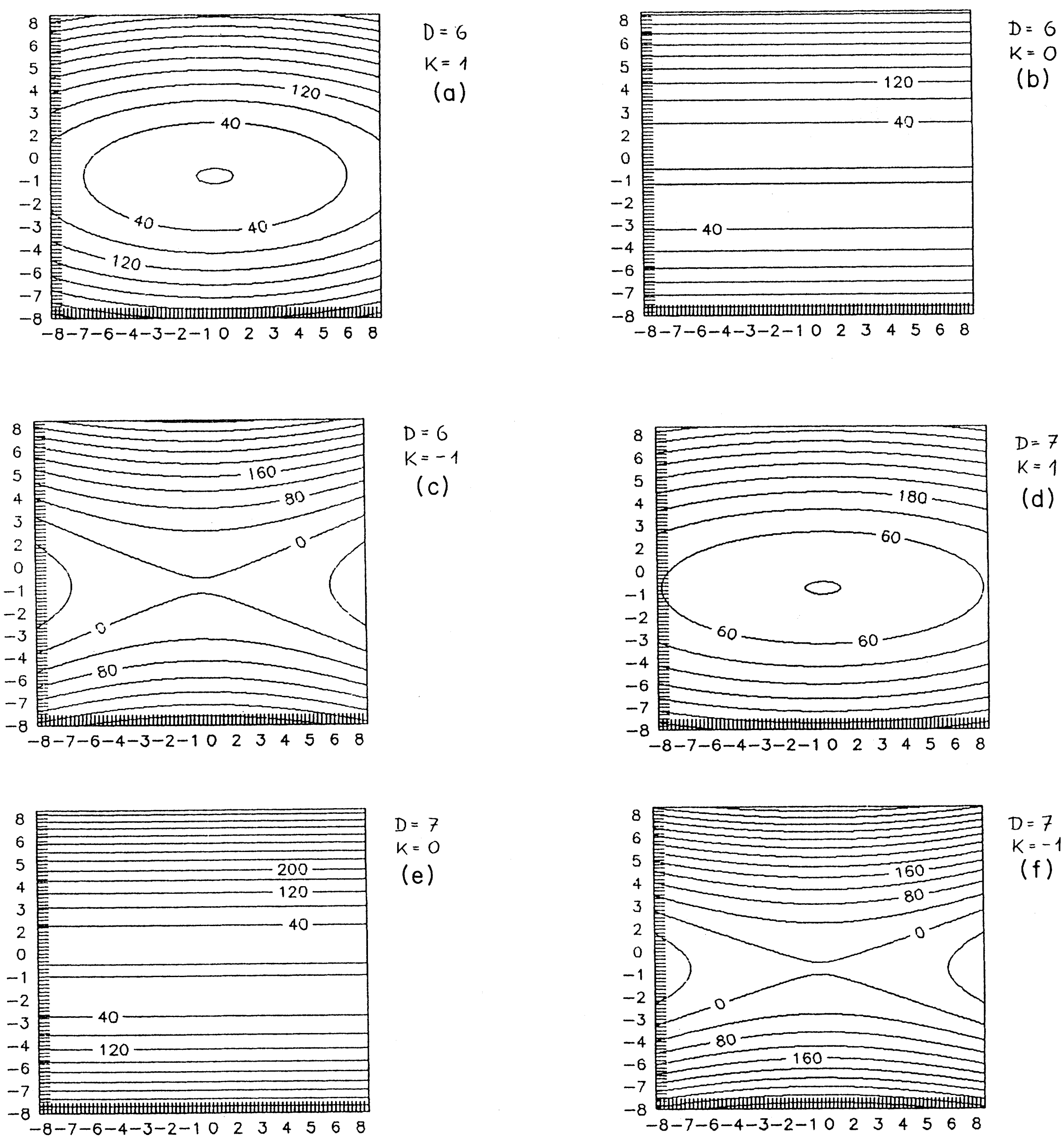

FIG. 3. Level surfaces of the deceleration parameter q [formula (8)] for different dimensions D of the internal space and curvatures of the physical space for multidimensional world models with topology FLRW$\times T^D$.

of Madsen and Ellis it is a pure effect of the hydrodynamic energy-momentum tensor.

(2) In both cases the value $\Omega \approx 1$ depends on the moment of observation (i.e., on the value of a parameter along the trajectory).

(3) In both cases, there exists a nonzero measure set of initial conditions leading to models with $\Omega \neq 1$ which remain in agreement with the actual observational data.

The essentially new effect that distinguished our case from the classical one is the existence of a mechanism which is responsible for an infinite series of inflationary phases.

To discuss the evolution of Ω from the phase diagrams in the (z,u) variables, the line $\Omega - 1 = 0$ is shown. This line can be determined from the condition $\rho = 0$. From (10) and (12) it follows that on the boundary $\rho = 0$ of the physical condition $\rho \geq 0$ one has $\Omega = 0$.

By a typical state of the metric in a neighborhood of the initial singularity we shall mean a repelling critical point such that the dimension of its invariant repelling manifold is equal to the dimension of the phase space. Analogously, by a typical state of the metric for $t \to \infty$ or for $t \to t_0$ (where t_0 is the final singularity) we shall mean an attracting critical point such that the dimension of its attracting invariant manifold is equal to the dimension of the phase space. From the phase portraits (Fig. 1) it can be seen that typical states of the metric are always situated on the boundary $\rho = 0$ of the physical condition $\rho \geq 0$. For the typical states of the metric for $t \to \infty$ or for $t \to t_0$, one has $\Omega = 0$ (and not $\Omega = 1$), i.e., the states with $\Omega = 0$ are attractors in the phase space.

Now, we shall turn our attention to particular models. World models with $D = K = 1$ do not undergo inflation, and Ω changes from zero to any arbitrary value. The state $\Omega = 1$ is reached by various trajectories at various times, i.e., whether or not an observer will see $\Omega = 1$ depends on the time the observation is performed.

Models with $D = 1$, $K = 0$ exhibit similar properties.

World models with $D = 1$, $K = -1$ undergo (a single) inflation along certain trajectories. Inflation leads to a constant size of the internal space. The state $\Omega = 1$ is reached by various trajectories at various times τ_1. Models with the internal space contracting and then expanding to a constant size reach twice the state $\Omega = 1$.

World models with $D = 6$, $K = 1$ have a nonzero measure set of trajectories which remain in the inflation region when the internal space contracts. The state $\Omega \approx 1$ is not an attractor, although it can be reached twice. Models with $D = 6$, $K = 0$ behave similarly.

World models with $D = 6$, $K = -1$ have a nonzero measure set of trajectories for which inflation takes place. The state $\Omega = 1$ is reached by trajectories along which the internal space either expands to a constant size [Fig. 1(f), trajectories a, b, c] or contracts and then expands to a constant size [Fig. 1(f), trajectory d].

Generally speaking, we can see that although the state $\Omega \approx 1$ is not an attractor in phase space, it can be reached by trajectories that do not pass through the inflation region. It can happen that the state $\Omega \approx 1$ is reached once or twice by trajectories for different time parameters. In the case when the internal space contracts, inflation does

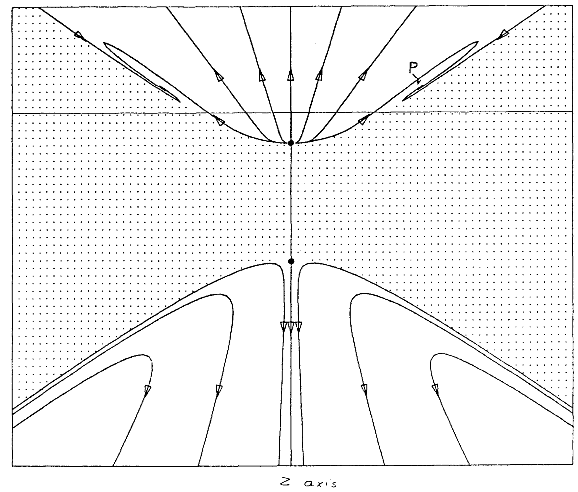

FIG. 4. Phase portraits of the world models $\text{FLRW}(k = 1) \times T^7$ with the Freund-Rubin ansatz. The critical point P is a stable focus. In its neighborhood phase trajectories enter into the inflation region infinitely many times.

guarantee that the state $\Omega \approx 1$ will be attained. For world models, in which there is a period during which the internal space expands, the region of the phase space corresponding to this period has a nonempty overlap with the inflation region [Figs. 1(c) and 1(f)] provided that $0.1 < \Omega < 10$ (see [4]) and $K = -1$. For $K = 0$ and $K = 1$, the assumption $0.1 < \Omega < 10$ does not imply that inflation took place in the past, i.e.,

$$\{\Omega\colon 0.1 < \Omega < 10\} \cap \{(z,u)\colon q < 0\} = \emptyset \ .$$

We have analyzed the evolution of Ω in FLRW $(k = \pm 1) \times T^D$ models with radiation. For all other cases of the energy-momentum tensor, listed in Table I, the behavior will be qualitatively the same (for this class of models) with the exception of the $\text{FLWR}(K = -1) \times T^D$ world model with the Freund-Rubin ansatz. In this model infinitely many inflation epochs occur which lead to the state $\Omega \approx 1$, and inversely, $0.1 < \Omega < 10$ for the present epoch implies that infinitely many inflations took place in the past. Let us notice that the typical state of the metric for this model is represented by a stable focus point P (Fig. 4). This example clearly shows that whether or not inflation solves the flatness problem in multidimensional cosmology depends on matter fields filling the model, and it does not depend on the number of extra dimensions (with the exception of the case $D = 1$). In models with the open physical space inflation solves the flatness problem.

ACKNOWLEDGMENTS

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