String instabilities in black hole spacetimes

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We study the emergence of string instabilities in D-dimensional black hole spacetimes (Schwarzschild and Reissner-Nordström), and de Sitter space (in static coordinates to allow a better comparison with the black hole case). We solve the first-order string fluctuations around the center-of-mass motion at spatial infinity, near the horizon, and at the spacetime singularity. We find that the time components are always well behaved in the three regions and in the three backgrounds. The radial components are *unstable:* imaginary frequencies develop in the oscillatory modes near the horizon, and the evolution is like $(\tau-\tau_0)^{-P}$ $(P>0)$ near the spacetime singularity $r\rightarrow 0$, where the world-sheet time $(\tau-\tau_0)\rightarrow 0$ and the proper string length grows infinitely. In the Schwarzschild black hole, the angular components are always well behaved, while in the Reissner-Nordström case they develop instabilities inside the horizon near $r\rightarrow 0$ where the repulsive effects of the charge dominate over those attractive of the mass. In general, whenever large enough repulsive effects in the gravitational background are present, string instabilities develop. In de Sitter space, all the spatial components exhibit instability. The infalling of the string to the black hole singularity is like the motion of a particle in a potential $\gamma(\tau-\tau_0)^{-2}$ where γ depends on the D spacetime dimensions and string angular momentum, with $\gamma>0$ for Schwarzschild and $\gamma<0$ for Reissner-Nordström black holes. For $(\tau-\tau_0)\rightarrow 0$ the string ends trapped by the black hole singularity.

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I. INTRODUCTION

The study of the string dynamics in curved spacetimes reveals new insights with respect to string propagation in flat spacetime (see for example Refs. [1–6]).

The equations of motion and constraints for strings in curved spacetimes are highly nonlinear (and, in general, not exactly solvable). In Ref. [1], a method was proposed (the "strong-field expansion") to study systematically (and approximately), the string dynamics in the strong curvature regime. In this method, one starts from an exact particular solution of the string equations in a given metric and then constructs a perturbative series around this solution. The space of solutions for the string coordinates is represented as

$$X^A(\sigma,\tau)=q^A(\sigma,\tau)+\eta^A(\sigma,\tau)+\xi^A(\sigma,\tau)+\cdots\ , \quad (1)$$

$A=0,\ldots,D-1$. Here $q^A(\sigma,\tau)$ is an exact solution of the string equations and $\eta^A(\sigma,\tau)$ obeys a linearized perturbation equation around $q^A(\sigma,\tau)$. $\xi^A(\sigma,\tau)$ is a solution of second perturbative order around $q^A(\sigma,\tau)$. Higher-order perturbations can be considered systematically. A physically appealing starting solution is the center-of-mass motion of the string, $q^A(\tau)$, that is, the point particle (geodesic) motion. The world-sheet time variable appears here naturally identified with the proper time of the center-of-mass trajectory. The spacetime geometry is treated *exactly,* and the string fluctuations around q^A are treated as perturbations. Even at the level of the zeroth-order solution, gravitational effects including those of the singularities of the geometry are fully taken into account. This expansion corresponds to low-energy excitations of the string as compared with the energy associated with the geometry. This corresponds to an expansion in powers of $(\alpha')^{1/2}$. Since $\alpha'=(l_{\rm Planck})^2$, the expansion parameter turns out to be the dimensionless constant

$$g=l_{\rm Planck}/R_c=1/(l_{\rm Planck}M)\ , \quad (2)$$

where R_c characterizes the spacetime curvature and M is its associated mass (the black hole mass, or the mass of a closed universe in cosmological backgrounds). The expansion is well suited to describe strings in strong gravitational regimes (in most of the interesting situations one has clearly $g\ll 1$). The constraint equations are also expanded in perturbations. The classical (mass)^2 of the string is defined through the center-of-mass motion (or Hamilton-Jacobi equation). The conformal generators (or world-sheet two-dimensional energy-momentum tensor) are bilinear in the fields $\eta^A(\sigma,\tau)$. [If this method is applied to flat spacetime, the zeroth-order plus the first-order fluctuations provide the exact solution of the string equations].

This method was first applied to cosmological (de Sit-

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ter) spacetimes. One of the results was that for a large enough Hubble constant, the frequency of the lower string modes, i.e., those with $|n| < \alpha' m H$ (α' being the string tension and m its mass), becomes imaginary. This was further analyzed [3,4] as the onset of a physical instability, in which the proper string size starts to grow (precisely like the expansion factor of the universe). The string modes couple with the background geometry in such a way that the string inflates with the Universe itself. The same happens for strings in singular gravitational planes waves [5,6] (see also Ref. [7]), and the results of this paper here show that this is a generic feature of strings near spacetime singularities.

For black hole spacetimes, such unstable features had not been yet explored. The string dynamics in black hole spacetimes is much more complicated to solve (even asymptotically and approximately). In Ref. [2], the study of string dynamics in a Schwarzschild black hole was started, and the scattering problem was studied for large impact parameters. The stable oscillatory behavior of the string was found for the transversal (angular) components; scattering amplitudes, cross section, and particle transmutation processes were described and explicitly computed in an expansion in $(R_S/b)^{D-3}$, R_S being the Schwarzschild radius and b the impact parameter. The aim of this paper (and of a subsequent one [8]), is to find, and then to describe, the *unstable sector* of strings in black hole backgrounds. By unstable behavior we mean here the following characteristic features: nonoscillatory behavior in time, or the emergence of imaginary frequencies for some modes, accompanied by an infinite stretching of the proper string length. In addition, the spatial coordinates (some of its components) can become unbounded. Stable string behavior means the usual oscillatory propagation with real frequencies (and the usual mode-particle interpretation), the fact that the proper string size does not blow up, and that the string modes remain well behaved.

We express the first-order string fluctuations η^μ $(\mu = 0, \ldots D-1)$ in D-dimensional Reissner–Nordström–de Sitter spacetime, as a Schrödinger-type equation for the amplitudes $\Sigma^\mu = q^R \eta^\mu$, q^R being the radial center-of-mass coordinate. We find the asymptotic behavior of the longitudinal and transverse string coordinates $(\Sigma^+, \Sigma^-, \Sigma^i)$ with $i = 2, \ldots, D-1$, at the spatial infinity, near the horizon and near the spacetime singularity. Plus and minus stand for the longitudinal (temporal and radial) components respectively, and i for the transverse (angular) ones. We analyze first a head-on collision (angular momentum $L=0$), that is, a radial infall of the string towards the black hole. Then, we analyze the full $L \neq 0$ situation. We consider Schwarzschild, Reissner-Nordström and de Sitter spacetimes (described here in static coordinates that allow a better comparison among the three cases). In all the situations (with and without angular momentum) and for the three cases we find the following results.

The time component Σ^+ is *always stable* in the three regions (near the infinity, the horizon and the singularity), and in the three cases (black holes and de Sitter spacetimes).

The radial component Σ^- is *always unstable* in the three regions and in the three backgrounds. In the Schwarzschild case, the instability condition for the radial modes, which develop imaginary frequencies near the horizon, can be expressed as

$$n < \frac{\alpha' m \sqrt{D-3}}{R_S} \left[D-2-\left[\frac{D-3}{2}\right]\frac{m^2}{E^2}\right]^2 , \qquad (3)$$

where α', m, and E are the string tension, string mass, and energy respectively. The quantity within the square brackets is always positive; thus the lower modes develop imaginary frequencies when the typical string size $\alpha' m \sqrt{D-3}$ is larger than the horizon radius. Notice the similarity with the instability condition in de Sitter space. $n < \alpha' m / r_H, r_H$ being the horizon radius.

In the Schwarzschild black hole, the transverse modes Σ^i are stable (well behaved) everywhere including the spacetime singularity at $q^R = 0$. In the Reissner-Nordström (RN) black hole, the transverse modes Σ^i are stable at infinity and outside the horizon. Imaginary frequencies appear, however, inside a region from $r_- < q^R < r_+$ to $q^R \to 0$, where $r_\pm = R_S[(1 \pm 4\beta)/2]^{1/(D-3)} R_S$ and β being given by Eqs. (5) and (34), respectively, in particular for $D=4$, $r_\pm = M \pm \sqrt{M^2 - Q^2}$, M and Q being the mass and charge of the black hole, respectively. For the extreme black hole ($Q=M$), instabilities do not appear. There is a critical value of the electric charge of a Reissner-Nordström black hole, above which the string passing through the horizon passes from an unstable to a stable regime. In the de Sitter spacetime, the only stable mode is the temporal one (Σ^+). All the spatial components exhibit instability, in agreement with the previous results in the cosmological context [1,3,4]. A summary of this analysis is given in Table I.

Imaginary frequencies in the transverse string coordinates (Σ^i) appear in the case in which the local gravity, i.e., $\partial_r a/2$, is negative (that is, repulsive effects). Here,

$$a(r) = 1 - (R_S/r)^{D-3} + (\tilde{Q}^2/r^2)^{D-3} + \frac{\Lambda}{3} r^2 , \qquad (4)$$

where

$$\tilde{Q}^{2(D-3)} = \frac{8\pi G Q^2}{(D-2)(D-3)} ,$$

and Λ is the cosmological constant. That is why the transverse modes (Σ^i) are well behaved in the Schwarzschild case, and outside the Reissner-Nordström event horizon. But close to $q^R \to 0$, $a'_{\mathrm{RN}} < 0$ (Reissner-Nordström spacetime has a repulsive inner horizon), and the gravitational effect of the charge overwhelms that of the mass; in this case instabilities develop. In the Reissner–Nordström–de Sitter spacetime, unstable string behavior appears far away from the black hole where the de Sitter solution dominates, and inside the black hole where the Reissner-Nordström solution dominates. For $M=0$ and $Q=0$, we recover the instability criterion [1,3] $\alpha' M \Lambda / 6 > 1$ for a large enough Hubble constant; this is in agreement with the criterion given in Ref. [9].

We find that in the black hole spacetimes, the transver-

TABLE I. Regimes of string stability in black hole and de Sitter spacetimes: Here stable means well-behaved string fluctuations and the usual oscillatory behavior with real frequencies. Unstable behavior corresponds to unbounded amplitudes ($\Sigma^{\pm},\Sigma^{i}$) with the emergence of nonoscillatory behavior or imaginary frequencies, accompanied by the infinite string stretching of the proper string length. Σ^{+}, Σ^{-}, and Σ^{i}, ($i=2,\ldots,D-1$), are the temporal, radial, and angular (or transverse) string components, respectively.

Region	Mode	Schwarzschild	Reissner-Nordström	de Sitter
$q^R\to 0$	Ξ^+	Stable	Stable	Stable
	Ξ^-	Unstable	Unstable	Unstable
	Ξ^i	Stable	Unstable	Unstable
$q^R\to q_H^R$	Ξ^+	Stable	Stable	Stable
	Ξ^-	Unstable	Unstable/Stable	Unstable
	Ξ^i	Stable	Stable	Unstable
$q^R\to\infty$	Ξ^+	Stable	Stable	Stable
	Ξ^-	Unstable	Unstable	Unstable
	Ξ^i	Stable	Stable	Unstable

sal first-order fluctuations (Σ^i) near the spacetime singularity $q^R=0$, obey a Schrödinger-type equation (with τ playing the role of a spatial coordinate), with a potential $\gamma(\tau-\tau_0)^{-2}$ (where τ_0 is the proper time of arrival to the singularity at $q^R=0$). The dependence on D and L is concentrated in the coefficient γ. Thus, the approach to the black hole singularity is like the motion of a particle in a potential $\gamma(\tau-\tau_0)^{-\beta}$, with $\beta=2$. And, then, like the case $\beta=2$ of strings in singular gravitational waves [5,6] (in which case the spacetime is simpler and the exact full string equations become linear). Here $\gamma>0$ for strings in the Schwarzschild spacetime, for which we have regular solutions Σ^i, while $\gamma<0$ for Reissner-Nordström, that is, in this case we have a singular potential and an unbounded behavior [negative powers in $(\tau-\tau_0)$] for $\Sigma^i_{\rm RN}$. The fact that the angular coordinates $\Sigma^i_{\rm RN}$ become unbounded means that the string makes infinite turns around the spacetime singularity and remains trapped by it.

For $(\tau-\tau_0)\to 0$, the string is trapped by the black hole singularity. In Kruskal coordinates $(u_k(\sigma,\tau),v_k(\sigma,\tau))$, for the Schwarzschild black hole we find

$$\lim_{(\tau-\tau_0)\to 0} u_k v_k=\exp[2KC(\sigma)(\tau-\tau_0)^P]\ ,$$

where $K=(D-3)/(2R_S)$ is the surface gravity, $P>0$ is a determined coefficient that depends on the D dimensions, and $C(\sigma)$ is determined by the initial state of the string. Thus $u_k v_k\to 1$ for $(\tau-\tau_0)\to 0$. The proper spatial string length at fixed $(\tau-\tau_0)\to 0$ grows like $(\tau-\tau_0)^{(D-1)P}$.

It must be noticed that in cosmological inflationary backgrounds, the unstable behavior manifests itself as nonoscillatory in $(\tau-\tau_0)$ [exponential for $(\tau-\tau_0)\to\infty$, powerlike for $(\tau-\tau_0)\to 0$]; the string coordinates η^i are constant (i.e., functions of σ only), while the proper amplitudes Σ^i grow like the expansion. In the black hole cases, and more generally, in the presence of spacetime singularities, all the characteristic features of string instability appear, but in addition the spatial coordinates η^i (or some of its components) become unbounded. That is, not only the amplitudes Σ^i diverge, but also the string coordinates η^i, which appears as a typical feature of the strings near the black hole singularities. A full description of the string behavior near the black hole singularity will be reported elsewhere [8]. This paper is organized as follows. In Sec. II we formulate the problem of string fluctuations in the Reissner–Nordström–de Sitter spacetime and express the equations of motion in a convenient Schrödinger-type equation. In Sec. III we treat the head-on collision. In Secs. IV and V we describe the full τ dependence and the noncollinear case, respectively, and discuss the conclusions and consequences of our results.

II. FORMUALTION OF THE PROBLEM

de Vega and Sánchez [1] have obtained the equations of motion of fundamental strings in curved backgrounds by expanding the fluctuations of the string around a given particular solution of the problem (for example, the center-of-mass motion). For the case of a black hole background [2] with mass M, charge Q, and cosmological constant Λ,

$$\begin{aligned} ds^2&=-a(r)(dX^0)^2+a^{-1}(r)dr^2+r^2d\Omega^2_{D-2}\ ,\\ a(r)&=1-(R_S/r)^{D-3}+(\tilde{Q}^2/r^2)^{D-3}+\frac{\Lambda}{3}r^2\ ,\\ R_S^{D-3}&=\frac{16\pi GM}{(D-2)\Omega_{D-1}},\quad \Omega_D=\frac{2\pi^{D/2}}{\Gamma(D/2)}\ ,\\ \tilde{Q}^{2(D-3)}&=\frac{8\pi GQ^2}{(D-2)(D-3)}\ , \end{aligned} \tag{5}$$

the equations of motion of the first-order fluctuations read

$$\left[\frac{d^2}{d\tau^2}+n^2\right]\eta^A_n(\tau)+2A^A_B(\tau)\dot{\eta}^B_n(\tau)+B^A_B(\tau)\eta^B_n(\tau)=0\ . \tag{6}$$

Here we have expanded the first-order string perturbations in a Fourier transform:

$$\eta^A(\sigma,\tau)=\sum_n e^{in\sigma}\eta^A_n(\tau)\ , \tag{7}$$

η^A being the vector

$$\eta=\begin{pmatrix}\eta^0\\ \eta^*\\ \eta^i\end{pmatrix},\quad i=2,3,\ldots,D-1 \tag{8}$$

and $A_B^A(\tau)$ and $B_B^A(\tau)$ the components of the matrices

$$A=\begin{pmatrix} -\frac{a'\dot{q}^R}{2a} & -\frac{\alpha' E a'}{2a} & 0\\ -\frac{\alpha' E a'}{2a} & -\frac{a'\dot{q}^R}{2a} & q^R\dot{q}^j\\ 0 & \frac{\dot{q}^i}{q^R}a & \frac{\dot{q}^R}{q^R}\delta^{ij}+q^i\dot{q}^j\end{pmatrix}, \tag{9}$$

$$B=\begin{pmatrix} 0 & -\frac{\alpha' E a''\dot{q}^R}{a} & 0\\ 0 & -S & 0\\ 0 & 2\dot{q}^i\left[\frac{a}{q^R}\right]^{\cdot} & \left[\frac{\alpha' L}{(q^R)^2}\right]^2\delta^{ij}\end{pmatrix}, \tag{10}$$

with

$$S=\frac{a''}{2a}[(q^R)^2+\alpha'^2E^2]=\left[\frac{\alpha' L}{(q^R)^2}\right]^2 a\ . \tag{11}$$

Here the dot stands for $\partial/\partial\tau$ and a prime denotes $\partial/\partial q^R$. The solution for the string coordinates is given by the expansion

$$X^\mu=q^\mu+\eta^\mu+\xi^\mu+\cdots,\quad \mu=0,1,\ldots,D-1\ , \tag{12}$$

$q^\mu(\tau)$ being the center-of-mass coordinates (zeroth-order solutions), which follow the geodesics of the background spacetime, i.e.,

$$\dot{q}^0=\frac{\alpha' E}{a},\quad (\dot{q}^i)^2=\left[\frac{\alpha' L}{(q^R)^2}\right]^2,$$
$$\frac{(\dot{q}^R)^2}{\alpha'^2 a}+\frac{L^2}{(q^R)^2}+m^2-\frac{E^2}{a}=0\ . \tag{13}$$

We have identified the proper time of the geodesic with the world-sheet τ coordinate.

To study the equation of motion of the first fluctuations, Eq. (6), it is convenient to apply a transformation to the vector η^A. Let us purpose

$$\eta_n=G\Xi_n\ , \tag{14}$$

where the matrix G is chosen to eliminate the term in the first derivative in Eq. (6), i.e.,

$$G=P\exp\left[-\int^\tau A(\tau')d\tau'\right]\ , \tag{15}$$

where P is a constant normalizing matrix. Thus, Eq. (6) transforms into

$$\ddot{\Xi}_n+G^{-1}(n^2+B-\dot{A}-A^2)G\Xi_n=0\ , \tag{16}$$

which is a Schrödinger-type equation with τ playing the role of the spatial coordinate.

III. HEAD-ON COLLISION

Let us start with the simpler case of a radial infall of a fundamental string towards a black hole. In this case the transversal components of the center-of-mass motion are zero, i.e., $q^i=\dot{q}^i=0$.

A. Transverse coordinates

They uncouple from the radial ones and from each others giving rise to the equation

$$\ddot{\eta}^i_n+n^2\eta^i_n+2\frac{\dot{q}^R}{q^R}\dot{\eta}^i_n=0\ . \tag{17}$$

By making the transformation

$$\Xi^i_n=q^R\eta^i_n\ , \tag{18}$$

Eq. (17) yields

$$\ddot{\Xi}^i_n+(n^2-\ddot{q}^R/q^R)\Xi^i_n=0\ . \tag{19}$$

And by use of the geodesics, Eqs. (13), we find

$$\ddot{\Xi}^i_n+\left[n^2+\frac{(\alpha' m)^2}{2}\frac{a'(q^R)}{q^R}\right]\Xi^i_n=0\ , \tag{20}$$

where a prime denotes $\partial/\partial q^R$.

From the form of this equation we can see that when the quantity in square brackets is greater than zero we will have the typical oscillatory motion of strings, but when the quantity in square brackets reaches negative values, this equation suggests the onset of instabilities. We can have imaginary frequencies only in the case in which the local gravity, i.e., $a'/2$, is negative (repulsive effects). We know that for Schwarzschild black holes we have

$$a_S'(q^R)=\frac{D-3}{q^R}\left[\frac{R_S}{q^R}\right]^{D-3}\ , \tag{21}$$

which is always greater than zero for $D>3$. Thus, we can conclude that the transversal modes are stable in the case of a radial infall towards a Schwarzschild black hole.

The case of a charged black hole gives the same result concerning the stability outside the event horizon, although, close to the singularity ($q^R\to 0$), we will have negative values of a' (it is known that the Reissner-Nordström solution has a repulsive inner horizon [10]). In fact,

$$a'_{\mathrm{RN}}(q^R)=\frac{D-3}{q^R}\left[\left[\frac{R_S}{q^R}\right]^{D-3}-2\left[\frac{\tilde{Q}}{q^R}\right]^{2(D-3)}\right]$$
$$\simeq -2\frac{D-3}{q^R}\left[\frac{\tilde{Q}}{q^R}\right]^{2(D-3)}\ . \tag{22}$$

We see that the onset of the instability appears within the event horizon. As the string falls towards the singularity $q^R\to 0$, the first mode ($n=1$), begins to suffer instabilities, then the second mode does too, and so on. In this Reissner-Nordström case, the presence of a second inner horizon (usually denoted as r_-), implies $a'<0$ from somewhere in the region $r_- < q^R < r_+$ to $q^R \to 0$, where

$$r_\pm = R_S \left[\frac{1 \pm \sqrt{1-4\beta}}{2} \right]^{1/(D-3)}$$

[see Eq. (34)]. Including a positive cosmological constant will again produce instabilities. In fact, for de Sitter space,

$$a'_{\rm dS}(q^R) = -\frac{\Lambda}{3} q^R . \tag{23}$$

And thus, in particular for $M=0$ and $Q=0$, we recover the results [1,3] about the onset of instabilities generated by the expansion of the de Sitter universe, for large enough cosmological constant, i.e., $\alpha' m \Lambda/6 > 1$. Obviously, the anti–de Sitter universe ($\Lambda<0$) will not generate instabilities. Thus, for the complete case of a Reissner-Nordström black hole immersed in a de Sitter universe, we will have the possibility of instabilities far from the black hole, where the cosmological solution dominates and inside the black hole, where the Reissner-Nordström solution dominates.

In the analysis above we have considered q^R as a parameter, but in general it will be τ dependent. However, in the approximation of first-order fluctuations we are considering we can take q^R as parametrizing the trajectory in an adiabatic approximation. We will see in the next section that this is indeed the case, when we include in detail the time dependence of q^R.

B. Radial coordinates

Let us study now the radial coordinates of the string. Here we have a coupled set of equations for the first-order perturbations η^0 and η^*. To eliminate the first time derivative appearing in the equations of motion (6), as we have seen, we can apply the matrix transformation G given by expression (15), which in our case, by use of the matrix $A(\tau)$, Eq. (9), takes the form

$$G = a^{1/2} \Bigg/ \left[\left| 1 - \left[\frac{\sqrt{E^2-m^2a}+E}{m} \right]^2 \right| \right]^{1/2} \begin{bmatrix} 1 & \dfrac{\sqrt{E^2-m^2a}+E}{m} \\ \dfrac{\sqrt{E^2-m^2a}+E}{m} & 1 \end{bmatrix} . \tag{24}$$

Here we have used the geodesic equations (13), as well as [11]

$$\int \frac{da}{a\sqrt{E^2-am^2}} = \frac{1}{E} \ln \left| \frac{\sqrt{E^2-m^2a}-E}{\sqrt{E^2-m^2a}+E} \right| . \tag{25}$$

It is not difficult to compute $\dot{A}$ and A^2 from expression (9). Thus, we have all the elements to write down explicitly the first-order perturbations equations (16):

$$\ddot{\Xi}_n + M \Xi_n = 0, \quad M = G^{-1}(n^2 + B - \dot{A} - A^2) G ; \tag{26}$$

the matrix M is given explicitly by

$$M = \frac{(\alpha')^2}{2(1-g^2)} \begin{bmatrix} c-e+n^2 & d-ge \\ d+e & c+ge+n^2 \end{bmatrix} , \tag{27}$$

where

$$c = [(2E^2-am^2)(a'/a)^2 + E^2(a''/a)](g^2-1) ,$$
$$d = \{E\sqrt{E^2-am^2}[2(a'/a)^2+(a''/a)]\}(g^2-1) , \tag{28}$$
$$e = (a''/a)[2E\sqrt{E^2-am^2} - g(2E^2-am^2)] ,$$

and

$$g = \frac{\sqrt{E^2-m^2a}+E}{m} . \tag{29}$$

Now, Eq. (26) represents a set of two coupled equations for the components of the vector

$$\Xi_n = \begin{bmatrix} \Xi_n^0 \\ \Xi_n^1 \end{bmatrix} . \tag{30}$$

In order to uncouple and better analyze Eq. (26), we apply the following unitary transformation, which diagonalizes the matrix M:

$$\tilde{M} = TMT^{-1} .$$

Thus, $\tilde{M}$ reads

$$\tilde{M} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} , \tag{31}$$

where the eigenvalues $\lambda_\pm$ are given by

$$\lambda_\pm = \tfrac{1}{2}\{D + A \pm [4BC + (D-A)^2]^{1/2}\} + n^2 , \tag{32}$$

and we have written, for the matrix M in Eq. (27),

$$M \doteq \begin{bmatrix} A & B \\ C & D \end{bmatrix} .$$

Thus, explicitly the eigenvalues (32) are given by

$$\lambda_\pm = \frac{(\alpha')^2}{4}[2(a'/a)^2(2E^2-am^2) + m^2a'' \pm \sqrt{16E^2(E^2-am^2)(a'/a)^4 + m^4(a'')^2}] + n^2 . \tag{33}$$

Now, this diagonalized expression allows us to analyze the stability of the fluctuations around the center of mass of the string. By looking for negative values of λ_+, we can scan the possibility of the onset of instabilities in the motion of strings. The analysis of Eq. (33) for every value of q^R, can be made by replacing there the expression for $a(q^R)$ given by Eq. (5).

It is convenient, in the Reissner–Nordström–(de Sitter) case, to write expressions in terms of the following variable

$$x^{-1} \doteq \left[\frac{R_S}{q^R}\right]^{D-3}, \quad \beta \doteq \left[\frac{\tilde{Q}}{R_S}\right]^{2(D-3)} . \tag{34}$$

Thus, Eq. (5) reads

$$a(x) = 1 - \frac{1}{x} + \frac{\beta}{x^2} + \left[-\frac{\Lambda}{3}(q^R)^2\right] . \tag{35}$$

It can be seen that, for $m=0$,

$$\lambda_+ = n^2 + 2\alpha'^2 \left[\frac{a'}{a}\right]^2 E^2 ,$$
$$\lambda_- = n^2 ; \tag{36}$$

that is, for $m=0$ the evolution is *stable* in the three cases: Schwarzschild, Reissner-Nordström, and de Sitter spacetimes. λ_+ (and then Σ^0) is always stable (this is so, even for $m \neq 0$) and λ_- is as in flat spacetime; the first-order fluctuations for the massless string always oscillate.

C. Analysis

To simplify the analysis, we will see what happens in three important regions of the black hole space: far away from the black hole, close to the event horizon, and approaching the singularity in the Schwarszchild, Reissner-Nordström, and de Sitter spacetimes.

(i) $q^R \to \infty$. In this case the terms proportional to a'' dominate over those proportional to $(a'/a)^2$ and thus we have

$$\lim_{q^R \to \infty} \lambda_\pm^\infty = \frac{(\alpha')^2}{4}[2(a'/a)^2(2E^2 - am^2) + m^2(a'' \pm |a''|)] + n^2 . \tag{37}$$

Thus, from expressions (34) and (35), we can conclude that the modes with coefficient λ_+, being definite positives, are stable in Schwarzschild and Reissner-Nordström black holes, i.e.,

$$\lim_{q^R \to \infty} \lambda_+^{\rm BH} = \frac{(\alpha')^2}{2}(D-3)^2 \frac{2E^2 - m^2}{(q^R)^2 x^2} + n^2 , \tag{38}$$

as well as in de Sitter spacetime, i.e.,

$$\lim_{q^R \to \infty} \lambda_+^{\rm ds} = 2(\alpha')^2 \left[\frac{m^2 \Lambda}{3} + \frac{2E^2 - m^2}{(q^R)^2}\right] + n^2 . \tag{39}$$

The other eigenvalue λ_-, however, would indicate the emergence of an instability for not so large values of q^R. In fact, in this case we have, for the Schwarzschild and Reissner-Nordström black holes,

$$\lim_{q^R \to \infty} \lambda_-^{\rm BH} = \frac{(\alpha')^2}{2}(D-3)(D-2)\frac{m^2}{(q^R)^2 x} + n^2 . \tag{40}$$

And for the de Sitter case,

$$\lim_{q^R \to \infty} \lambda_-^{\rm dS} = -\frac{(\alpha')^2}{3} m^2 \Lambda + n^2 . \tag{41}$$

We observe that in λ_- appears a fundamental difference with respect to λ_+: The term proportional to m^2 is here negative. This allows the possibility of the onset of instabilities for values of m^2 large enough to unstabilize successively the modes $n=1,2,\ldots$ and so on. In the case of the Schwarzschild or Reissner-Nordström black holes, for large values of q^R, we have stability, even for the λ_- modes, as one would have expected due to the asymptotic flatness of the spacetime.

(ii) $q^R \to q^R_{\rm horizon}$. In this case the terms proportional to $(a'/a)^2$ dominate over those proportional to a'', and we have

$$\lim_{q^R \to q^R_H} \lambda_\pm^H = n^2 + \frac{(\alpha')^2}{4}[2(a'/a)^2(2E^2 - am^2 \pm 2E\sqrt{E^2 - am^2}) + m^2 a''] . \tag{42}$$

Again, we will see that the modes λ_+ yield stable fluctuations around the center of mass. In fact, for black holes and de Sitter cases,

$$\lim_{q^R \to q^R_H} \lambda_+^H = n^2 + (\alpha')(a'/a)^2(2E^2 - am^2) , \tag{43}$$

which is always a positive quantity.

The other mode λ_- carries again the possibility for the emergence of instabilities. For the Schwarzschild black hole we have

$$\lim_{q^R \to q^R_H} \lambda_-^S = n^2 + \frac{(\alpha')^2(D-3)}{4}\frac{m^2}{R_s^2}\left[2 - D + \frac{D-3}{2}\frac{m^2}{E^2}\right]$$
$$= n^2 - \frac{(\alpha')^2 m^2}{2R_s^2}\left[1 - \frac{m^2}{4E^2}\right] \quad \text{for } D=4 . \tag{44}$$

As the term between the square brackets is always negative for $D \geq 3$, this indicates that we have the possibility of an unstable regime (depending on the value of m^2/R_s^2) for the first excited modes. The instability condition can be written as

$$n < \frac{S}{R_S}\left[D - 2 - \frac{D-3}{2}\frac{m^2}{E^2}\right]^{1/2} , \tag{45}$$

where the quantity between the square brackets is always positive and $S = \alpha' m \sqrt{D-3}$ is a measure of the string size. Thus, if the string is larger than the horizon radius, it becomes unstable. This is similar to the instability criterion in de Sitter space, i.e., $n < \alpha' m / r_H$, $r_H = H^{-1}$ being the horizon radius.

This is not the case, however, for the extreme Reissner-Nordström black hole. In fact,

$$\lim_{q^R\to q_H^R} \lambda_-^{\rm ERN}=(\alpha')^2 2^{(D+1)/(3-D)}(D-3)^2\frac{m^2}{R_S^2}+n^2 \ . \qquad (46)$$

which is always a positive quantity and does not develop instabilities at first perturbative order. This shows that there is a critical value of the electric charge of the Reissner-Nordström black hole above which the string on the horizon passes from the unstable to the stable regime. This critical value can be found by making vanish λ_-^H in Eq. (42).

For the de Sitter space

$$\lim_{q^R\to q_H^R} \lambda_-^{\rm dS}=\frac{(\alpha')^2}{6}m^2\Lambda\left[\frac{m^2}{E^2}-1\right]+n^2 \ . \qquad (47)$$

As the quantity between the parentheses is always negative for the radial orbits we are studying, we could have the onset of instabilities, again this depending on having large enough values of $m^2\Lambda$.

(iii) $q^R\to 0$. This limit gives the approach to the singularity. Here, terms proportional to a'' dominate over those proportional to $(a'/a)^2$. We have, then, in this limit

$$\lim_{q^R\to 0} \lambda_\pm^0=\frac{(\alpha')^2}{4}[2(a'/a)^2(2E^2-am^2) +m^2(a''\pm|a''|)]+n^2 \ . \qquad (48)$$

We will see, again, that as $q^R\to 0$, λ_+^0 remains positive, thus giving stable first-order fluctuations.

In fact, for Schwarzschild black holes

$$\lim_{q^R\to 0} \lambda_+^S=\frac{(\alpha')^2}{2}\frac{(D-3)^2m^2}{(q^R)^2x}+n^2 \qquad (49)$$

which is always positive. Thus, producing stable first-order fluctuations.

The same happens for the Reissner-Nordström black hole:

$$\lim_{q^R\to 0} \lambda_+^{\rm RN}=\frac{(\alpha')^2}{2}\frac{(D-3)(4D-10)\beta m^2}{(q^R)^2x^2}+n^2 \ , \qquad (50)$$

and also for the case of the de Sitter space,

$$\lim_{q^R\to 0} \lambda_+^{\rm dS}=\tfrac{2}{9}(\alpha')^2\Lambda^2(q^R)^2(2E^2-m^2)+n^2 \ , \qquad (51)$$

the first-order oscillations will be bounded.

Again, the situation changes when we study the mode λ_-. For the Schwarzschild black hole we have

$$\lim_{q^R\to 0} \lambda_-^S=-\frac{(\alpha')^2}{2}\frac{(D-3)(D-2)m^2}{(q^R)^2x}+n^2 \ , \qquad (52)$$

which suggests the onset of instabilities. We remark here about the similarity between this expression and Eq. (40) for the case $q^R\to\infty$.

The Reissner-Nordström case yields also the possibility of instabilities:

$$\lim_{q^R\to 0} \lambda_-^{\rm RN}=-2(\alpha')^2\frac{(D-3)^2\beta m^2}{(q^R)^2x^2}+n^2 \ . \qquad (53)$$

And finally, the de Sitter space-time also gives an unstable solution:

$$\lim_{q^R\to 0} \lambda_-^{\rm dS}=-\tfrac{1}{3}(\alpha')^2\Lambda m^2+n^2 \ . \qquad (54)$$

We remark again the similarity with the result Eq. (41), valid for $q^R\to\infty$.

Several remarks are worth making here. First, we observe from Table I that the longitudinal modes λ_+ always give stable fluctuations, while the λ_- modes almost always suggest the existence of instabilities. By choosing an appropriate gauge in the Ξ^0 and Ξ^1 coordinates, we can interpret the mode λ_+ as the temporal coordinate and the mode proportional to λ_- as the radial coordinate. As for the cosmological backgrounds [3], the string time coordinate is well behaved.

We have already remarked that in the Schwarzschild and de Sitter spacetimes the modes λ_- and Ξ^i are quite similar in the limit $q^R\to 0$ and $q^R\to\infty$. This property has also been found in cosmological backgrounds [12].

We have studied separately the cases of black holes and de Sitter space. We have done so for the sake of simplicity and for the importance of de Sitter space in itself, but it is straightforward to observe the regimes in which one case predominates over the other when we study a black hole embedded in the de Sitter spacetime: At large distances from the hole, the cosmological term dominates, while not far from the black hole its contribution can be neglected.

Here, as remarked in Refs. [1,3], in order to discuss the constraints imposed to the string equations of motion, one has to go to second-order string fluctuations. To first order, constraints are satisfied consistently with the equations of motion only for stable modes.

It is important to stress that the perturbative analysis of the equations of motion we have done, is strictly valid in the stable regimes and allows us to discover the presence of instabilities. In the unstable cases, one has to describe the unstable nonlinear regime, nonperturbatively. Asymptotic solutions describing the highly unstable string regime are under study by the present authors and will be published elsewhere [8].

IV. EVOLUTION WITH TIME

In the last section we have considered q^R, the coordinate of the center of mass as a parameter. This allowed us to analyze the stability of the fluctuations of fundamental strings. Now, in order to see the evolution with the proper time τ, let us consider $q^R(\tau)$ and integrate the resulting differential equation for $\Xi(\tau)$:

$$\ddot{\Xi}_n^\mu+[n^2+\lambda^\mu]\Xi_n^\mu=0 \ , \qquad (55)$$

where

$$\Xi_n^\mu=(\Xi_n^+,\Xi_n^-,\Xi_n^i) \ , \quad \lambda^\mu=(\lambda^+,\lambda^-,\lambda^i) \ , \qquad (56)$$

and

$$\lambda^\pm=\lambda_\pm-n^2, \quad \lambda^i=\frac{(\alpha' m)^2}{2}\frac{a'(q^R)}{q^R} \ . \qquad (57)$$

We can obtain the time dependence of q^R from the geodesics equations, (13). Thus,

$$\alpha'(\tau-\tau_0)=\int_{q_0^R}^{q^R}\frac{dq^R}{\sqrt{E^2-m^2a(q^R)}} . \tag{58}$$

Let us now study the different curved backgrounds we are interested in.

A. de Sitter space

This case is the simplest for analyzing, due to the constancy of λ^i:

$$\lambda^i_{\rm dS}=-\frac{(\alpha' m)^2}{6}\Lambda . \tag{59}$$

Thus, Eq. (55) for the transverse modes can be easily solved in terms of exponentials: i.e.,

$$\Xi^i_n(\tau)=\exp\left\{\pm i\left[n^2-\frac{(\alpha' m)^2}{6}\Lambda\right]^{1/2}\tau\right\} . \tag{60}$$

We see that for a cosmological constant positive and bigger than $6/(\alpha' m)$, the first mode $\Xi^i_1(\tau)$ will begin to grow exponentially with time. For even bigger values of Λ further modes can be excited. Negative values of Λ (anti–de Sitter space) give bounded fluctuations.

The analysis for the longitudinal modes is quite more complicated, because of the time dependence of $\lambda^{\pm}_{\rm dS}$. Thus, we will consider the two asymptotic regions $q^R\to 0$ and $q^R\to\infty$.

$q^R(\tau)$ can be obtained explicitly by integrating [11] Eq. (58),

$$\alpha'(\tau-\tau_0)=\begin{cases}\dfrac{1}{m}\left[\dfrac{3}{\Lambda}\right]^{1/2}\operatorname{arcsinh}\left[\dfrac{m}{\sqrt{E^2-m^2}}\left[\dfrac{\Lambda}{3}\right]^{1/2}q^R\right] & \text{for } \Lambda>0,\\[2ex] -\dfrac{1}{m}\left[\dfrac{3}{-\Lambda}\right]^{1/2}\arcsin\left[\dfrac{m}{\sqrt{E^2-m^2}}\left[\dfrac{-\Lambda}{3}\right]^{1/2}q^R\right] & \text{for } \Lambda<0 ,\end{cases} \tag{61}$$

where τ_0 is the proper time of arrival to $q^R=0$.

In the limit $q^R\to\infty$, the eigenvalues $\lambda^{\pm}_{\rm dS}$ are given by Eqs. (39) and (41) and thus, have constant values:

$$\lambda^+_{\rm dS}\to\frac{2(\alpha' m)^2}{3}\Lambda , \qquad \lambda^-_{\rm dS}\to-\frac{(\alpha' m)^2}{3}\Lambda . \tag{62}$$

With these constant values, Eq. (55) can be easily solved for the $\Xi^{\pm}$ coordinates, again in terms of exponentials. The same analysis and the same condition of instability as for the transversal modes apply here for λ^-. This is so because we have negative values of λ^- for $\Lambda>0$. For λ^+, instead, we have bounded solutions.

In the limit $q^R\to 0$ we must use expressions (51) and (54). Again λ^- gives a constant negative value, which indicates the emergence of instability. Ξ^- behaves like

$$\Xi^-_{n,{\rm dS}}\to\exp\left\{\pm i\left[n^2-\frac{(\alpha' m)^2}{3}\Lambda\right]^{1/2}\tau\right\} . \tag{63}$$

On the other hand, from Eqs. (51) and (61), we find that

$$\lambda^+_{\rm dS}\simeq\tfrac{2}{9}(\alpha')^4\Lambda^2(E^2-m^2)(\tau-\tau_0)^2\doteq C(\tau-\tau_0)^2 \tag{64}$$

which inserted into Eq. (55) gives a regular behavior for Ξ^+_n in terms of a power series of $(\tau-\tau_0)$, i.e.,

$$\Xi^+_{n,{\rm dS}}(\tau)=\sum_{j=0}^{\infty}K_j(\tau-\tau_0)^j . \tag{65}$$

Here K_j can be found from the recursive formula

$$(j+2)(j+1)K_{j+2}+n^2K_j+CK_{j-2}=0 , \qquad j=0,1,2\ldots,\quad K_j=0 \text{ for } j<0 , \tag{66}$$

and $K_0=\Xi_0$ and $K_1=p^+$ are the initial data.

B. Schwarzschild black hole

This case is somewhat more complicated algebraically than the de Sitter one, but it can be analyzed in the two interesting asymptotic regions $q^R\to 0$ and $q^R\to\infty$.

When $q^R\to 0$, the geodesic path can be approximated by

$$q^R_S(\tau)\to\left[\frac{(D-1)}{2}\alpha' R_S^{(D-3)/2}m(\tau-\tau_0)\right]^{2/(D-1)} , \qquad q^0_S(\tau)\sim(\tau-\tau_0)^{2(D-1)/(D+1)}\to 0 , \tag{67}$$

where τ_0 is the proper time of arrival to the singularity at $q^R=0$.

Replacing this path, $q^R_S(\tau)$, into Eq. (55), we have for the transversal coordinates

$$\ddot{\Xi}^i_n+\left[n^2+\frac{2(D-3)}{(D-1)^2}(\tau-\tau_0)^{-2}\right]\Xi^i_n=0 , \tag{68}$$

for which we find the power-law solution

$$\Xi^i_{n,S}(\tau)\sim[n(\tau-\tau_0)]^P,\quad P^i_S=\frac{1}{2}\pm\left[\frac{1}{4}-\frac{2(D-3)}{(D-1)^2}\right]^{1/2} . \tag{69}$$

For $D\geq 4$, P is always real and positive. Thus, in the regime studied, $(\tau-\tau_0)$ small, we have regular and bounded solutions Σ^i. This confirms that there are not instabilities in this region.

From expressions (49) and (52), we observe that $\lambda^{\pm}$ produce equations analogous to Eq. (68) and powerlike solutions as in expression (69). Only the value of P changes:

$$P_S^+=\frac{1}{2}\pm\left[\frac{1}{4}-2\left[\frac{D-3}{D-1}\right]^2\right]^{1/2} . \tag{70}$$

This gives complex solutions for $D>5$, but anyway bounded as $(\tau-\tau_0)\rightarrow 0$.

On the other hand,

$$P_S^-=\frac{1}{2}\pm\left[\frac{1}{4}+\frac{2(D-3)(D-2)}{(D-1)^2}\right]^{1/2} \tag{71}$$

gives a solution, which with the minus sign in front of the square root is unbounded as $(\tau-\tau_0)\rightarrow 0$. Thus, being consistent with the results of the last section [see comments made after Eq. (52) about the possibility of having instabilities in this longitudinal mode of the string].

The same kind of analysis can be made in the region $q^R\rightarrow\infty$. The geodesic equations in this case yields

$$q_S^R(\tau)\rightarrow\alpha'\sqrt{E^2-m^2}(\tau-\tau_0) . \tag{72}$$

Thus, the equation for the transversal modes is

$$\ddot{\Xi}{}_n^i+[n^2+D^i(\tau-\tau_0)^{1-D}]\Xi_n^i=0 , \tag{73}$$

where

$$D^i=\frac{D-3}{2}\frac{m^2}{(E^2-m^2)}\left[\frac{R_s}{\alpha'\sqrt{E^2-m^2}}\right]^{D-3}$$

while for the longitudinal modes we have

$$\ddot{\Xi}{}_n^-+[n^2+D^-(\tau-\tau_0)^{1-D}]\Xi_n^-=0 , \tag{74}$$

with

$$D^-=-\frac{D-3}{2}\frac{(D-2)m^2}{(E^2-m^2)}\left[\frac{R_S}{\alpha'\sqrt{E^2-m^2}}\right]^{D-3}$$

and

$$\ddot{\Xi}{}_n^++[n^2+D^+(\tau-\tau_0)^{4-2D}]\Xi_n^+=0 , \tag{75}$$

where

$$D^+=\frac{(D-3)^2}{2}(\alpha')^{6-2D}\frac{2E^2-m^2}{(E^2-m^2)^{D-2}}R_S^{2(D-3)} .$$

Equations (73)–(75) can be solved in terms of Laurent series [negative power series of $(\tau-\tau_0)$], that in the limit $(\tau-\tau_0)\rightarrow\infty$ makes them convergent solutions. Only the negative value in D^- suggests that for $(\tau-\tau_0)$ not so large, some kind of instabilities could occur.

C. Reissner-Nordström black hole

For charged black holes, in the limit of $q^R\rightarrow 0$, the gravitational effect of the charge overwhelms that of the mass. In that limit, from Eqs. (5) and (58), we obtain

$$q^R(\tau)\rightarrow[\alpha'(D-2)m(\tau-\tau_0)\tilde{Q}^{D-3}]^{1/(D-2)} . \tag{76}$$

Plugging this expression into Eq. (55) and using (22) and (57), we obtain

$$\ddot{\Xi}{}_n^i+\left[n^2-\frac{D-3}{(D-2)^2}(\tau-\tau_0)^{-2}\right]\Xi_n^i=0 , \tag{77}$$

this is as Eq. (68) for the transversal oscillations in a Schwarzschild black hole, but the coefficient in front of $(\tau-\tau_0)^{-2}$ now has a negative value. This means that the solution

$$\Xi_{n,\mathrm{RN}}^i(\tau)\sim[n(\tau-\tau_0)]^P,\quad P_{\mathrm{RN}}^i=\frac{1}{2}\pm\left[\frac{1}{4}+\frac{D-3}{(D-1)^2}\right]^{1/2} , \tag{78}$$

allows a solution (that with minus sign in front of the square root) which is unbounded as $(\tau-\tau_0)\rightarrow 0$, and indicates an instability as seen in the analysis of the preceding section.

From Eqs. (50) and (53) for $\lambda^{\pm}$ we see that the longitudinal coordinates of the string in the Reissner-Nordström background, as they fall toward the singularity at $r=0$, will behave like a power $P_{\pm}$ of the proper time:

$$P_{\mathrm{RN}}^+=\frac{1}{2}\pm\left[\frac{1}{4}-\frac{(D-3)(4D-10)\beta}{2(D-1)^2}\right]^{1/2} . \tag{79}$$

For $D\geq 4$, this exponent becomes complex (for big enough β), but still produces the fluctuations Ξ^+ to vanish as $(\tau-\tau_0)\rightarrow 0$.

The case for Ξ^- is different because

$$P_{\mathrm{RN}}^-=\frac{1}{2}\pm\left[\frac{1}{4}+2\frac{(D-3)^2\beta}{(D-1)^2}\right]^{1/2} \tag{80}$$

allows again one solution that is unbounded as $(\tau-\tau_0)\rightarrow 0$, thus confirming our analysis of the last section for the Reissner-Nordström black hole in this region.

The time evolution in the intermediate region between the two asymptotic ones studied can be analyzed from the results obtained in the last section. In fact, on the event horizon we have the behavior

$$\Xi_{n,H}^{\mu}(\tau)\sim\exp\{\pm i[n^2+\lambda_H^{\mu}]^{1/2}(\tau-\tau_0)\} , \tag{81}$$

where λ_H^{μ} can be found from Eqs. (43)–(47). λ_H^+ is always positive, while λ_H^- and λ_H^i can take negative values depending on the string and black hole parameters. We can thus draw the same conclusions as before about the conditions of stability: Near the horizon, the string behavior is stable and oscillatory for the time coordinate Σ^+ and for the higher n modes of the radial and transversal components; the lower Σ^- and Σ^i modes being, however, unstable.

In summary, the time-dependent analysis confirms completely the results about stability and about the onset of instabilities obtained in the last section on the grounds of an adiabatic study.

V. NONCOLLINEAR COLLISION AND DISCUSSION

It is interesting to investigate how the picture changes when the infalling string has an orbit with nonzero impact parameter. It is simple to analyze the first-order fluctuations in the transversal coordinates. In fact, for $i>2$, the matrices A and B given by Eqs. (9) and (10) are diagonal,

$$A^{ij}=\frac{\dot{q}^R}{q^R}\delta^{ij}\ ,\quad B^{ij}=\left[\frac{\alpha' L}{(q^R)^2}\right]^2\delta^{ij}\ , \tag{82}$$

then the equations for the first-order fluctuations are

$$\ddot{\Xi}^i_n+\left[n^2+\left[\frac{\alpha' L}{(q^R)^2}\right]^2-\frac{\ddot{q}^R}{q^R}\right]\Xi^i_n=0\ . \tag{83}$$

This is the generalization of Eq. (19) for $L\neq 0$.

By use of the geodesic equations (13) we can rewrite Eq. (83) as

$$\ddot{\Xi}^i_n+\left\{n^2+\frac{(\alpha')^2}{2}\left[\frac{a'(q^R)m^2}{q^R}+2\left[\frac{L}{(q^R)^2}\right]^2(1-a+\tfrac{1}{2}a'q^R)\right]\right\}\Xi^i_n=0\ . \tag{84}$$

Thus we are now able to analyze particular cases we are interested in

A. Schwarzschild black hole

Plugging Eqs. (5) and (21) into (84) we obtain

$$\lambda^i_S=\frac{(\alpha')^2}{2}R_S^{D-3}(q^R)^{1-D}\left[(D-3)m^2+(D-1)\left[\frac{L}{q^R}\right]^2\right]\ . \tag{85}$$

We observe that for $D\geq 4$, $\lambda^i_S>0$, thus producing stable first-order fluctuations. In fact, when we study the time evolution we obtain

$$\lim_{q^R\to 0}\lambda^i_S=\frac{2(D-1)}{(D+1)^2}(\tau-\tau_0)^{-2}\ , \tag{86}$$

where we have integrated the expression (13)

$$\alpha'(\tau-\tau_0)=\int_{q_0^R}^{q^R}\frac{dq^R}{\sqrt{E^2-m^2a(q^R)-a(L/q^R)^2}}\ , \tag{87}$$

in order to obtain the behavior

$$q_S^R(\tau)\to\left[\frac{D+1}{2}\alpha' R_S^{(D-3)/2}L(\tau-\tau_0)\right]^{2/(D+1)}\ . \tag{88}$$

Equation (84) has, then, a powerlike solution in the limit $q^R\to 0$,

$$\Xi^i_{n,S}(\tau)\sim[n(\tau-\tau_0)]^P,\ P^i_S=\frac{1}{2}\pm\left[\frac{1}{4}-\frac{2(D-1)}{(D+1)^2}\right]^{1/2}\ , \tag{89}$$

That vanishes for $(\tau-\tau_0)\to 0$.

B. de Sitter spacetime

This case is very interesting because replacing expressions (5) and (23) into the first-order fluctuations equations with $L\neq 0$, Eq. (84), we obtain that the L dependence disappears. This gives, in fact, the equation

$$\ddot{\Xi}^i_n+\left[n^2-\frac{(\alpha' m)^2}{6}\Lambda\right]\Xi^i_n=0\ . \tag{90}$$

The solutions to this equation are exponentials in the proper time τ and are given by Eq. (60). The same conclusions about the possibility of instabilities given after Eq. (60) can be drawn here.

The fact that the solutions should be L independent could have been guessed from the symmetries of the de Sitter space. This metric has no preferred point to which to refer the angular moment as in the black hole cases (there is no singularity at $r=0$). This allows us to say that the components Σ^-, Σ^+, and Σ^i will behave as the ones already studied for the case $L=0$.

C. Reissner-Nordström black hole

From the metric coefficients (5) and its first derivative we find that Eq. (84) reads

$$\ddot{\Xi}^i_n+\left\{n^2+\frac{(\alpha')^2}{2}\left[\frac{(D-3)m^2}{(q^R)^2x}\left[1-\frac{2\beta}{x}\right]+\frac{L^2}{x(q^R)^4}\left[(D-1)-\frac{2\beta}{x}(D-2)\right]\right]\right\}\Xi^i_n=0\ . \tag{91}$$

We observe that as $q^R\to\infty$, the term proportional to L^2 vanishes faster than the other terms. Thus, in this limit we recover the $L=0$ results. As we approach to the black hole, and arrive to the horizon, we have, for the Schwarzschild black hole,

$$\lim_{q^R\to q_H^R}\lambda^i_{H,S}=\frac{(\alpha')^2}{2}\left[\frac{(D-3)m^2}{R_S^2}+(D-1)\frac{L^2}{R_S^4}\right]\ , \tag{92}$$

which is definite positive, and does not produce instabilities.

For the extreme Reissner-Nordström black hole we have

$$\lim_{q^R\to q_H^R}\lambda^i_{H,\mathrm{ERN}}=(\alpha')^2 2^{4/(D-3)}\frac{L^2}{R_S^4}\ . \tag{93}$$

Again, it is always positive. Thus, we see that the instabilities do not appear yet. The time-dependence of the solutions close to the horizon will be oscillatory with the squared frequency given by $\lambda^i_{H,\mathrm{ERN}}$.

However, the picture changes when we go closer to the singularity. For the Reissner-Nordström black hole, when $q^R \to 0$, we have

$$\lim_{q^R\to 0} \lambda^i_{0,\mathrm{RN}} = -(\alpha')\frac{(D-2)\beta}{x^2}\frac{L^2}{(q^R)^4} . \tag{94}$$

As this squared frequency takes negative values, this allows the possibility that instabilities develop in the string transversal coordinates. In order to find the time dependence in the coordinates we integrate first the center of mass motion, Eq. (13). Then, as $q^R \to 0$, we obtain

$$\alpha'(\tau-\tau_0) \sim \frac{x}{L\sqrt{\beta}}\frac{(q^R)^2}{(D-1)} . \tag{95}$$

Plugging this expression into Eq. (94), we have

$$\lim_{q^R\to 0} \lambda^i_{0,\mathrm{RN}} \to -\frac{D-2}{(D-1)^2}(\tau-\tau_0)^{-2} . \tag{96}$$

And the solution of the first-order fluctuations is again a powerlike one:

$$\Xi^i_{n,\mathrm{RN}}(\tau) \sim [n(\tau-\tau_0)]^P ,$$
$$P^i_{\mathrm{RN}} = \frac{1}{2} \pm \left[\frac{1}{4} + \frac{(D-2)}{(D-1)^2}\right]^{1/2} . \tag{97}$$

We here have that the solution with the minus sign in front of the square root produces an unbounded solution as $(\tau-\tau_0) \to 0$, thus, suggesting the existence of instabilities.

It is worth remarking that the solutions (97) and (89) for the time dependence of the transversal coordinates for the Reissner-Nordström and Schwarzschild black holes, respectively, are independent of L. They are, however different from those of the case $L=0$ [Eqs. (78) and (70)]. This is so because even if the L dependence cancels out from the final Eqs. (97) and (108), the approach to the singularity, $q^R \to 0$, is different if $L \neq 0$, thus producing different final coefficients.

Another interesting feature of the equations for the transversal first-order modes is that for black hole cases (Schwarzschild or Reissner-Nordström; orbit of the string center of mass with or without angular momentum) the time dependence of λ^i appears to be $(\tau-\tau_0)^{-2}$ as $q^R \to 0$. The behavior of λ^i as a function of q^R is different for each case, but the τ dependence of the orbit in each case exactly compensates for such difference.

Thus, for the linearized string fluctuations, the approach to the black hole singularity corresponds to the case $\beta=2$ of the motion of a particle in a potential $\gamma(\tau-\tau_0)^{-\beta}$. This is like the case of strings in singular plane-wave backgrounds [5,6]. In fact, the linearized first-order string fluctuations produce a one-dimensional Schrödinger equation, with τ playing the role of a spatial coordinate. The potential term in Eq. (84), can be written fully τ dependent, as we have seen, by plugging into it the center-of-mass trajectory $q^R(\tau)$. (In the case of gravitational plane waves, the spacetime is simpler than in the black hole one, and the exact full string equations become linear.)

The solution of Eq. (84) with a potential proportional to $\gamma(\tau-\tau_0)^{-2}$ can be given in terms of Bessel functions:

$$\Xi^i_n(\tau) = \sqrt{(\tau-\tau_0)}\{V^i_n J_\nu[n(\tau-\tau_0)] + W^i_n J_{-\nu}[n(\tau-\tau_0)]\} , \tag{98}$$

where V^i_n and W^i_n are arbitrary constants coefficients and

$$\nu = \sqrt{\tfrac{1}{4}-\gamma}, \quad \text{i.e.,} \quad \nu = P^i - \tfrac{1}{2} , \tag{99}$$

where $P^i(D,\beta)$'s are defined in Sec. IV.

For $\gamma<0$ we have Bessel functions (those with negative index) with a divergent behavior as $(\tau-\tau_0) \to 0$, indicating the existence of string instabilities. We would also like to stress that for black holes, what determines the possibility of instabilities is not the type of singularity (Reissner-Nordström or Schwarzschild), nor how it is approached ($L=0$ or $L \neq 0$), because we have seen that the time dependence of the potential is always $(\tau-\tau_0)^{-2}$, but the sign of the coefficient γ in front of it, i.e., the attractive character of the potential $(\tau-\tau_0)^{-2}$. Thus, we can conclude that whenever we have big enough repulsive effects in a gravitational background, instabilities in the propagation of strings on the spacetime background will appear. The coefficient γ is given by

$$\gamma_S = \begin{cases} \dfrac{2(D-3)}{(D-1)^2}, & L=0, \\ \dfrac{2(D-1)}{(D+1)^2}, & L \neq 0 , \end{cases}$$

for the Schwarzschild black hole, and

$$\gamma_{\mathrm{RN}} = \begin{cases} \dfrac{-2(D-3)}{(D-2)^2}, & L=0 \\ \dfrac{-2(D-2)}{(D-1)^2}, & L \neq 0 \end{cases}$$

for the Reissner-Nordström black hole.

Near the spacetime singularity, the dependence on the D spacetime dimensions is concentrated in γ. Notice the attractive singular character of the potential $\gamma(\tau-\tau_0)^{-2}$, for the Reissner-Nordström black hole, in agreement with the singular behavior of the string near $q^R=0$, while for the Schwarzschild black hole γ is positive, and the string solutions Σ^i are well behaved there.

The approach to the black hole singularity is better analyzed in terms of the Kruskal coordinates (u_k, v_k):

$$u_k = e^{Ku_S}, \quad v_k = e^{Kv_S}$$

u and v being null coordinates, and K the surface gravity of the black hole [$K=(D-3)/(2R_s)$ for Schwarzschild]. From Eqs. (69)–(71) we have [$C^{\pm}(\sigma)$ being coefficients determined by the initial state of the string]:

$$u_k(\sigma,\tau)=\exp\{K[C^+(\sigma)(\tau-\tau_0)^{P^+} - C^-(\sigma)(\tau-\tau_0)^{-|P^-|}]\}\ ,$$
$$v_k(\sigma,\tau)=\exp\{K[C^+(\sigma)(\tau-\tau_0)^{P^+} + C^-(\sigma)(\tau-\tau_0)^{-|P^-|}]\}\ , \tag{100}$$

and thus, $u_k v_k \to 1$ for $(\tau-\tau_0)\to 0$. That is, for $(\tau-\tau_0)\to 0$, the string approaches the spacetime singularity $u_k v_k = 1$, and it is trapped by it. The proper spatial length element of the string at fixed $(\tau-\tau_0)\to 0$, between (σ,τ) and $(\sigma+d\sigma,\tau)$, stretches infinitely as

$$ds^2_{(\tau-\tau_0)\to 0} \to [C^-(\sigma)]'^2 d\sigma^2 (\tau-\tau_0)^{-(D-1)|P^-|}\ , \tag{101}$$

where P^- is given by Eq. (71). Here, τ_0 is the (finite) proper falling time of the string into the black hole singularity. Stretching of the string also occurs for uniformly accelerated strings in flat space time [13].

The fact that the angular coordinates Σ^i become unbounded in the Reissner-Nordström case means that the string makes infinite turns around the spacetime singularity and remains trapped by it.

The same conclusions can be drawn for the quantum propagation of strings. The τ dependence is the same because this is formally described by the same Schrödinger equation with a potential $\gamma(\tau-\tau_0)^{-2}$, the coefficients of the solutions being quantum operators instead of c numbers. The τ evolution of the string near the black hole singularity is fully determined by the spacetime geometry, while the σ dependence (contained in the overall coefficients) is fixed by the state of the string.

For the modes Ξ^-, Ξ^+, and Ξ^i we can conclude that they should behave as in the case $L=0$ far from the black hole, where the influence of the angular momentum vanishes. Then, Ξ^+ and Ξ^i will oscillate with bounded amplitude outside the horizon, while Ξ^- will present an unbounded behavior. The approach to the singularity with $L\neq 0$ should not change qualitatively from the picture for $L=0$. The analysis can also be made in terms of the geodesic orbit followed by the center of mass of the string. For a given energy E, there is a critical impact parameter b_c that determines whether the string will fall into the black hole or not. From our results here for $L=0$ (see Table I), and for $L\neq 0$, we can draw the following picture: For large impact parameters $b > b_c$, the transversal Ξ^i and temporal Ξ^+ modes will be stable, while the radial modes Ξ^- begin to suffer instabilities. For small impact parameter, $b < b_c$, the string will fall into the black hole and, for a Reissner-Nordström background, also the transversal coordinates suffer instabilities.

It can be noticed that in cosmological inflationary backgrounds, for which unstable string behavior appears when $(\tau-\tau_0)\to 0$, the string coordinates η^i, remain bounded. In the black hole cases, all the characteristic features of string instabilities appear, but, in addition, the string coordinates η^i become unbounded near the $r=0$ singularity. This happens to be a typical behavior of strings near spacetime singularities, describing the fact that the string is trapped by it.

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