Dilaton quantum cosmology in two dimensions

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We consider a renormalizable two-dimensional model of dilaton gravity coupled to a set of conformal fields as a toy model for quantum cosmology. We discuss the cosmological solutions of the model and study the effect of including the back reaction due to quantum corrections. As a result, when the matter density is below some threshold new singularities form in a *weak*-coupling region, which suggests that they will not be removed in the full quantum theory. We also solve the Wheeler-DeWitt equation. Depending on the quantum state of the Universe, the singularities may appear in a quantum region where the wave function is not oscillatory, i.e., when there is not a well-defined notion of classical spacetime.

I. INTRODUCTION

Two situations in which quantum gravity effects are expected to be important are the late stages of black hole evaporation and the early Universe. The analysis of these problems in realistic models is a very difficult task. In particular, all the attempts to address the questions of the final state of black holes, or the evolution of the Universe near Planck scale, have to deal with the nonrenormalizability of quantum gravity in $3+1$ dimensions.

For this reason it is of interest to consider solvable toy models in which some of the difficulties of the realistic problem are not present. In the last months there has been considerable progress in the understanding of Hawking radiation and black-hole evaporation. The fundamental observation, done by Callan, Giddings, Harvey, and Strominger [1], is that the $(1+1)$-dimensional renormalizable[1] theory of gravity coupled to a dilaton field ϕ and N conformal fields f_i,

$$S_0 = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[e^{-2\phi}[R + 4(\nabla\phi)^2 + 4\lambda^2] - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2 \right], \quad (1)$$

contains black holes and Hawking radiation. Some interesting discussions on this model can be found in [5–9] (for a review and more references see Ref. [10]).

In this paper we will focus on the cosmological problem. A generalization of the theory Eq. (1) to D dimen-

sions has been considered before by Tseytlin and Vafa [11]. They coupled the theory to stringy matter and study the cosmological solutions exploiting the duality symmetry of the strings. The classical solutions were further investigated by Tseytlin [12]. Our aim here is different. We will consider the two-dimensional theory defined by Eq. (1) as a toy model to understand the influence of quantum effects in cosmological situations in higher dimensions.

In fact, a similar model can be obtained by restricting the four-dimensional (4D) action of general relativity to the metrics with spherical symmetry:

$$ds^2 = g_{ab}(x^0,x^1)dx^a dx^b + e^{-2\phi(x^0,x^1)}d\Omega_2, \quad a,b=1,2. \quad (2)$$

The reduced Einstein-Hilbert action reads

$$S_{\text{red}} = \frac{1}{2\pi} \int d^2x \sqrt{-g}\, e^{-2\phi}[R + 2(\nabla\phi)^2 - 2\Lambda + 2e^{2\phi}], \quad (3)$$

where Λ is the (four-dimensional) cosmological constant. Note that the dilaton field in the two-dimensional theory is related to the radius of the two-sphere in the four-dimensional metric, and that the cosmological constant gives an exponential contribution to the dilaton potential. We will consider model 1 because the classical and semiclassical analysis is simpler. Moreover, unlike the reduced action 3, the theory 1 is renormalizable. Model 1 is analogous to the reduced model 3 with $\Lambda \leq 0$. Indeed, as far as the black-hole physics is concerned, the semiclassical behavior of the Callan-Giddings-Harvey-Strominger (CGHS) model was shown to be very similar to the semiclassical physics of 4D Schwarzschild black holes [10]. On the other hand, the model given by action 1 with the opposite sign in the λ^2 term is analogous to model 3 with $\Lambda > 0$, i.e., standard 4D de Sitter gravity.

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[1]Discussions on the renormalization of this model can be found in [2–5].

Although in this case there is no black-hole formation in the Universe of model 1, the cosmology presents some interesting features which deserve some attention.

An interesting particular class of spherically symmetric metrics is the Kantowski-Sachs minisuperspace

$$ds^2 = a^2(t)(-dt^2 + dx^2) + b^2(t)d\Omega^2 , \tag{4}$$

which describes a Universe with a $S^1 \times S^2$ spatial geometry. This minisuperspace has been investigated both at the classical and quantum levels in Ref. [13]. However, because of the nonrenormalizability of general relativity, it is difficult to include quantum effects due to

matter and metric fluctuations. As we will see, it is possible to include these effects in the model Eq. (1), which can be considered as a *toy* Kantowski-Sachs cosmological model after identifying the dilaton with the radius of the two-sphere.

The paper is organized as follows. In the next section we discuss the exact solutions to the back reaction problem. That is, we find the time-dependent solutions of the equations of motion which follow from the one-loop effective action. In Sec. III we quantize the effective theory. We find different solutions to the Wheeler-DeWitt equation and discuss whether or not they predict classical behavior.

II. THE BACK REACTION PROBLEM

In the conformal gauge $g_{++} = g_{--} = 0, g_{+-} = -\frac{1}{2}e^{2\rho}$, the classical action S_0 becomes

$$S_0 = \frac{1}{\pi} \int d^2x \left[e^{-2\phi}(2\partial_+\partial_-\rho - 4\partial_+\phi\partial_-\phi + \lambda^2 e^{2\rho}) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] . \tag{5}$$

The classical equations of motion can be put into the form

$$\partial_+\partial_- f_i = 0 , \quad \partial_+\partial_-\rho = \partial_+\partial_-\phi , \quad 2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi = \lambda^2 e^{2\rho} , \tag{6}$$

$$0 = e^{-2\phi}(4\partial_\pm\rho\partial_\pm\phi - 2\partial_\pm^2\phi) + \frac{1}{2} \sum_{i=1}^{N} \partial_\pm f_i \partial_\pm f_i , \tag{7}$$

where we have included the equations of motion associated to the vanishing metric components g_{++} and g_{--}. The general solution to the classical equations is [1]

$$
\begin{aligned}
f_i &= f_{i+}(x^+) + f_{i-}(x^-) , \\
e^{-2\phi} &= j - \lambda^2 \int dx^+ \int dx^- e^{2h} , \\
\rho &= \phi + h , \\
0 &= \frac{1}{2} \sum_{i=1}^{N} f'_{i\pm} f'_{i\pm} + j''_\pm - 2j'_\pm h'_\pm ,
\end{aligned}
\tag{8}
$$

where $h = h_+(x^+) + h_-(x^-)$ and $j = j_+(x^+) + j_-(x^-)$ are free fields.

A particular one-loop quantum effective action including the conformal anomaly was introduced in Ref. [7]:

$$S = S_0 - \frac{\kappa}{8\pi} \int d^2x \left[R\frac{1}{\nabla^2}R - 2\phi R \right] , \tag{9}$$

where $\kappa = (N-24)/12$ is assumed to be positive. The first (nonlocal) quantum correction in Eq. (9) is the usual anomaly term. The second term is local and covariant and makes the theory exactly solvable.[2]

By using a free field representation [9], it is easy to find the general solution to the semiclassical equations of motion. We will follow closely Ref. [7]. Introducing new fields

$$\Omega = \frac{\kappa}{2}\phi + e^{-2\phi} , \quad \chi = \kappa\rho - \frac{\kappa}{2}\phi + e^{-2\phi} , \tag{10}$$

the effective action becomes

$$S = \frac{1}{\pi} \int d^2x \left[\frac{1}{\kappa}\partial_+\chi\partial_-\chi + \frac{1}{\kappa}\partial_+\Omega\partial_-\Omega + \lambda^2 e^{2(\chi-\Omega)/\kappa} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] , \tag{11}$$

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[2]In the conformal gauge this term renders the effective action a conformal invariant field theory with vanishing central charge, which is a consistency requirement if the path integral is to be regularized by a naive, field-independent cutoff [14] (discussions on this point in the present context can be found, e.g., in Refs. [2,9,10,15]).

and the semiclassical equations of motion are

$$\partial_+\partial_-\Omega=\partial_+\partial_-\chi=-\lambda^2 e^{2(\chi-\Omega)/\kappa} \ , \tag{12}$$

$$t_\pm=-\frac{1}{\kappa}\partial_\pm\chi\partial_\pm\chi+\partial_\pm^2\chi+\frac{1}{\kappa}\partial_\pm\Omega\partial_\pm\Omega+\frac{1}{2}\sum_{i=1}^N\partial_\pm f_i\partial_\pm f_i \ . \tag{13}$$

The arbitrary functions $t_\pm(x^\pm)$ reflect the nonlocality of the effective action. They are fixed by the boundary conditions necessary to define univocally the two-point function $1/\nabla^2$ (see below).

The general solution to the above equations can be written in terms of two free fields h and j:

$$\Omega=j-\lambda^2 \int dx^+ \int dx^- e^{2(h-j)/\kappa} \ ,$$

$$\chi=h-\lambda^2 \int dx^+ \int dx^- e^{2(h-j)/\kappa} \ , \tag{14}$$

$$t_\pm=\frac{1}{2}\sum_{i=1}^N f'_{i\pm}f'_{i\pm}+h''_\pm+\frac{1}{\kappa}(j'^2_\pm-h'^2_\pm) \ .$$

Note that the terms proportional to λ^2 have canceled out in the constraint.

From the general classical and semiclassical solutions [Eqs. (8) and (14)], it is easy to find time-dependent cosmological solutions. Let us consider coordinates $\sigma^\pm=\tau\pm\sigma$ and a two-dimensional metric of the form

$$ds^2=-e^{2\rho(\tau)}d\sigma^+d\sigma^- \ . \tag{15}$$

An homogeneous matter distribution is $f_i=f_i(\tau)$. From the equation of motion for the f_i fields, $\partial_+\partial_- f_i=0$, it follows that $f_i=p_i\tau+b_i$, $p_i,b_i=$ constants. The functions $t_\pm$ depend on the quantum state of the matter fields. The natural choice here is to impose the matter energy-momentum tensor to vanish in Minkowski space [$\rho(\tau)=0$]. This gives $t_\pm=0$ in $\sigma^\pm$ coordinates.[3] From Eqs. (8) and (14) we find

$$e^{-2\phi}=-e^{2\lambda\tau}+2m^2\lambda\tau+\frac{M}{\lambda} \ , \quad \rho=\phi+\lambda\tau \ , \quad m^2\equiv\frac{1}{8\lambda^2}\sum_i p_i^2 \ , \tag{16}$$

$$\Omega=-e^{2\lambda\tau}+2\varepsilon\lambda\tau+\frac{M}{\lambda} \ , \quad \chi=\Omega+\kappa\lambda\tau \ , \quad \rho=\phi+\lambda\tau \ , \tag{17}$$

where $\varepsilon=m^2-\kappa/4$ and the parameter M is arbitrary. We have chosen the coordinate τ in such a way that $\rho=\phi+\lambda\tau$ both in the classical and semiclassical solutions.[4]

As can be seen from the structure of the propagators, quantum loop corrections to the above solutions will be suppressed by the coupling $g=e^{2\phi}$, or by the effective coupling

$$g^2_{\text{eff}}=\frac{e^{2\phi}}{|1-(\kappa/4)e^{2\phi}|}$$

if the path integration is performed by using the effective action which includes the anomaly term (which implies a resummation of diagrams of standard perturbation theory; see, e.g., [2]). Thus solutions (16) and (17) can be trusted only in the regions in which g^2 and g^2_{eff} are small.

Equation (10) implicitly defines ϕ in terms of $\Omega(x^+,x^-)$. There is a critical point $\Omega_c=(\kappa/4)[1-\ln(\kappa/4)]$ at which $d\Omega/d\phi=0$. This transcendental equation (10) has no solution for $\Omega<\Omega_c$, and two solutions for $\Omega>\Omega_c$, the "Liouville theory" branch and the "string theory" branch. In the Liouville branch $\phi\in(\phi_c,\infty)$, and hence the anomaly term in the effective action dominates over the classical "string effective action" kinetic terms. Far from ϕ_c, $g^2_{\text{eff}}\sim 4/\kappa$ and hence the $1/N$ expansion is applicable. In the "string theory" branch $\phi\in(-\infty,\phi_c)$, so that the anomaly term is always dominated by the classical kinetic terms. Far away from ϕ_c, $g^2_{\text{eff}}\sim e^{2\phi}$, and the weak-field expansion is applicable.

We will now analyze the physical behavior of classical solutions (16) and semiclassical solutions (17). We will restrict ourselves to the case $M>0$; other cases can be studied in a similar way.

Let us first consider the classical solutions (16). We can distinguish two cases.

(i) $m^2=0$. The solution describes an expanding Universe, evolving from $\tau=-\infty$ to τ_0. The curvature and the coupling $g^2=e^{2\phi}$ are regular at $t=-\infty$, but the scale $a^2=e^{2\rho}$ is zero. At $\tau=\tau_0$, all the curvature, the scale, and the coupling $e^{2\phi}$ diverge. The corresponding Penrose diagram is illustrated by Fig. 1(a). This

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[3]This condition is not correct if the spatial coordinate is compact, i.e., if $0<\sigma<L$. In this case, the matter energy-momentum tensor contains a vacuum polarization term even when $\rho=0$. This term is given by $t_\pm=N/48L^2$ and will appear in the next section where the theory is quantized on a cylinder.

[4]As $\rho-\phi$ is a free field we have $\rho-\phi=a\tau+b$. The above choice ρ is possible as long as a is different from zero. If $a=0$, Ω is quadratic in τ. We will not consider this particular solution in what follows.

geometry can also be interpreted as the interior of a black hole.

(ii) $m^2 > 0$. The solution begins at a finite value of $\tau = \tau_1$, at which there is a curvature singularity, and the scale and the coupling $e^{2\phi}$ diverge. The Universe contracts, the coupling becomes weaker, and then reexpands. At finite time $\tau = \tau_2$ there is another curvature singularity, with infinite values for the coupling and the scale as well [Fig. 1(b)].

The semiclassical solutions can be separated in three different classes.

(i) $\varepsilon = 0$, $M > \lambda\Omega_c$. Replacing g^2 by g^2_{eff}, this case is qualitatively the same as the classical case $m^2 = 0$, but the scale is finite at $\tau_0 = \tau_c$ [the two branches of Eq. (10) behave in fact in a similar way].

(ii) $\varepsilon > 0$, $M > \lambda\Omega_c + \lambda\varepsilon(1 - \ln\varepsilon)$: This case behaves in a way that is similar to the classical case $m^2 > 0$, with g_{eff} diverging at initial and final times $\tau_{1,2}$, except by the fact the scale takes finite values at $\tau_{1,2}$.

(iii) $-\kappa/4 < \varepsilon < 0$. Now the two solutions given by the two branches of Eq. (10) must be separated.

"Liouville theory" branch. At $\tau = -\infty$, $g^2_{\text{eff}} = 4/\kappa$, $a^2 \sim e^{2\alpha\lambda\tau}$, $\alpha = 4m^2/\kappa$, and $R \sim \text{const} \times e^{2(1-2\alpha)\lambda\tau}$, so the scale goes to zero ($m^2 \neq 0$) and there is a curvature singularity if $\alpha > \frac{1}{2}$. At $\tau = \tau_c$ the curvature is infinity, the scale a is finite, but the coupling g^2_{eff} diverges.

"String theory" branch. At $\tau = -\infty$, $a = 0$, $R = -\infty$,

$g^2_{\text{eff}} = 0$. At $\tau = \tau_c$ the curvature is infinity, the scale a is finite, but the coupling g^2_{eff} diverges.

A disappointing fact is that in both cases there is a curvature singularity in a region of *weak* coupling. Therefore in this case it is very unlikely that this singularity will be removed in the full quantum theory. These kind of singularities could only be removed by an *ad hoc* restriction of allowed initial conditions (see below).

It is interesting to repeat the analysis for the model with the opposite sign in the λ^2 term. These solutions are similar to the static solutions in the CGHS model interchanging σ and τ. There are solutions where the Universe expands, the coupling goes to zero, and the metric approaches to Minkowski metric as $\tau \to \infty$ [see Figs. 2(a) and 2(b)]. In some solutions ϕ never reaches ϕ_c. In particular, there is an interesting case with $-\kappa/4 < \varepsilon < 0$: at $\tau = -\infty$, $a = 0$, $\phi = -\infty$, and $R = \infty$; at $\tau = \infty$, $a = 1$, $R = 0$, and $\phi = -\infty$ [Fig. 2(a)]. Also, here the coupling goes to zero even in the region near the

FIG. 2. Penrose diagrams for semiclassical solutions of the model with $\lambda^2 < 0$. (a) A typical solution in which ϕ never reaches the critical value. (b) The solution starts from ϕ_c where, generically, there is a curvature singularity. Natural boundary conditions select the only solution with regular curvature at $\phi = \phi_c$.
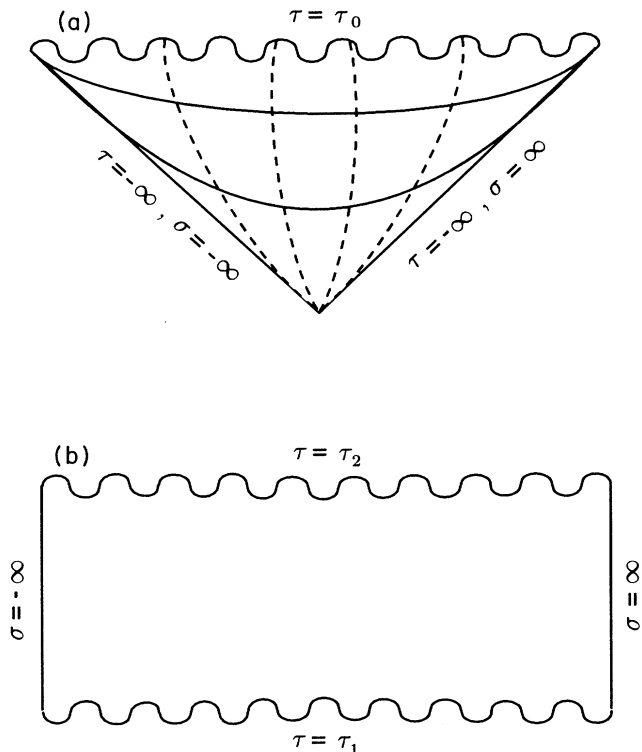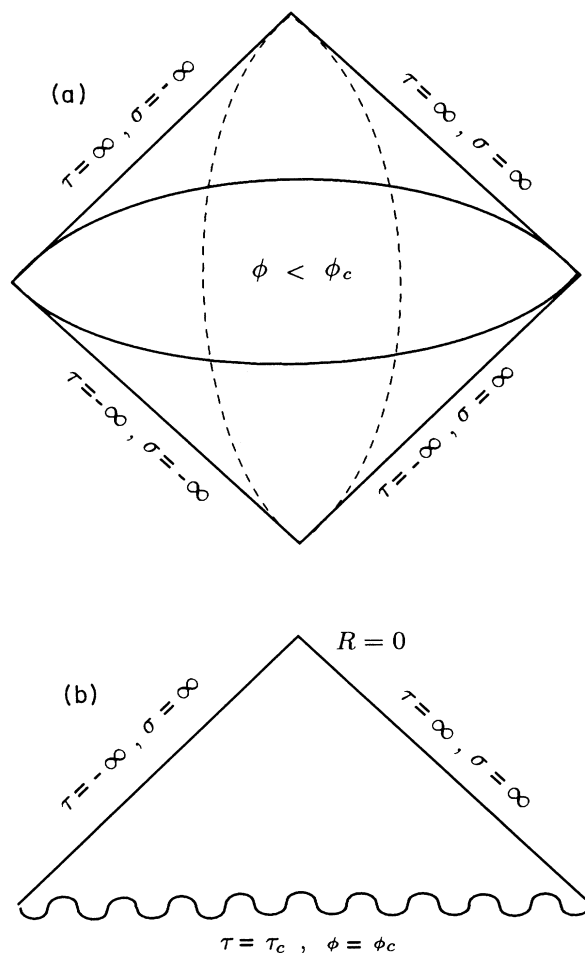
FIG. 1. (a) Penrose diagram for the classical solution (i). Solid and dashed lines indicate curves of constant τ and σ respectively. (b) Penrose diagram for the classical solution (ii).

singularity, which makes it very unlikely that more quantum effects could wash this singularity away.

Now we can state the more important effects of including the matter loop correction or back reaction.

(I) There is a doubling in the number of solutions, due to the presence of the "Liouville theory" and "string theory" branches. All the semiclassical solutions with $\lambda^2 > 0$ reach the critical point $\phi = \phi_c$ that separates both branches. There is a strong-coupling singularity there. Some of the solutions with the opposite sign of λ^2 never reach this point; i.e., they are always in a weak-coupling region.

(II) There appears a threshold in the matter density m^2, $m_0^2 = \kappa/4$. Above the threshold the cosmologies are essentially similar to the classical cosmologies, and the singularities appear in the strong-coupling regime. Below the threshold, $-\frac{1}{4} < \varepsilon < 0$, singularities also occur in a weak-coupling region. This suggests that they will still be present in the full quantum theory.

It is important to see whether these singularities can be removed by proper boundary conditions. In Ref. [16] it was shown that timelike, naked singularities at $\phi = \phi_c$ can be removed by the reflecting-type boundary conditions on the matter energy-momentum tensor. The analysis is simpler in "Kruskal" coordinates $x^{\pm}$, where $\phi = \rho$ and $t_{\pm} = \kappa/4x^{\pm 2}$ (see, e.g., [7]). Since

$$R = \frac{4}{1 - (k/4)e^{2\phi}} [\lambda^2 - (\nabla\phi)^2] \ ,$$

it can be finite at $\phi = \phi_c$ only if $\partial_+ \Omega|_{\phi=\phi_c} = \partial_- \Omega|_{\phi=\phi_c} = 0$. To see under which circumstances these boundary conditions can be applied it is interesting to consider solutions with general matter distributions f_i. They are given by Eq. (14). Note that $h_{\pm} = j_{\pm}$ in Kruskal coordinates. Regularity of the curvature at $\phi = \phi_c$ requires

$$j'_+(x^+) = \lambda^2 x^- \ , \quad j'_-(x^-) = \lambda^2 x^+ \ ,$$

$$\forall x^{\pm}: \quad \phi(x^+, x^-) = \phi_c \ . \tag{18}$$

Equation (18) defines the shape of the boundary curve $x^+ = \hat{x}^+(x^-)$ and implies a relation between the left- and right-moving components of the matter energy-momentum tensor. Using Eq. (18) we find

$$\lambda^2 \frac{d\hat{x}^+}{dx^-} = j''_-(x^-) = \frac{\lambda^4}{j''_+(\hat{x}^+)} \ .$$

Here the boundary is spacelike, therefore it is possible to have a regular curvature at $\phi = \phi_c$ only if $j''_{\pm} < 0$, i.e. [see Eq. (14)],

$$\frac{\kappa}{4x^{\pm 2}} < \frac{1}{2} \sum_{i=1}^{N} f'_{i\pm} f'_{i\pm} \ . \tag{19}$$

For homogeneous solutions, this is satisfied in the case $\varepsilon > 0$ discussed above. But condition (18) implies $M = \lambda \Omega_c + \lambda \varepsilon(1 - \ln\varepsilon)$. In this limiting value of M, τ_1 is equal to τ_2 and there is no cosmological evolution.

This result can be generalized to arbitrary matter contributions. The conclusion is that these type of boundary conditions cannot be implemented on spacelike boundaries. In contrast, in the model with the opposite sign of the λ^2 term this type of boundary condition can in fact be implemented on spacelike curves. This permits cosmologies free of singularities starting at $\phi = \phi_c$, in which the Universe expands up to $a = 1$, and the curvature and the coupling constant go to zero as $\tau \rightarrow \infty$ [see Fig. 2(b)].

More problematic, however, is the appearance of a singularity in a weak-coupling region. To eliminate this singularity, a possibility is to postulate that only matter densities with $\varepsilon \geq 0$ are allowed. Then no statement could be made about the presence of singularities at this semiclassical level; the only remaining singularities would be in a strong-interaction region where additional quantum effects will be important.

III. THE WHEELER-DeWITT EQUATION

We will now quantize the effective theory defined by Eq. (11). In doing this we will ignore boundary effects due to the fact that the transformation (10) is defined for $\Omega \geq \Omega_c$. It is presently unclear what are the correct boundary conditions to impose (see Ref. [16]).

Assuming that the spatial coordinate σ is periodic we can expand the fields and their derivatives as

$$\Omega(0,\sigma) = \Omega_+(0,\sigma) + \Omega_-(0,\sigma) \ ,$$

$$\Omega_{\pm}(0,\sigma) = \frac{1}{2}\Omega_0 \pm \frac{1}{2}P_\Omega \sigma - i \sum_{n \neq 0} \frac{1}{n} \Omega_n^{\pm} e^{\pm in\sigma} \ , \tag{20}$$

$$\partial_{\pm}\Omega(0,\sigma) = \sum_{n=-\infty}^{\infty} \Omega_n^{\pm} e^{\pm in\sigma} \ ,$$

with similar expressions for χ and f_i. The commutation relations are ($\Omega_0^{\pm} = \frac{1}{2}P_\Omega$)

$$[\Omega_0, P_\Omega] = i\frac{\kappa}{2} \ , \quad [\Omega_n^{\pm}, \Omega_m^{\pm}] = -\frac{\kappa}{2}\delta_{n,-m} \ ,$$

$$[\chi_0, P_\chi] = -i\frac{\kappa}{2} \ , \quad [\chi_n^{\pm}, \chi_m^{\pm}] = \frac{\kappa}{2}\delta_{n,-m} \ , \tag{21}$$

$$[f_{i0}, P_{f_i}] = i \ , \quad [f_{in}^{\pm}, f_{im}^{\pm}] = -n\delta_{n,-m}$$

[we are omitting a factor π which comes from the global factor in the action (11)].

The Virasoro generators $L_n^{\pm}$ are defined as usual as the Fourier modes of $T_{\pm\pm} - \kappa/2$, where $T_{\pm\pm}$ are defined as twice the right-hand side of Eq. (13). Using the equations of motion to eliminate the second time derivatives we find

$$L_n^{\pm} = \sum_m \left[\frac{2}{k}(\Omega_m^{\pm}\Omega_{n-m}^{\pm} - \chi_m^{\pm}\chi_{m-n}^{\pm}) + f_{im}^{\pm}f_{in-m}^{\pm} \right] + 2in\chi_n^{\pm} - \frac{\kappa}{2}\delta_{n0} - \frac{\lambda^2}{\pi}\int_0^{2\pi} d\sigma \, e^{\mp in\sigma}e^{2(\chi-\Omega)/\kappa} \, . \tag{22}$$

Assuming the ghosts to be in their ground state $|0\rangle_{\text{gh}}$, the physical states must satisfy

$$L_n^{\pm}|\text{phys}\rangle = \delta_{n0}|\text{phys}\rangle \, , \quad n \geq 0 \, . \tag{23}$$

This set of equations is equivalent to the functional Hamiltonian and supermomentum constraints [17]. The nonvanishing right-hand side in the physical state condition for $n = 0$ [combined with the shift $-(\kappa/2)\delta_{n0}$ in the Virasoro generators] represents the vacuum polarization of the physical degrees of freedom on the cylinder.

The zero modes χ_0 and Ω_0 do not decouple from the other modes in the constraints. To find the physical states, we will adopt here the minisuperspace approximation, and neglect the coupling between the modes. It should be stressed that this is an *improved* minisuperspace approximation, since we are performing the approximation into the effective action Eq. (11), not in the classical action Eq. (5) [15]. Assuming the quantum matter fields f_i to be in their ground states, the physical states are given by $|\text{phys}\rangle = |0\rangle \otimes |\Psi\rangle$, where $|0\rangle$ is the state annihilated by $\chi_n^{\pm}$, $\Omega_n^{\pm}$, and $f_{in}^{\pm}$, for $n > 0$, and the state $|\Psi\rangle$ satisfies the Wheeler-DeWitt (WD) equation

$$\left[\frac{\kappa}{4}\frac{\partial^2}{\partial\chi_0^2} - \frac{\kappa}{4}\frac{\partial^2}{\partial\Omega_0^2} - \frac{1}{2}\sum_i \frac{\partial^2}{\partial f_{i0}^2} - 4\lambda^2 e^{2(\chi_0-\Omega_0)/\kappa} - \kappa - 2 \right]\Psi = 0 \, . \tag{24}$$

This equation can also be obtained by linearizing the tachyon β function, as observed in Ref. [18] in the context of noncritical theory. In that paper the authors point out that nonlinear terms may play an important role in the cosmological constant problem.

At this point one would like to have a criterion to choose a particular solution to the WD equation and consider this particular state as the "quantum state of the Universe." Although there are several proposals to select a wave function [19–21] and to extract physical predictions from it [22], none of them is completely satisfactory. Here we will analyze some particular solutions which are the two-dimensional analogs of some of the wave functions proposed in $3+1$ quantum cosmology. We will consider the simplest interpretation of the wave function; i.e., we will assume that it predicts classical behavior only in the regions where it is oscillatory. The classical trajectories associated with a wave function of the form $\Psi = e^{iS}$ are obtained through the identification

$$\frac{\partial S}{\partial\chi_0} = -\frac{2}{\kappa}\dot{\chi}_0 \, , \quad \frac{\partial S}{\partial\Omega_0} = \frac{2}{\kappa}\dot{\Omega}_0 \, , \quad \frac{\partial S}{\partial f_{i0}} = \dot{f}_{i0} \, , \tag{25}$$

where the dot denotes derivative with respect to τ. It is important to notice that, in this context, this is the *definition* of classical time τ in terms of the quantum degrees of freedom [23,24].

A basis for the solutions to Eq. (24) is given by $\Psi_p = \exp(i\sum p_i f_{i0})\Psi_\alpha$, where $[\chi_+ = \chi_0 + \Omega_0, \chi_- = (\chi_0 - \Omega_0)/\kappa]$

$$(\partial_+\partial_- - \alpha - 4\lambda^2 e^{2\chi_-})\Psi_\alpha = 0 \, , \quad \alpha \equiv \frac{N}{12} - \frac{1}{2}\sum p_i^2 \, , \tag{26}$$

i.e.,

$$\Psi_\alpha = e^{i\omega} \, , \quad \omega \equiv p_-\chi_+ + p_+\chi_- - \frac{2\lambda^2}{p_-}e^{2\chi_-} \, , \tag{27}$$

with $p_+p_- = -\alpha$. When the numbers $p_i, p_\pm$ are large and real, this wave function is rapidly oscillating and implies classical behavior. The classical trajectories can be found from Eq. (25). In terms of $\chi_\pm$ this reads $\partial_\pm S = -\dot{\chi}_\mp$. From $S = p_i f_i + \omega$ we find

$$\chi_- = -p_-\tau \, , \quad f_i = p_i\tau + b_i \, ,$$
$$\chi_+ = -p_+\tau - \frac{2\lambda^2}{p_-^2}e^{-2p_-\tau} + \text{const} \, . \tag{28}$$

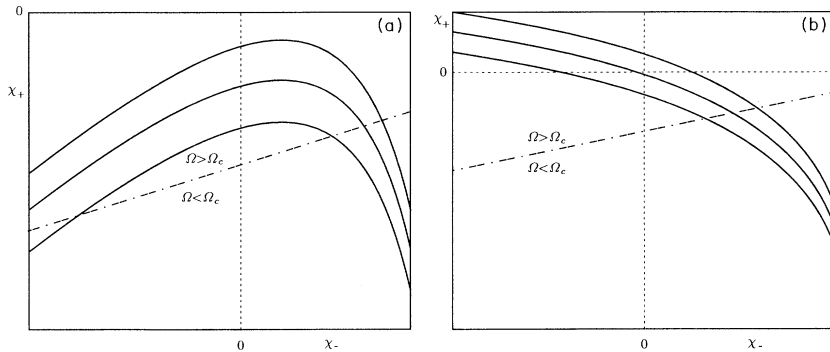Equation (28) reproduces the semiclassical solutions (17)

FIG. 3. Classical trajectories defined by Eq. (28) (solid lines). The dashed-dotted line corresponds to $\Omega = \Omega_c$. The trajectories are physical above this line. (a) Trajectories for $\alpha < 0$. (b) For $\alpha > 0$.

studied in the previous section. The only difference is the vacuum polarization contribution $N/12$ contained in α. Figure 3 shows the classical trajectories in the plane (χ_+,χ_-).

Next we consider a popular choice: the Hartle-Hawking state $|HH\rangle$, defined by the Euclidean path integral with only one boundary [19]. As noted in Refs. [25,26], the two-dimensional analog of this state is the $|sl(2,C)\rangle$ state, defined by

$$L_n^\pm|sl(2,C)\rangle=0 \; , \quad n \geq -1 \; . \tag{29}$$

Let us consider the case $\lambda=0$. Taking into account that

$$iL_{-1}^\pm=\chi_{-1}^\pm\left[2+\frac{\partial}{\partial\chi_0}\right]+\Omega_{-1}^\pm\frac{\partial}{\partial\Omega_0}+\sum_i f_{i-1}^\pm\frac{\partial}{\partial f_{i0}}+\cdots \; , \tag{30}$$

it is easy to show that [25]

$$|sl(2,C)\rangle=|0\rangle e^{-2\chi_0} \; . \tag{31}$$

This state is not physical, because it satisfies a different $n=0$ condition. However, it becomes a physical state in the limit $N\to\infty$. Indeed, in this limit, it coincides with

$$|HH\rangle=|0\rangle\exp\left[-\frac{2}{\sqrt{1-24/N}}\chi_0\right] \tag{32}$$

which is a particular exponential solution to the WD equation. In fact, this corresponds to the solution Ψ_{pHH} $(\lambda=0)$, with

$$p_{HH}=\{p_i=0,p_-=i\sqrt{\alpha/\kappa},p_+=i\sqrt{\alpha\kappa}\} \; .$$

This is our analog of the Hartle-Hawking state, which is naturally extended to the $\lambda\neq0$ case by the solution Ψ_{pHH}. Note that this wave function is a *real* exponential. As a

consequence it does not predict classical behavior. If this were the physical state our toy universe would never exit the quantum era!

Let us now consider other classes of physical states, which are similar to the tunneling wave functions defined in terms of the Lorentzian path integral in Kantowski-Sachs cosmology [27]. They are solutions of the form

$$\Psi_\alpha=J(z) \; , \quad z^2=-4(\chi_+-a)(\alpha\chi_-+2\lambda^2e^{2\chi_-}-b) \; . \tag{33}$$

Inserting the ansatz (33) into the WD equation it is easy to show that $J(z)$ must be a linear combination of the Bessel functions $J_0(z)$ and $Y_0(z)$. This class of wave functions is oscillating when z is large and real. Indeed, using the asymptotic form of the Bessel functions we find

$$\sqrt{z}J(z)\simeq\alpha e^{iz}+\beta e^{-iz} \; , \quad |z|\gg1 \; , \tag{34}$$

i.e., in this region of the plane (χ_+,χ_-) the wave function is a linear combination of WKB solutions with a Hamiltonian-Jacobi function $S=\pm z$. From Eq. (25) we see that both WKB components of the wave function (33) are associated with the family of classical trajectories:

$$\frac{\partial_-S}{\partial_+S}=\frac{d\chi_+}{d\chi_-}=\frac{(\chi_+-a)(\alpha+4\lambda^2e^{2\chi_-})}{\alpha\chi_-+2\lambda^2e^{2\chi_-}-b} \; , \tag{35}$$

i.e.,

$$(\chi_+-a)=C(\alpha\chi_-+2\lambda^2e^{2\chi_-}-b) \; , \tag{36}$$

where C is an arbitrary *negative* constant (it must be negative because in the classical region $z^2>0$). Figure 4 shows these trajectories in the plane (χ_+,χ_-). The plane is divided into classical $(z^2\gg1)$ and forbidden $(z^2\leq1)$ regions.[5] We see that in general the classical trajectories leave, at some time, the classical region. We should also
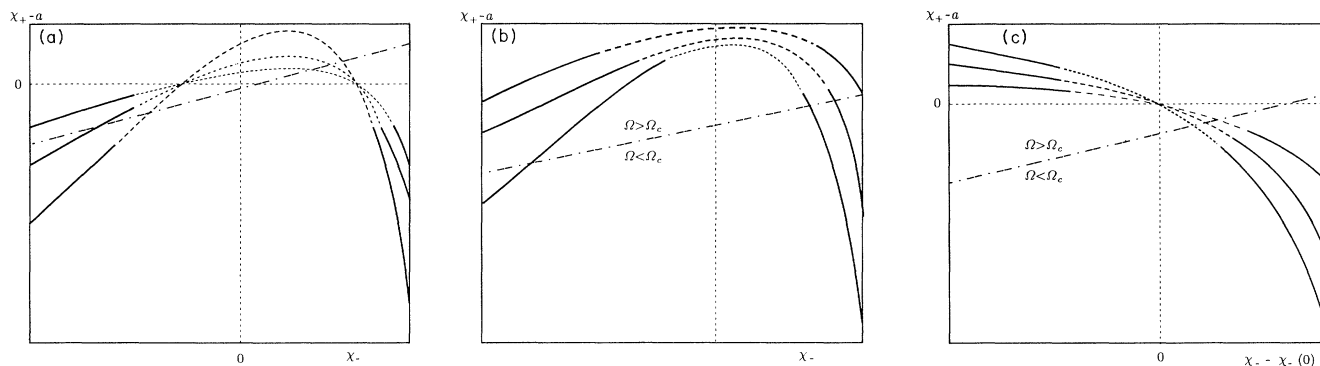

FIG. 4. The classical trajectories defined by Eq. (36). The solid lines are the part of the classical trajectories contained in the classical region. The dashed lines are the part contained in the forbidden region. Same conventions for the line $\Omega=\Omega_c$. (a) The case $\alpha<0$, $b<(\alpha/2)[\ln(-\alpha/4\lambda^2)-1]$. (b) The case $\alpha<0$ and $b>(\alpha/2)[\ln(-\alpha/4\lambda^2)-1]$. (c) The case $\alpha>0$. $\chi_-^{(0)}$ is the zero of the function $\alpha\chi_-+2\lambda^2e^{2\chi_-}-b$. A typical case where the line $\Omega=\Omega_c$ is completely contained in the forbidden region.

[5]A more precise definition of the forbidden region is $z^2\leq j_{01}^2$ or y_{01}^2, where j_{01}^2 and y_{01}^2 are, respectively, the first zero of the Bessel function J_0 and Y_0. For a similar discussion see Ref. [28].

stress that, due to the relation between $\chi_{\pm}$ and the original variables ϕ and ρ, the above trajectories are meaningful only in the region $\Omega > \Omega_c$, i.e., when $\chi_+ > \kappa\chi_- + 2\Omega_c$. This is also illustrated in Fig. 4.

The semiclassical solutions of the previous section contain two different classes of singularities: weak-coupling singularities at $\tau = -\infty$ ($\chi_{\pm} = -\infty$) and strong-coupling singularities at the critical point $\Omega = \Omega_c$ ($\chi_+ = \kappa\chi_- + 2\Omega_c$). From Figs. 4(a) and 4(b), and comparing with the results of Sec. II, we see that the singularities that occur at weak coupling [see (iii)] are present in the classical region where the wave function is oscillatory. However, the strong-coupling singularity may be avoided in the case $\alpha > 0$. Indeed, as shown in Fig. 4(c), for particular values of the constants a and b in Eq. (33), the curve $\Omega = \Omega_c$ is contained completely in the classically forbidden region.

To summarize, we have shown extreme examples where the whole plane is classically forbidden (Hartle-Hawking state), or allowed [exponential solutions in Eq.

(27)]. We also found more interesting solutions where there are both classically allowed and forbidden regions (tunneling solutions). For some wave functions, the semiclassical singularities are present only in the forbidden regions. It is in this way that the choice of the quantum state may solve the problem of the singularities; they may take place where there is not a well-defined notion of spacetime.

Note added. We have received a paper in which the Wheeler-DeWitt equation is studied in the same model in the context of black holes [29].

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