Unitary equivalence of the metric and holonomy formulations of (2+1)-dimensional quantum gravity on the torus

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Recent work on canonical transformations in quantum mechanics is applied to transform between the Moncrief metric formulation and the Witten-Carlip holonomy formulation of (2+1)-dimensional quantum gravity on the torus. A nonpolynomial factor ordering of the classical canonical transformation between the metric and holonomy variables is constructed which preserves their classical modular transformation properties. An extension of the definition of a unitary transformation is briefly discussed and is used to find the inner product in the holonomy variables which makes the canonical transformation unitary. This defines the Hilbert space in the Witten-Carlip formulation which is unitarily equivalent to the natural Hilbert space in the Moncrief formulation. In addition, gravitational θ states arising from "large" diffeomorphisms are found in the theory.

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I. INTRODUCTION

The theory of (2+1)-dimensional quantum gravity for the spatial topology of the torus is quantum mechanical rather than field theoretic. As a consequence, it can be exactly solved and serves as an excellent toy model in which to study time in quantum gravity. Among the several treatments of 2+1 quantum gravity, there are essentially two contrasting classes of formulation: that of Witten [1,2] in terms of holonomies and that of Moncrief [3,4] in terms of the Arnowitt-Deser-Misner (ADM) metric decomposition and the York extrinsic time. The most striking distinction between them is that Witten's formulation is dynamically sterile—the holonomies are constants of the motion—while, in Moncrief's formulation, the two-metric evolves as the York time progresses. This difference highlights the ambiguity in defining time in quantum gravity. The natural question is whether the two theories are equivalent, and, if so, what this means about time.

Carlip [5] has proven the classical equivalence of the two formulations by defining variables which reflect the holonomy of the torus and then giving the canonical transformation between his variables and the metric variables of Moncrief. He finds that the canonical transformation is a time-dependent one which trivializes Moncrief's Hamiltonian and thereby transforms the time-dependent metric variables to time-independent ones. Carlip goes on to discuss the quantum equivalence of the formulations by undertaking the formidable task of factor ordering the classical canonical transformation. This procedure is not well controlled; it is difficult to find

an ordering that takes one between two chosen Hamiltonians. Indeed, Carlip does not obtain the Moncrief Hamiltonian, but rather one close to it, which he calls a Dirac square root [6].

The purpose of this paper is to use recent work on canonical transformations in quantum mechanics [7] to prove the unitary equivalence of the metric and holonomy formulations. A general canonical transformation can be implemented in quantum mechanics as a product of elementary canonical transformations, each of which has a well defined quantum implementation. This is briefly reviewed in Sec. II. In general, the sequence of elementary canonical transformations that takes one between two Hamiltonians classically is not the same sequence which does so quantum mechanically because of terms that arise from factor ordering. This implies that, in general, there is no simple (i.e., polynomial) factor ordering of the classical general canonical transformation between two Hamiltonians which relates their quantum variables. This is the reason that attempting to factor order a classical result is uncontrolled: the practical restriction to polynomial factor orderings means that one can only relate those Hamiltonians for which the classical and quantum sequence of elementary canonical transformations are essentially the same.

The quantum canonical transformation between the metric and holonomy variables of 2+1 gravity on the torus will be found explicitly. First, in Sec. III, the classical solution of the metric super-Hamiltonian and the classical transformation to Carlip's holonomy variables is reviewed. In Sec. IV, a sequence of elementary canonical transformations which transform the metric super-Hamiltonian to a massive relativistic free particle is given. The sequence of transformations which complete the trivialization of the super-Hamiltonian is then given in Sec. V, followed in Sec. VI by the sequence which transforms the variables of the trivialized super-Hamiltonian into those of Carlip. A nonpolynomial fac-

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tor ordering of the classical canonical transformation is found. Carlip [5,6] emphasizes the importance of the modular transformation properties of the metric and holonomy variables. In Sec. VII, the quantum canonical transformation between the metric and holonomy variables is shown to preserve their classical modular transformation properties.

In Sec. VIII, the familiar definition of a unitary transformation as a linear norm-preserving isomorphism of a Hilbert space onto itself is extended to transformations between Hilbert spaces. Given a canonical transformation and the measure density for the inner product of the original Hilbert space, a transformed measure density defining the inner product of the new Hilbert space is found which preserves the values of inner products between states. This is used to construct the measure density of the inner product in the holonomy variables from the natural measure density in the metric variables. This defines the Hilbert space in the Witten-Carlip holonomy formulation. With this choice of Hilbert space, the metric and holonomy formulations of $(2+1)$-dimensional quantum gravity on the torus are unitarily equivalent.

In Sec. X, the role of modular transformations is considered from the standpoint of the wave function. It is found that, in analogy to the θ vacua of Yang-Mills theory, there are θ states in $2+1$ gravity on the torus which arise from "large" diffeomorphisms. Finally, in the conclusion, the implications about the nature of time following from the quantum equivalence of the metric and holonomy formulations of $2+1$ gravity on the torus are discussed. In an appendix, solutions of the metric super-Hamiltonian constraint are constructed using the canonical transformations found in Sec. IV.

II. CANONICAL TRANSFORMATIONS

The use of canonical transformations in quantum mechanics has been recently systematized [7]. This allows quantum systems to be solved by transforming to simpler systems whose solutions are known. Since time-dependent canonical transformations are of interest, it is useful to work with the super-Hamiltonian in extended phase space, where (q_0, p_0) have been adjoined to the spatial variables. A canonical transformation P is an operator transformation between two super-Hamiltonians:

$$P \mathcal{H}^{(0)} P^{-1} = \mathcal{H}' \ .$$

This gives the solutions of $\mathcal{H}^{(0)} \psi^{(0)} = 0$ in terms of those of $\mathcal{H}' \psi' = 0$ as $\psi^{(0)} = P^{-1} \psi'$. It has been conjectured that all integrable super-Hamiltonians $\mathcal{H}^{(0)}$ can be reduced to trivialized form, where $\mathcal{H}' = p_0$. There may be more than one independent reduction to triviality, and together these give all independent solutions of $\mathcal{H}^{(0)}$.

Canonical transformations exist independent of the choice of Hilbert space. All solutions of one super-Hamiltonian, including non-normalizable ones, are transformed to solutions of the other super-Hamiltonian. The conditions under which a canonical transformation is a unitary transformation are discussed in Sec. VIII. Canonical transformations are useful even when they are not unitary [8]. In such cases, it is most natural to im-

pose boundary conditions and normalizability at the level of $\mathcal{H}^{(0)}$, rather than upon the solutions of $\mathcal{H}'$.

For completeness, a brief review of the use of canonical transformations in quantum mechanics follows. In principle, a general quantum canonical transformation can be implemented as a product of three elementary canonical transformations [7]: similarity (gauge) transformations, point canonical transformations, and the interchange of coordinates and momenta. Each of these is characterized by its transformation of the phase-space variables and its action on the wave function. Since the transformations are canonical, the transformed variables may simply be substituted in the Hamiltonian for the original variables, respecting the original operator ordering.

A similarity transformation is a shift of the momentum by a function of the coordinate

$$p = p^a - i f(q^a)_{,q^a}, \quad q = q^a \ , \tag{1}$$

where the comma indicates differentiation with respect to the subscript and latin alphabet superscripts indicate the generation of the transformation. Under this transformation, the wave function changes by a factor

$$\psi^{(a)}(q) = e^{-f(q)} \psi^{(0)}(q) \ . \tag{2}$$

This transformation is often referred to as a gauge transformation because of its role in the theory of a particle interacting with an electromagnetic field.

A point canonical transformation is a change of variables

$$q = f(q^a), \quad p = \frac{1}{f(q^a)_{,q^a}} p^a \ . \tag{3}$$

This transformation cannot always be expressed in terms of a single exponential of a function, so it is denoted symbolically by its action on the coordinate as $P_{f(q)}$. Its action on the wave function is

$$\psi^{(a)}(q) = P_{f(q)} \psi^{(0)}(q) = \psi^{(0)}(f(q)) \ . \tag{4}$$

Each occurrence of q is simply replaced by $f(q)$.

The interchange of coordinate and momentum is

$$p = -q^a, \quad q = p^a \ . \tag{5}$$

This is implemented by the Fourier transform operator

$$I = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dq \, e^{iqq^a} \ , \tag{6}$$

and the wave function is transformed as

$$\psi^{(a)}(q) = I \psi^{(0)}(q)$$
$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dq' e^{iq'q} \psi^{(0)}(q') \ . \tag{7}$$

Using the interchange operator, analogues of the similarity and point canonical transformations which involve functions of the momentum are obtained by conjugation. The composite similarity transformation is

$$q = q^a + i f(p^a)_{,p^a}, \quad p = p^a \ , \tag{8}$$

and it acts on the wave function

$$\psi^{(a)}(q) = e^{-f(p)}\psi^{(0)}(q) = I e^{-f(q)} I^{-1}\psi^{(0)}(q) \ . \tag{9}$$

The composite point canonical transformation is

$$p = f(p^a), \quad q = \frac{1}{f(p^a)_{,p^a}} q^a \tag{10}$$

and is given by $P_{f(p)} = I P_{f(q)} I^{-1}$.

Many-variable transformations can be constructed by using many-variable functions in the elementary canonical transformations above. Since independent variables commute, all the variables not being acted on may be treated as constant parameters. For example, the point canonical transformation $\exp(i\alpha q_1 p_1)$ generates the conformal scaling

$$p_1 = e^{-\alpha}p_1^a, \quad q_1 = e^{\alpha}q_1^a \ , \tag{11}$$

where

$$\psi^{(a)}(q_1, q_2) = \exp(i\alpha q_1 p_1)\psi^{(0)}(q_1, q_2)$$
$$= \psi^{(0)}(e^{\alpha}q_1, q_2) \ .$$

If $\alpha = \ln q_2$, then

$$p_1 = \frac{p_1^a}{q_2^a}, \quad q_1 = q_2^a q_1^a \ . \tag{12}$$

When this transformation acts on (q_2, p_2), it is a similarity transformation

$$p_2 = p_2^a - \frac{q_1^a p_1^a}{q_2^a}, \quad q_2 = q_2^a \ . \tag{13}$$

The wave function is transformed:

$$\psi^{(a)}(q_1, q_2) = \exp[i(\ln q_2)q_1 p_1]\psi^{(0)}(q_1, q_2)$$
$$= \psi^{(0)}(q_1 q_2, q_2) \ . \tag{14}$$

III. CLASSICAL SOLUTION

To understand the relationship between the quantum and classical canonical transformations between the metric and holonomy formulations of $2+1$ gravity on the torus, it is important to have both the quantum and classical solutions at hand. The classical evolution has been discussed in detail from a different perspective by Hosoya and Nakao [9].

The super-Hamiltonian in the metric variables [4] is the analogue of the Wheeler-DeWitt equation for $2+1$ gravity on the torus:

$$\mathcal{H}^{(0)} = -(q_0 p_0)^2 + q_2^2(p_1^2 + p_2^2) = 0 \ . \tag{15}$$

The ADM Hamiltonian

$$H = \frac{\sqrt{q_2^2(p_1^2 + p_2^2)}}{q_0} \tag{16}$$

found by Moncrief [3] is a square root of this super-Hamiltonian in a sense discussed by Carlip [6].

The classical solution of the metric super-Hamiltonian follows from the equations of motion

$$\dot{q}_0 = -2q_0^2 p_0, \quad \dot{p}_0 = 2q_0 p_0^2 \ ,$$
$$\dot{q}_1 = 2q_2^2 p_1, \quad \dot{p}_1 = 0 \ , \tag{17}$$
$$\dot{q}_2 = 2q_2^2 p_2, \quad \dot{p}_2 = -2q_2(p_1^2 + p_2^2) \ ,$$

where the dot signifies differentiation with respect to the affine parameter t. Clearly, p_1 is a constant. The general solution for the other variables is found to be

$$q_0 = e^{-2c_0(t-t_0)} \ ,$$
$$p_0 = c_0 e^{2c_0(t-t_0)} \ ,$$
$$q_1 = \frac{c_0}{p_1}\tanh[2c_0(t-t_0) + c_2] + c_1 \ ,$$
$$q_2 = \frac{c_0}{p_1\cosh[2c_0(t-t_0) + c_2]} \ , \tag{18}$$
$$p_2 = -p_1\sinh[2c_0(t-t_0) + c_2] \ .$$

There are only five constants of integration because in addition to the six equations of motion there is one constraint, and this fixes one constant of integration. (Note that the degenerate solutions of the equations of motion, for which the momenta vanish and the coordinates are constant, are included in this solution as the special case $p_1 \to 0$, $c_0 \to 0$, $c_0/p_1 \to$ const.) By eliminating t in favor of q_0, these become

$$p_0 = \frac{c_0}{q_0} \ ,$$
$$q_1 = \frac{c_0}{p_1}\frac{1 - \beta^2 q_0^2}{1 + \beta^2 q_0^2} + c_1 \ ,$$
$$q_2 = \frac{2\beta c_0 q_0}{p_1(1 + \beta^2 q_0^2)} \ , \tag{19}$$
$$p_2 = -\frac{p_1(1 - \beta^2 q_0^2)}{2\beta q_0} \ ,$$

where $\beta = e^{-c_2}$. Comparing with Carlip [5], the correspondence between the integration constants and Carlip's holonomy variables is

$$p_1 = -2a\lambda \ ,$$
$$c_0 = -(a\mu - \lambda b) \ ,$$
$$c_1 = \frac{1}{2}\left[\frac{\mu}{\lambda} + \frac{b}{a}\right] \ , \tag{20}$$
$$c_2 = \ln\frac{\lambda}{a} \ .$$

Carlip defines (a, μ) and $(b, -\lambda)$ to be pairs of canonically conjugate variables. Using the Poisson brackets defined in terms of the holonomy variables,

$$\{f, g\}_h = \partial_a f \ \partial_\mu g - \partial_\mu f \ \partial_a g - \partial_b f \ \partial_\lambda g + \partial_\lambda f \ \partial_b g \ ,$$

the brackets among the integration constants in (20) can be calculated. It is found that $\{c_1, p_1\}_h = 2, \{c_2, c_0\}_h = 2$ and all other brackets vanish. This implies that the

transformation is an extended canonical transformation [10]. That is, the transformation is canonical, but the momenta have been rescaled during the transformation, so the brackets between conjugate variables are not equal to one.

IV. METRIC SUPER-HAMILTONIAN TO FREE PARTICLE

There is a straightforward sequence of elementary transformations which reduce the metric super-Hamiltonian (15) to a massive relativistic free particle [in $(1+1)$-dimensional Minkowski space]. This allows the solutions of $\mathcal{H}^{(0)}$ to be constructed from the plane-wave solutions of the relativistic free particle. Of course, the constraint $\mathcal{H}^{(0)}\psi^{(0)}=0$ can be solved directly by separation of variables, but it is instructive to construct the wave function by canonical transformation. This will be done in the Appendix.

Here, the sequence of transformations and the super-Hamiltonian after each transformation will be given. For convenience, only the variable(s) changed by a transformation will be stated. A superscript indicating the generation of the transformation will change for all variables.

Beginning with the metric super-Hamiltonian, the time dependence is simplified by a point canonical transformation

$$p_0 = e^{-q_0^a}p_0^a, \quad q_0 = e^{q_0^a} ,$$

and a linear term in $q_2^a p_2^a$ is introduced by shifting the momentum

$$p_2 = p_2^a - \frac{i}{2q_2^a}$$

giving

$$\mathcal{H}^{(a)} = -p_0^{a^2} + q_2^{a^2}p_1^{a^2} + q_2^{a^2}p_2^{a^2} - iq_2^a p_2^a + \tfrac{1}{4} . \tag{21}$$

The wave function is

$$\psi^{(a)} = q_2^{-1/2}P_{e^{q_0}}\psi^{(0)} . \tag{22}$$

A conformal canonical transformation is made to absorb the momentum p_1^a into the coordinate q_2^a:

$$q_1^a = q_1^b - \frac{q_2^b p_2^b}{p_1^b}, \quad p_2^a = p_1^b p_2^b, \quad q_2^a = \frac{1}{p_1^b}q_2^b .$$

This gives the super-Hamiltonian

$$\mathcal{H}^{(b)} = -p_0^{b^2} + q_2^{b^2} + q_2^{b^2}p_2^{b^2} - iq_2^b p_2^b + \tfrac{1}{4} . \tag{23}$$

The wave function is transformed

$$\psi^{(b)} = e^{-i(\ln p_1)q_2 p_2}\psi^{(a)} . \tag{24}$$

As a differential operator, the spatial part of (23) is the operator in the modified Bessel equation. It can be transformed to the operator in a Gegenbauer equation by making the interchange

$$p_2^b = -q_2^c, \quad q_2^b = p_2^c .$$

The super-Hamiltonian becomes

$$\mathcal{H}^{(c)} = -p_0^{c^2} + p_2^{c^2}(1+q_2^{c^2}) + ip_2^c q_2^c + \tfrac{1}{4} . \tag{25}$$

The new wave function is given by the Fourier transform

$$\psi^{(c)} = I_2\psi^{(b)} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dq_2^b e^{iq_2^b q_2^c}\psi^{(b)} , \tag{26}$$

where the subscript on I_2 indicates on which variable the transform acts.

The super-Hamiltonian has not been reordered with the momentum operators on the right to facilitate the application of the transformation:

$$q_2^c = q_2^d + \frac{i}{p_2^d} .$$

As a quantum operator, this can be expressed as

$$q_2^c = \frac{1}{p_2^d}q_2^d p_2^d . \tag{27}$$

This is an especially useful composite similarity transformation. It is the realization of a first-order intertwining operator as a canonical transformation [7], and it is an example of a transformation whose behavior is simpler in the quantum than the classical context. The transformed super-Hamiltonian is

$$\mathcal{H}^{(d)} = -p_0^{d^2} + (1+q_2^{d^2})p_2^{d^2} - iq_2^d p_2^d + \tfrac{1}{4} . \tag{28}$$

The wave function is

$$\psi^{(d)} = p_2^{-1}\psi^{(c)} , \tag{29}$$

where p_2^{-1} is the integral operator inverse to $p_2 = -i\partial_{q_2}$.

The point canonical transformation

$$p_2^d = \frac{1}{\cosh q_2^e}p_2^e, \quad q_2^d = \sinh q_2^e$$

reduces the super-Hamiltonian to the massive relativistic free particle in Minkowski space

$$\mathcal{H}^{(e)} = -p_0^{e^2} + p_2^{e^2} + \tfrac{1}{4} . \tag{30}$$

The new wave function is

$$\psi^{(e)} = P_{\sinh q_2}\psi^{(d)} . \tag{31}$$

The original wave function is found by inverting the sequence of canonical transformations. It is given by

$$\psi^{(0)} = P_{\ln q_0}q_2^{1/2}e^{i(\ln p_1)q_2 p_2}I_2^{-1}p_2 P_{\text{arcsinh}q_2}\psi^{(e)} . \tag{32}$$

This will be evaluated in the Appendix.

The accumulated canonical transformation from the original super-Hamiltonian to the relativistic free particle is

$$q_0 = \exp(q_0^e) \ ,$$

$$p_0 = \exp(-q_0^e)p_0^e \ ,$$

$$q_1 = \frac{\tanh(q_2^e)}{p_1^e}p_2^e + q_1^e \ ,$$

$$p_1 = p_1^e \ ,$$

$$q_2 = \frac{1}{p_1^e \cosh(q_2^e)}p_2^e \ ,$$

$$p_2 = -p_1^e \sinh(q_2^e) - \frac{3i}{2q_2} \ .$$

(33)

Note that a nonclassical term involving q_2 appears in p_2. This will be discussed below.

V. TRIVIALIZING THE FREE PARTICLE

The next step in the transformation to the holonomy variables is to trivialize the relativistic free particle super-Hamiltonian. This can be done in many ways, but the one which leads to the most classical transformation from the metric variables to triviality is accomplished by the sequence

$$p_0^e = (p_0^f)^{1/2}, \quad q_0^e = 2(p_0^f)^{1/2}q_0^f \ ,$$

$$p_0^f = p_0^g + p_2^{g^2} + \tfrac{1}{4}, \quad q_2^f = q_2^g - 2p_2^g q_0^g \ ,$$

$$p_0^g = -p_0^h, \quad q_0^g = -q_0^h \ .$$

(34)

(In Sec. VII, this will also be found to be the transformation which preserves the classical modular transformation properties of the variables.) These make the full transformation

$$p_0^e = (-p_0^h + p_2^{h^2} + \tfrac{1}{4})^{1/2} \ ,$$

$$q_0^e = -2(-p_0^h + p_2^{h^2} + \tfrac{1}{4})^{1/2}q_0^h \ ,$$

$$p_2^e = p_2^h \ ,$$

$$q_2^e = q_2^h + 2p_2^h q_0^h \ .$$

(35)

The resulting super-Hamiltonian is

$$\mathcal{H}^{(h)} = p_0^h \ .$$

(36)

As an aside, note that after the constraint $p_0^h = 0$ is applied, Eq. (35) implies that

$$p_0^e = \sqrt{p_2^{e^2} + \tfrac{1}{4}} \ .$$

(37)

This signifies that the canonical transformation used here is producing the solutions to this part of the full constraint. This corresponds classically to the ADM Hamiltonian (16). A second reduction would produce the solutions to

$$p_0^e = -\sqrt{p_2^{e^2} + \tfrac{1}{4}} \ .$$

That more than one reduction is needed to obtain all solutions is evident because the second-order operator (30) has been reduced to one of first order (36).

Substituting (35) into (33) gives

$$q_0 = \exp[-2(-p_0^h + p_2^{h^2} + \tfrac{1}{4})^{1/2}q_0^h] \ ,$$

$$p_0 = \exp[2(-p_0^h + p_2^{h^2} + \tfrac{1}{4})^{1/2}q_0^h](-p_0^h + p_2^{h^2} + \tfrac{1}{4})^{1/2} \ ,$$

$$q_1 = \frac{\tanh(q_2^h + 2p_2^h q_0^h)}{p_1^h}p_2^h + q_1^h \ ,$$

$$p_1 = p_1^h \ ,$$

$$q_2 = \frac{1}{p_1^h \cosh(q_2^h + 2p_2^h q_0^h)}p_2^h \ ,$$

$$p_2 = -p_1^h \sinh(q_2^h + 2p_2^h q_0^h) - \frac{3i}{2q_2} \ .$$

(38)

There is a clear correspondence with (18) with the identifications

$$q_0^h = t - t_0 \ ,$$

$$p_1^h = p_1 \ ,$$

$$p_2^h = c_0 \ ,$$

$$q_1^h = c_1 \ ,$$

$$q_2^h = c_2 \ .$$

(39)

The most evident differences between (18) and (38) are a nonclassical term in p_2 and the dependence of q_0 and p_0 on p_2^h ($=c_0$). Consider the latter first. When the super-Hamiltonian constraint $p_0^h = 0$ is imposed on physical states, the argument of the exponential in the formula for q_0 becomes $-2(p_2^{h^2} + \tfrac{1}{4})^{1/2}q_0^h$, instead of the $-2p_2^h q_0^h$ expected from (18) after the identification $p_2^h = c_0$. The difference lies in the quantum shift of $\tfrac{1}{4}$ which arose because the metric super-Hamiltonian is equivalent to a massive rather than a massless free particle. The effect of this shift is to make time flow differently in the quantum theory than in the classical.

The $\tfrac{1}{4}$ shift also contributes $(2iq_2)^{-1}$ to the nonclassical term in p_2. When this contribution is removed, the remaining $(iq_2)^{-1}$ is the quantity needed to adjust the momentum p_2 so that it becomes self-adjoint in the Petersson metric on moduli space. This is argued in Secs. X and XI to be the natural inner product for $2+1$ gravity in a torus universe. Arguably, the self-adjoint momentum operator, being observable, is the quantity which should be compared with the classical momentum. Thus, one sees that the $\tfrac{1}{4}$ shift introduces a quantum modification to the physically observable momentum.

Further effects on the flow of time in the quantum theory arise because of the noncommutativity of q_2^h and p_2^h. If one wants to eliminate the affine parameter q_0^h in favor of the physical time q_0, it is necessary to factor the exponentials in the hyperbolic functions defining the spatial coordinates and momentum. The Baker-Campbell-Hausdorff (BCH) formula may be used to find

$$\exp(q_2^h + 2p_2^h q_0^h) = \exp(q_2^h)\exp(-iq_0^h)\exp(2p_2^h q_0^h) \ .$$

(40)

An additional factor in the affine parameter has arisen. Eliminating the affine parameter in favor of q_0 gives

$$\exp(q_2^h + 2p_2^h q_0^h) = \exp(q_2^h) q_0^{(i/2 - p_2^h)(p_2^{h\,2} + 1/4)^{-1/2}} \; . \qquad (41)$$

(Note that in this formula p_0^h has been set to zero. This implies that the formula is valid only when applied to physical states which satisfy the super-Hamiltonian constraint. This restriction is important because the presence of p_0^h would prevent the simple device of raising q_0 to a power to replace q_0^h.)

Equation (41) can be used to obtain a quantum analogue of (19), but the result gives no additional insight. The point is that the modified relation between q_0 and q_0^h, together with the BCH factor, has significantly complicated the quantum analogue of (19), giving the dependence of the spatial variables on q_0. It is the nonpolynomial nature of these complications that stand in the way of factor ordering the classical result, as Carlip attempted.

There is an important conclusion about factor ordering to be inferred here. It deserves emphasis and will benefit from restatement: When one attempts to factor order a classical solution of a problem, one implicitly assumes that the time dependence of the quantum version is essentially unchanged from the classical. This is naively justifiable because time is a c number and commutes with the quantum position and momentum operators one is ordering. Closer inspection of time-dependent canonical transformations in quantum theory reveals however that details of the super-Hamiltonian sensitively affect the relation between the affine parameter and the physical time. For the time dependence of the ordered quantum operator to be the same as it is classically, the quantum relation between the affine parameter and the physical time must be the classical relation. This occurs only for special super-Hamiltonians.

There are of course additional obstacles to factor ordering involving nonpolynomial orderings of the spatial variables. Generally, both of these problems arise when the sequence of canonical transformations trivializing a super-Hamiltonian are different classically and quantum mechanically. As a rule, one cannot rely on the naive approach of looking for a polynomial factor ordering of a classical formula to find a quantum version.

VI. TRANSFORMATION TO HOLONOMY VARIABLES

To continue the transformation to holonomy variables, it is necessary first to rescale the momenta. It was observed classically that the transformation between the metric and holonomy variables is an extended canonical transformation for which the Poisson brackets among variables were not preserved but multiplied by a factor of 2. Rescaling of the momenta is considered a trivial canonical transformation by Landau and Lifshiftz [10], but it deserves a brief discussion because the quantum implementation differs from the conventional classical treatment.

Classically, when the momenta are rescaled by a constant factor $P = kp, Q = q$, the Hamiltonian is rescaled by the same factor [10]:

$$k(p\,dq - H\,dt) = P\,dQ - H'dt \; . \qquad (42)$$

The result is that Hamilton's equations do not change, and the transformation is canonical. An alternative procedure is to leave the Hamiltonian unchanged but to rescale the Poisson brackets:

$$\{q,p\} = 1, \quad \{Q,P\} = k \; . \qquad (43)$$

Clearly, this has the same effect of preserving the equations of motion. Quantum mechanically, however, only the second procedure is consistent. The canonical commutation relations induce a relationship between the original coordinate and momentum operators which must be respected. Since the coordinate does not change, $p = -i\partial_q = -i\partial_Q$ and therefore $P = kp = -ki\partial_Q$. As well, the form of the differential operator for the super-Hamiltonian is unchanged when passing from q to Q, so wave functions are preserved under the momentum rescaling

$$\psi(q) = \psi'(Q) \; . \qquad (44)$$

In the transformation to the holonomy variables, before making the rescaling, an interchange is used on p_2^h:

$$p_2^h = -q_2^i, \quad q_2^h = p_2^i \; .$$

This enables the rescaling transformation to act on the coordinate q_2^h. The rescaling transformation

$$p_1^i = 2p_1^j, \quad p_2^i = 2p_2^j \; ,$$

with the coordinates unchanged leaves the super-Hamiltonian unchanged:

$$\mathcal{H}^{(j)} = p_0^j \; . \qquad (45)$$

Note the commutation relations for the new variables are now $[q_\alpha^j, p_\alpha^j] = i/2$ $(\alpha = 1, 2)$. An inverse interchange

$$p_2^j = q_2^k, \quad q_2^j = -p_2^k$$

switches the coordinate back to a momentum.

The next series of transformations serve to rearrange the variables. They establish the correspondence with Carlip's holonomy variables. There is a large amount of freedom in choosing the canonical variables that are associated with a given super-Hamiltonian. The classical correspondence (20) between the integration constants and the holonomy variables guides the choice. The transformations are

$$p_1^k = -q_1^l, \quad q_1^k = p_1^l \; ,$$

$$p_1^l = e^{-q_1^m} p_1^m, \quad q_1^l = e^{q_1^m} \; ,$$

$$p_1^m = p_1^n + \tfrac{1}{2} p_2^n, \quad q_2^m = q_2^n - \tfrac{1}{4} q_1^n \; ,$$

$$q_1^n = q_1^o + q_2^o, \quad p_2^n = p_2^o - p_1^o \; ,$$

$$q_1^o = \ln q_1^p, \quad p_1^o = q_1^p p_1^p \; ,$$

$$q_2^o = \ln q_2^p, \quad p_2^o = q_2^p p_2^p \; ,$$

$$q_2^p = -p_2^q, \quad p_2^p = q_2^q \; .$$

The accumulated transformations from the h variables are

$$p_1^h = 2q_1^q p_2^q ,$$

$$q_1^h = \frac{1}{2}\left[\frac{q_2^q}{q_1^q} - \frac{p_1^q}{p_2^q}\right] ,$$

$$p_2^h = -(q_1^q p_1^q + p_2^q q_2^q) ,$$

$$q_2^h = \ln(-p_2^q/q_1^q) .$$

(46)

Using (39), this agrees with the classical correspondence (20) between the integration constants and Carlip's holonomy variables, where

$$q_1^q = a, \quad p_1^q = \mu, \quad q_2^q = b, \quad p_2^q = -\lambda .$$

(47)

Using these in (38) and eliminating the affine parameter in favor of the physical time with (41) gives the full transformation from the metric variables to the holonomy variables. For notational convenience, let

$$\alpha = \left[p_2^h - \frac{i}{2}\right](p_2^{h^2} + \tfrac{1}{4})^{-1/2} ,$$

$$\beta = (p_2^h - i)(p_2^{h^2} + \tfrac{1}{4})^{-1/2} .$$

One finds

$$p_1 = 2q_1^q p_2^q ,$$

$$q_1 = \frac{1}{q_1^{q^2} + p_2^{q^2} q_0^{-2\beta}}(q_1^q q_2^q - p_1^q p_2^q q_0^{-2\beta}) ,$$

$$p_2 = -q_0^\alpha(q_1^{q^2} - p_2^{q^2} q_0^{-2\beta}) - \frac{3i}{2q_2} ,$$

$$q_2 = \frac{1}{q_1^{q^2} + p_2^{q^2} q_0^{-2\beta}}q_0^{-\alpha}(q_1^q p_1^q + p_2^q q_2^q) .$$

(48)

In the classical limit, with the $\tfrac{1}{4}$ shift term dropped, both α and β go to 1, so the transformation agrees with that found by Carlip [5]. Quantum mechanically, α and β are nonpolynomial ordering terms modifying the time dependence as discussed above. Also note that, classically, if one drops the quantum shift $\tfrac{1}{4}$ from the constraint (37), one finds

$$p_0 = \frac{-(a\mu - \lambda b)}{q_0} .$$

(49)

This is the classical expression that Carlip found for the ADM-Moncrief Hamiltonian (16) in terms of the holonomy variables.

VII. MODULAR TRANSFORMATIONS

The metric super-Hamiltonian is invariant under the symmetry of modular transformations of the upper-half plane. This symmetry has its origin in the diffeomorphism invariance of the original $(2+1)$-gravity theory, as will be discussed in the next section. Defining $q = q_1 + iq_2$ and $p = p_1 + ip_2$, classically the modular transformations on the metric variables are generated by

$$T_m : q \to q+1, \quad T_m : p \to p ,$$

$$S_m : q \to -1/q, \quad S_m : p \to q^{\dagger 2}p .$$

(50)

These correspond to the transformations on the classical holonomy q-variables:

$$T_h : (q_1^q, p_1^q) \to (q_1^q, p_1^q - p_2^q) ,$$

$$T_h : (q_2^q, p_2^q) \to (q_1^q + p_2^q, p_2^q) ,$$

$$S_h : (q_1^q, p_1^q) \to (q_2^q, p_2^q) ,$$

$$S_h : (q_2^q, p_2^q) \to (-q_1^q, -p_1^q) .$$

(51)

In the quantum theory, it is evident that the modular transformations of q induce the correct corresponding transformations of p, in the coordinate representation where $p_1 = -i\partial_{q_1}$ and $p_2 = -i\partial_{q_2}$. Thus, the classical modular symmetry is consistently implemented quantum mechanically.

It is not obvious however that the quantum q-variables defined in terms of the transformations above have their classical modular transformation properties. Modular invariance of the quantum super-Hamiltonian is clearly maintained, but since the holonomy super-Hamiltonian is trivialized, it imposes no direct condition on the transformation properties of its variables. Carlip's primary requirement [5,6] in his approach to factor ordering the classical solution was to find the quantum transformation which preserves the classical modular transformation properties of the holonomy variables. This, together with the practical restriction to polynomial orderings, led him to conclude that the metric super-Hamiltonian had to be modified. Above, a nonpolynomial ordering was found which transforms from the unmodified metric super-Hamiltonian to the holonomy version. It remains to check that the modular transformation properties have been preserved.

The modular transformations (50) and (51) are themselves canonical transformations. By finding their explicit representation as a product of elementary canonical transformations, the problem of comparing them is greatly simplified. The canonical transformations producing the modular transformations in the holonomy variables are not difficult to find because they are linear canonical transformations:

$$T_h = e^{i2q_1^q p_2^q} ,$$

$$S_h = e^{i2q_2^q p_1^q} e^{-i2q_1^q p_2^q} e^{i2q_2^q p_1^q} .$$

(52)

In the metric variables, in complex form $q = q_1 + iq_2$, $p = p_1 + ip_2$, the canonical transformations are again linear. There is a subtlety because the momentum operator conjugate to q is $p^\dagger = -2i\partial_q$. Recognizing that $(q, p^\dagger)$ and $(q^\dagger, p)$ are two independent (commuting) sets of variables, one finds

$$T_m = e^{ip_1} ,$$

$$S_m = e^{-iq^{\dagger 2}p/2} e^{-ip/2} e^{-iq^{\dagger 2}p/2} e^{-iq^2 p^\dagger/2} e^{-ip^\dagger/2} e^{-iq^2 p^\dagger/2}$$

$$= \exp\{ -i[(q_1^2 - q_2^2)p_1 + 2q_1 q_2 p_2] \} e^{-ip_1} \exp\{ -i[(q_1^2 - q_2^2)p_1 + 2q_1 q_2 p_2] \} . \tag{53}$$

Comparing these, the modular transformations are the same if

$$p_1 = 2q_1^q p_2^q \tag{54}$$

and

$$(q_1^2 - q_2^2)p_1 + 2q_1 q_2 p_2 = -2q_2^q p_1^q . \tag{55}$$

Both conditions are satisfied by the transformation from the metric to holonomy variables. The first is obvious as it is one of the transformations. The second follows from a computation most easily done by first passing from the metric variables to the h variables and then to the q variables. One finds

$$(q_1^2 - q_2^2)p_1 + 2q_1 q_2 p_2 = q_1^h p_1^h q_1^h - \frac{p_2^{h^2}}{p_1^h}$$

$$= -2q_2^q p_1^q . \tag{56}$$

Thus, the quantum canonical transformation given here from the metric to the holonomy variables preserves the modular transformation properties of each. A word of caution should be raised: it is not difficult to find other canonical transformations to other variables which have the correct classical correspondence and which preserve modular invariant quantities, but do not have the desired modular transformation properties. This shows that there is important information contained in the transformation properties of noninvariant quantities.

VIII. UNITARY EQUIVALENCE

Having constructed the canonical transformation between the metric and holonomy variables which preserves their classical modular transformation properties, it is necessary to determine if the transformation is unitary. If it is, this will complete the proof of the unitary equivalence of the two quantum theories.

The familiar definition of a unitary transformation is a linear norm-preserving isomorphism of one Hilbert space onto itself. This definition can be naturally extended to a linear norm-preserving isomorphism from one Hilbert space to another [8]. Canonical transformations are not in themselves unitary as they are defined independent of the Hilbert space structure and transform all solutions, not just normalizable ones, of one super-Hamiltonian constraint to solutions of another. For a canonical transformation to define a unitary equivalence, when restricted to act on the Hilbert space of states of one theory, it must be a unitary transformation to the Hilbert space of states of the other theory.

In proving the unitary equivalence of the metric and holonomy formulations of $2+1$ gravity on the torus, there is a difficulty because Witten (and his successors)

did not derive the inner product which defines the Hilbert space of states in the holonomy variables. The holonomy-variable inner product for which the two formulations are unitarily equivalent will be constructed by requiring that the value of transition amplitudes computed in the metric-variable inner product be preserved through the canonical transformation. After checking that the kernel of the canonical transformation does not lie in the Hilbert spaces, it is concluded that the canonical transformation is a norm-preserving isomorphism of the Hilbert spaces. The two theories are then equivalent. If one were to choose a different modular invariant inner product in the holonomy formulation, the two theories would be unitarily inequivalent.

An inner product is characterized by its measure density $\mu(\mathbf{q},\mathbf{p})$ [$\mathbf{q} = (q,q_0)$, $\mathbf{p} = (p,p_0)$, where q,p stand for all of the spatial variables]:

$$\langle \phi | \psi \rangle_\mu \equiv \int dq\, \phi(q)^* \mu(\mathbf{q},\mathbf{p}) \psi(q) , \tag{57}$$

where $\mathbf{p}$ acts to the right. Note that, in general, the measure density may be operator valued and time dependent. The standard inner product with trivial measure density is given by $\mu = 1$.

In general, when one makes a canonical transformation, if the value of the inner product is to be preserved, the measure density will transform. If one has the canonical transformation between solutions,

$$\psi^{(0)} = C \psi^{(a)} ,$$

then formally one has

$$\langle \phi^{(0)} | \psi^{(0)} \rangle_{\mu^{(0)}} = \langle C\phi^{(a)} | \mu^{(0)} | C\psi^{(a)} \rangle_1$$

$$= \langle \phi^{(a)} | C^{\dagger_1} \mu^{(0)} C | \psi^{(a)} \rangle_1$$

$$= \langle \phi^{(a)} | \psi^{(a)} \rangle_{\mu^{(a)}} . \tag{58}$$

The transformed measure density is

$$\mu^{(a)}(\mathbf{q},\mathbf{p}) = C^{\dagger_1} \mu^{(0)}(\mathbf{q},\mathbf{p}) C . \tag{59}$$

Here, $C^{\dagger_1}$ is the adjoint of C in the trivial measure density.

For functions of q and p, the adjoint in the trivial measure density is the operator formed by complex conjugation and integration by parts (all boundary terms are assumed to vanish, though this must be checked in specific examples). The "adjoints" of the interchange operator and the point canonical transformation can be computed by direct manipulation of inner products in which they appear. They are found to be

$$P_{f(q)}^{\dagger_1} \equiv f^{-1}(q)_{,q} P_{f^{-1}(q)} ,$$

$$I^{\dagger_1} \equiv I^{-1} . \tag{60}$$

[The factor $f^{-1}(q)_{,q}$ in the point canonical transformation adjoint arises from the transformation of the dq in the measure. This is the one-dimensional form.] For canonical transformations involving p_0, the adjoint cannot be taken because the inner product does not involve an integration over dq_0. Point canonical transformations of the time simply redefine the variable one uses for the time label and are implemented directly in the measure density by changing its time dependence accordingly.

In common practice, one usually only considers unitary transformations between a Hilbert space and itself. In these cases, the measure density does not change, and one finds from (59) that $C^\dagger C = 1$, where $C^\dagger = \mu^{(0)-1} C^\dagger{}_! \mu^{(0)}$ is the adjoint in the measure density $\mu^{(0)}$ of the Hilbert space. It should be emphasized that canonical transformations which are not naively unitary, such as multiplication by a real function of the coordinate, become so when the measure density is appropriately transformed.

There is a natural inner product in the metric formulation of $2+1$ gravity on the torus given by the measure density for the Petersson metric on the upper-half plane, $\mu^{(0)} = q_2^{-2}$,

$$\langle \phi^{(0)} | \psi^{(0)} \rangle_{\mu^{(0)}} = \int \frac{dq_1 dq_2}{q_2^2} \phi^{(0)*} \psi^{(0)} . \tag{61}$$

The canonical transformation from the holonomy to metric variables is summarized by

$$\psi^{(0)} = C \psi^{(q)} . \tag{62}$$

It is convenient for presentation to decompose C into three transformations

$$C = C_1 C_2 C_3 ,$$

where

$$\psi^{(0)} = C_1 \psi^{(e)} ,$$
$$\psi^{(e)} = C_2 \psi^{(h)} , \tag{63}$$
$$\psi^{(h)} = C_3 \psi^{(q)} .$$

The canonical transformations are then

$$C_1 = P_{\ln q_0} q_2^{1/2} e^{i(\ln p_1)q_2 p_2} I_2^{-1} p_2 P_{\mathrm{arcsinh} q_2} ,$$

$$C_2 = P_{p_2} e^{i(p_2^2 + 1/4)q_0} e^{\pi q_0 p_0} , \tag{64}$$

$$C_3 = I_2^{-1} R_{1/2} I_2 I_1^{-1} P_{\ln q_1} e^{ip_2 q_1/2} e^{-ip_1 q_2} P_{e^{q_1}} P_{e^{q_2}} I_2 .$$

Here, $R_{1/2}$ scales the momenta by a factor of $\frac{1}{2}$. Its "adjoint" would act to double any momenta appearing in the measure density.

Using (59), one can compute the transformed measure density one transformation at a time. The final result is that

$$\mu^{(q)} = -(q_1^q p_1^q + p_2^q q_2^q) . \tag{65}$$

The inner product in the holonomy variables which preserves the value of inner products (61) in the metric variables is then

$$\langle \phi^{(q)} | \psi^{(q)} \rangle_{\mu^{(q)}} = -\int dq_1^q dq_2^q \phi^{(q)*} (q_1^q p_1^q + p_2^q q_2^q) \psi^{(q)} . \tag{66}$$

Before concluding that the metric variable theory with inner product (61) is unitarily equivalent to the holonomy variable theory with this inner product, one must confirm that no states in the Hilbert space are in the kernel of the transformation C. All of the transformations in C are invertible except for p_2 in C_1 whose kernel is spanned by the function 1. Transforming this function to find its expression in the holonomy variables, one finds a function which is not modular invariant. It is therefore not a member of the Hilbert space. As this is the only function which is annihilated by the transformation C, all modular invariant functions in the holonomy variables are mapped to modular invariant functions in the metric variables and vice versa. (The preservation of modular invariance is guaranteed by the considerations of Sec. VII.)

The canonical transformation C is a (linear) norm-preserving isomorphism of the Hilbert spaces defined by the inner products (61) and (66). Therefore, one may conclude that the metric and holonomy formulations of $2+1$ gravity on the torus are unitarily equivalent. If one chooses a different modular invariant inner product in the holonomy variables, for example, a trivial measure $\mu^{(q)} = 1$, then the metric and holonomy formulations would not be unitarily equivalent.

This emphasizes the important point that a quantum theory is not complete until the Hilbert space is specified. Note that once one allows operator-valued measure densities, it is not clear what criteria one uses to choose among those in which the Hamiltonian is self-adjoint and which are invariant under the necessary symmetries. This introduces a new ambiguity into quantization.

IX. METRIC FROM HOLONOMY WAVE FUNCTIONS?

Having proven the unitary equivalence of the metric and holonomy formulations of $2+1$ gravity on the torus, one might hope to use the holonomy wave functions to construct the metric wave functions by simply applying the canonical transformation C. This would be a significant achievement because the metric wave functions are the weight-zero Maass forms and are of interest to number theorists [11]. In principle, this procedure is straightforward. There are few subtleties in applying C, even with its time dependence—the time is simply a parameter in the transformations. To illustrate the use of a canonical transformation, in the Appendix the transformation C_1 is applied to plane-wave solutions of the relativistic free particle to obtain the corresponding (nonmodular invariant) solutions of the metric super-Hamiltonian.

Unfortunately, there is no "free lunch." The trouble is that the canonical transformation transforms all solutions to the holonomy super-Hamiltonian, that is, all time-independent functions, to solutions of the metric super-Hamiltonian. The solutions of interest, however, are only the normalizable modular invariant functions. It is straightforward to define what one means by a modular invariant function in the holonomy variables using the

transformations (51), but to the author's knowledge, these functions are not known explicitly. Hence, they cannot be transformed to give explicit representations of the Maass forms. Furthermore, it is likely that, were they known, their expression would not be in closed form, but in the form of infinite series. Evaluation of the various Fourier transforms involved in C would then result in a series expansion. It is not obvious that the subtleties of Maass forms would be more transparent in this form. Granted, it is an improvement to be able to work with modular invariant functions which are not constrained to satisfy a differential equation, but further investigation is required.

X. θ STATES FOR 2+1 QUANTUM GRAVITY

In deriving the metric super-Hamiltonian, the configuration space arises from gauge fixing the diffeomorphism invariance of 2+1 gravity. The reduction from the space of all metrics on the torus to moduli space is made by observing that every metric on the torus is conformal to one of constant zero curvature. In particular, the metric can be expressed in the form $ds^2 = e^\rho |dx + \tau dy|^2$, where $\tau = q_1 + iq_2$ is a complex parameter. Restricting attention to the tori of zero curvature having the metric $ds^2 = |dx + \tau dy|^2$, fixes the "small" diffeomorphisms, i.e., those that are continuously deformable to the identity. The zero-curvature tori are classified by their moduli $\tau = q_1 + iq_2$ and the Teichmüller space of the torus is the Poincaré upper-half plane H. From this, it follows that the wave function will be a function of the moduli, and it is natural to expect that the inner product will be that on the upper-half plane.

In addition to small diffeormorphisms, there are also "large" diffeomorphisms which are not fixed by restricting to zero-curvature tori. These are diffeomorphisms which are not continuously deformable to the identity and correspond to Dehn twists: the action of cutting open the torus along a homotopically nontrivial loop and twisting the end before gluing the manifold back together again.

Large diffeomorphisms are the analog of the large gauge transformations in Yang-Mills theory that give rise to the θ vacua [12]. The possibility of θ states in gravity has been discussed in the past [13,14]. Until recently [15], no explicit examples were known. Gravitational θ states are present in (2+1)-quantum gravity on the torus, and they are constructed below. Their existence was overlooked in earlier treatments [5,6,9].

Since two tori related by a large diffeomorphism are physically equivalent, their moduli do not represent distinct configurations. The action of the large diffeomorphisms on moduli space is given by the mapping class group, $\Gamma = SL(2,Z)/Z_2$. The physically distinct moduli lie in a fundamental domain H/Γ which, under the action of the mapping class group, tesselates the upper-half plane.

A solution $\psi^{(0)}$ of the metric super-Hamiltonian constraint on Teichmüller space must be "periodized" to account for the physical equivalence of moduli in different copies of the fundamental domain. This is accomplished

by the method of images in which the wave function $\Psi^{(0)}(q)$ at a point $q = q_1 + iq_2$ in the fundamental domain is found by summing the solution $\psi^{(0)}$ evaluated at every image of q under the mapping class group, each weighted by some factor χ_α,

$$\Psi^{(0)}(q) = \sum_{\alpha \in \Gamma} \chi_\alpha \psi^{(0)}(\alpha q) . \tag{67}$$

Contrary to naive expectation [5,9], the wave function is not required to be invariant under the action of the mapping class group, but rather it must transform as a representation of the group. This determines the possible weights χ_α.

The situation is analogous to that first considered by Laidlaw and DeWitt [16]. They were studying the propagator for a particle in a multiply connected configuration space, but a related argument works for the wave function and applies to covering groups of nontopological origin. In the sum over images, the fundamental domain is a particular coset representative H/Γ that has been arbitrarily chosen. If a second coset representative were selected to be the fundamental domain by acting with an element of the mapping class group, the new wave function must be unitarily equivalent to the original wave function. Thus, if the wave function $\Psi^{(0)}$ is to be a (modular weight zero) scalar, it can change at most by a scalar phase

$$\Psi^{(0)}(\beta q) = e^{-i\phi(\beta)} \Psi^{(0)}(q) , \tag{68}$$

and every image in the sum must be changed by the same phase

$$\sum_{\alpha \in \Gamma} \chi_\alpha \psi^{(0)}(\alpha\beta q) = \sum_{\alpha \in \Gamma} e^{-i\phi(\beta)} \chi_{\alpha\beta} \psi^{(0)}(\alpha\beta q) .$$

This is true (though possibly not in the most general way) if $e^{i\phi(\beta)}\chi_\alpha = \chi_{\alpha\beta}$. Assuming without loss of generality that the weight in the fundamental domain is one, $\chi_e = 1$, one finds $\chi_\beta = e^{i\phi(\beta)}$. Since the mapping class group is non-Abelian while phases commute, the phases must form a one-dimensional unitary representation of the Abelianization of the mapping class group.

The mapping class group is generated by the two fundamental modular transformations

$$S: q \rightarrow -q^{-1} ,$$

$$V: q \rightarrow -(q+1)^{-1} ,$$

where $V = TS$, in terms of T used above. The modular transformations satisfy the relations: $S^2 = 1$, $V^3 = 1$. Every element of the mapping class group can be represented as an element of the free product of S and V. By assigning a phase to S and V that is consistent with their relations, a phase is assigned is consistent with their relations, a phase is assigned to each element of the mapping class group. This phase is a character of the Abelianization of the mapping class group, $Z_2 \times Z_3$. The possible phases are $(1, e^{i\pi})$ for S and $(1, e^{2\pi i/3}, e^{4\pi i/3})$ for V. Thus, there are potentially five nontrivial scalar θ states consistent with modular transformations, in addition to the modular invariant wave function. Two of these are not physically distinct, however, as the states formed

with the V phase $e^{4\pi i/3}$ are the complex conjugate of those formed with $e^{2\pi i/3}$. There are three nontrivial scalar θ states.

Note that from (68) the wave function $\Psi^{(0)}$ must vanish at the fixed points of a modular transformation if the phase associated with that transformation is not unity. This defines the boundary conditions that are associated with each of the θ states. These boundary conditions are analogous to those that arise, say, at a reflecting wall where a wave function must vanish because $\phi(x) = -\phi(-x)$.

θ states were found by dropping the requirement that the wave function be modular invariant. Further generalizations are possible by weakening other assumptions. If the wave function $\psi^{(0)}$ is not a scalar, but has a vector or spinor index, as the spinor wave function of Carlip's Dirac square root does, non-Abelian weights are possible. If the weight is a matrix of the same dimension as that of the vector (spinor), by contracting it with the wave function's index, a vector (spinor) of the same dimension is obtained, and the above argument may be repeated. It is found that the weights must be in a unitary matrix representation of the mapping class group of dimension equal to the dimension of the vector (spinor). There are more unitary representations in higher dimensions than in one, so there will be more θ states, and many will have the novel feature of being non-Abelian. This does not happen in Yang-Mills theory and is (so far) unique to gravitational θ states.

In principle, one can also use non-Abelian weights in higher-dimensional unitary representations of the mapping class group even when $\psi^{(0)}$ is a scalar (or is of different dimension than the weight). This possibility was raised by Hartle and Witt [14]. The result of doing so is that the wave function $\Psi^{(0)}$ is no longer a scalar but carries a group label, forming a higher-dimensional representation of the mapping class group. The inner product must accordingly be adjusted to remain a modular invariant, so that the group label does not appear in transition amplitudes. The physical significance of such an extended wave function is not clear.

XI. OBSERVABLES

Since the eigenvalues of a complete set of independent observables characterize the quantum state, it is useful to consider the observables for $2+1$ gravity in a toroidal universe. An observable is an operator which commutes with all of the constraints, up to a function of the constraints. This definition may be unfamiliar because one is not used to quantum mechanics in the presence of constraints [17]. If constraints are present, they must be preserved. Suppose a wave function Ψ satisfies the constraints C_i, and $[A, C_i] = B_i$, then

$$[A, C_i]\Psi = -C_i A\Psi = B_i \Psi .$$

If $A\Psi$ is to continue to satisfy the constraints, B_i must be a function of the constraints.

Here, the observables must commute with the super-Hamiltonian and be modular invariant. Because the super-Hamiltonian is modular invariant, there are no ad-

ditional constraints. Inspection reveals that p_1 and $H^2 = q_2^2(p_1^2 + p_2^2)$ commute with the metric super-Hamiltonian. Since the configuration space is two dimensional, they are complete. Unfortunately, p_1 is not modular invariant. This means that its eigenvalues will not be quantum numbers of the "periodized" quantum states (67). They are nevertheless useful because they characterize the quantum states $\psi^{(0)}$, appearing in the sum over images, which are constructed in the Appendix. It is an open problem (to the author's knowledge) to find a complete set of modular invariant observables.

Following the $*$-algebra approach [18], an inner product of a constrained system can be constructed by requiring that each of the observables be self-adjoint. Lacking a complete set of modular invariant observables, a physical inner product for the periodized quantum states $\Psi^{(0)}$ cannot be constructed directly. It is instructional, however, to construct the inner product for $\psi^{(0)}$ before imposing the symmetry of modular invariance. Since the coordinates and momenta satisfy the ordinary commutation rules in the coordinate representation, they have the usual conjugation properties. Assume that the measure density is solely a function of the coordinates. Then, for p_1 to be self-adjoint, the measure density must be independent of q_1 while, for H^2 to be self-adjoint, the measure density must be q_2^{-2}. The inner product is thus determined to be

$$\int \frac{1}{q_2^2} dq_1 dq_2 \psi^* \phi . \tag{69}$$

This is the standard inner product on the Poincaré upper-half plane with the Petersson metric, as it must be since H^2 is the covariant Laplace operator on this space. Note that the measure is modular invariant, even though this is not *a priori* obvious from the construction. This implies that this is in fact the physical inner product. This construction may seem superfluous since it was already known that the configuration space is the moduli space of the torus and thus has this as its natural inner product. In more general problems, however, an understanding of the configuration space may not precede the construction of observables.

XII. CONCLUSION

A sequence of elementary canonical transformations has been found that trivializes the metric super-Hamiltonian of $(2+1)$-dimensional quantum gravity on the torus. A further sequence establishes the quantum canonical transformation between the metric variables and the holonomy variables of Carlip. The full transformation is a nonpolynomial factor ordering of the classical canonical transformation which preserves the classical modular transformation properties of the metric and holonomy variables. The procedure used here enabled a systematic derivation of the transformation.

The definition of a unitary transformation was extended to apply to transformations between Hilbert spaces having inner products with different (operator-valued) measure densities. Requiring that the canonical transformation be such a unitary transformation from the natural

Hilbert space in the metric formulation to the Hilbert space in the holonomy formulation gave a construction of the inner product (66) in the holonomy formulation. This proves that the Witten-Carlip holonomy formulation of $(2+1)$-dimensional quantum gravity on the torus with this Hilbert space is unitarily equivalent to the Moncrief metric formulation with its natural Hilbert space. If a different inner product were used in the holonomy variables, the formulations would be inequivalent.

The primary motivation for studying $2+1$ gravity on the torus is to understand the different natures of time in the metric and holonomy formulations. There is no mystery here, but rather an important lesson. Among all possible variables equivalent under canonical transformation, Witten fortuitously chose a set that was time-independent while Moncrief did not. Mathematically, both choices are equally valid and equivalent. Physically, it becomes a question of what one can measure with experimental apparatus. Set aside the inconvenient fact that strictly speaking neither the metric nor the holonomy variables are observable since they are not modular invariant. If one had at hand a device which measures holonomy, then as one went out into one's $(2+1)$-dimensional world, there would appear to be no dynamics. If, on the other hand, one were to measure the moduli of the manifold, one would find that they change as the volume of the Universe grows. If one went out with a device which measures something else, one would find yet different dynamics.

The lesson is first a familiar one from classical mechanics: when solving a problem, one is free to choose the variables which make it convenient to solve; the physics does not depend on the choice of variables. Second, it is a reminder that "time" is not a coordinate label, but a perception that follows from physical observation. While it does not matter what variables one chooses to formulate a theory, it makes all the difference what quantities one measures experimentally. It is easy to confuse measurable quantities with the variables that represent them most conveniently, but those same quantities can be expressed in different variables with no loss of information. Time, as a quantity whose passage is inferred indirectly from the change of other measured quantities, is no different: it is not the coordinate q_0 which appears in the super-Hamiltonian. The challenge is to understand the connection between measurable quantities and the physical passage of time so that this connection can be taken over into the theory and be preserved as a relation amongst variables. Properly executed, the relation will persist no matter what the choice of variables.

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APPENDIX

The formal solution of the metric super-Hamiltonian can be constructed by applying the canonical transforma-

tion C_1 of (64) to the relativistic free particle wave function. As we are interested in the stationary states of the metric super-Hamiltonian, we begin with the familiar positive-frequency plane-wave solutions of the relativistic free particle, multiplied by a plane wave in q_1:

$$\psi_{k,n}^{(e)}(q_1,q_2,q_0)=\exp(ikq_2-i\omega q_0)\exp(2\pi inq_1)\,, \tag{A1}$$

where $\omega=(k^2+\tfrac{1}{4})^{1/2}$. Strictly, any function of q_1 gives a solution of $\mathcal{H}^{(e)}$; the plane wave is chosen so that $\psi^{(e)}$ is an eigenstate of the (partial) observable p_1, discussed in Sec. XI. The original wave function is given

$$\psi^{(0)}(q_1,q_2,q_0)=P_{\ln q_0}q_2^{1/2}e^{i\ln p_1 q_2 p_2}I_2^{-1}p_2$$
$$\times P_{\mathrm{arcsinh}q_2}\psi^{(e)}(q_1,q_2,q_0)\,. \tag{A2}$$

Focusing on the q_2-dependent part (suppressing the subscript 2), the transformation

$$I^{-1}pP_{\mathrm{arcsinh}q}e^{ikq}=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dq'e^{-iqq'}(-i\partial_{q'})$$
$$\times\exp(ik\,\mathrm{arcsinh}q') \tag{A3}$$

is to be evaluated. Doing the $\partial_{q'}$ derivative and using

$$\mathrm{arcsinh}q'=\ln[q'+(q'^2+1)^{1/2}]$$

gives

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dq'e^{-iqq'}k\frac{[q'+(q'^2+1)^{1/2}]^{ik}}{(1+q'^2)^{1/2}}\,. \tag{A4}$$

The change of variables $u=q'+(q'^2+1)^{1/2}$ gives an integral which integrates to a modified Bessel function, up to a constant N,

$$\frac{k}{(2\pi)^{1/2}}\int_0^{\infty}du\exp\left[-iq\left[u-\frac{1}{u}\right]\bigg/2\right]u^{ik-1}$$
$$=NK_{ik}(q)\,. \tag{A5}$$

Applying $\exp[i\ln(p_1)q_2p_2]$ to this scales q_2 by a factor of p_1, giving

$$\exp[i\ln(p_1)q_2p_2]K_{ik}(q_2)\exp(2\pi inq_1)$$
$$=K_{ik}(q_2p_1)\exp(2\pi inq_1)\,. \tag{A6}$$

Acting with the operator p_1, this becomes

$$K_{ik}(2\pi nq_2)e^{2\pi inq_1}\,. \tag{A7}$$

Including the final transformations, the wave function is

$$\psi_{n,k}^{(0)}=Nq_2^{1/2}K_{ik}(2\pi nq_2)e^{2\pi inq_1}q_0^{i\omega}\,. \tag{A8}$$

From Sec. X, the full periodized wave function is

$$\Psi^{(0)}(q_1,q_2,q_0)=\sum_{\alpha\in\Gamma}\chi_\alpha\psi_{n,k}^{(0)}(\alpha q_1,\alpha q_2,q_0)\,, \tag{A9}$$

where χ_α is a unitary one-dimensional representation of

$Z_2 \times Z_3$. Thus, the solutions of the metric super-Hamiltonian have been constructed using canonical transformations. It should be noted that the condition of normalizability has not been applied, and one does not expect every periodized sum (A9) to correspond to a normalizable wave function. The normalizable wave functions are, however, expected to be among those obtained by the periodizing procedure.

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