Time-asymmetric structure of gravitational radiation

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Gravitational radiation reaction effects in the dynamics of an isolated system arise from the use of retarded potentials for the radiation field, satisfying time-asymmetric boundary conditions imposed at past-null infinity. Part one of this paper investigates the "antisymmetric" component, a solution of the wave equation of the type retarded minus advanced, of the linearized gravitational field generated by an isolated system in the exterior region of the system. At linearized order such a component is well defined and is "time odd" in the usual post-Newtonian (PN) sense. We introduce a new linearized coordinate system which generalizes the Burke and Thorne coordinate system both in its space-time domain of validity, which is no longer limited to the near zone of the source, and in the post-Newtonian smallness of the linear antisymmetric ("time-odd") component of the metric, for all multipolarities of antisymmetric waves. These waves (as viewed in the near zone) define a generalized radiation reaction four-tensor potential $V^{\alpha\beta}_{\rm react}$ of the linear theory. At the 2.5 post-Newtonian approximation, the tensor potential reduces to the standard Burke-Thorne scalar potential of the lowest-order local radiation reaction force. At the 3.5 PN approximation, the potential involves scalar ($V^{00}_{\rm react}$) and vector ($V^{0i}_{\rm react}$) components which are associated with subdominant radiation reaction effects such as the recoil effect. At the higher-order PN approximations, the potential is intrinsically tensorial. A nonlinear exterior metric is iteratively constructed from the new linearized metric by the method of a previous work. Part two of this paper is devoted to the near-zone reexpansion of the nonlinear iterations of the exterior metric. We use a very convenient decomposition of the integral of the retarded potentials into a particular solution involving only "instantaneous" potentials, and a homogeneous solution of the antisymmetric type. The former particular solution is "even" in the sense that it explicitly contains only even powers of c^{-1}. The latter homogeneous solution defines a component of the exterior metric which is associated with radiation reaction effects of nonlinear origin. This decomposition of the retarded integral enables us to control the occurrence and the magnitude of "odd" terms in any nonlinear iterations of the metric, and to compute explicitly the radiation reaction potential of the nonlinear theory up to the 3.5 PN approximation. Finally we recover and complete a previous work concerning the hereditary modification, of quadratic nonlinear origin, of the radiation reaction potential at the 4 PN approximation.

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I. INTRODUCTION

A. Motivation and summary

The problem of computing the gravitational forces acting on an isolated system in reaction to the emission of gravitational radiation is an outstanding problem in general relativity. Gravitational radiation reaction forces, such as electromagnetic ones, arise from the use of retarded potentials for the radiation field, satisfying *time-asymmetric* boundary conditions, for instance, no-incoming-radiation conditions imposed at past-null infinity. The gravitational problem is made technically more difficult than the electromagnetic problem by the nonlinearity of Einstein's equations. Its importance lies in the exciting present day possibility of comparing the theory with astrophysical observations.

For the moment, the only direct evidence that gravitational radiation reaction forces are at work in the real world is in the observed dynamics of the Hulse-Taylor [1] binary pulsar PSR 1913+16. The orbital period P of this pulsar around its companion has been observed since 1974 to steadily decrease at a rate $\dot{P}$ which is perfectly consistent with the expectation that the binary system loses energy by gravitational radiation [2]–[6]. The general relativistic formula for $\dot{P}$, which now numerically agrees to within 0.5% with the observations [6], can be derived heuristically by an energy balance argument based on the Einstein quadrupole equation [7]–[10] or, more rigorously, by a study of the dynamics of a system of two neutron stars up to the level where radiation reaction effects appear [11]–[13]. Thus, the binary system of the pulsar and its companion is actually emitting gravitational radiation, and the radiation is reacting back on the system (it obeys time-asymmetric boundary conditions).

In the future, gravitational-wave astronomy will open a new window on our Universe, and we expect to observe systems in which gravitational radiation reaction plays an important role. This is of course the case for coalescing compact binary systems, whose dynamics is driven by radiation reaction, but also for supernova explosions which may undergo a non-negligible net change of total linear momentum (recoil) in reaction to the emission of waves [14].

The first explicit expression of the local radiation reaction force in general relativity was obtained by Burke [15] and Thorne [16] using a method of matched asymptotic expansions (see also Refs. [17–19]). This force, which involves the fifth time derivative of the quadrupole moment of the source, can be viewed as the gravitational analogue of the Lorentz damping force of electromagnetism. Its interpretation as a reaction force rests on the fact that it appears as the first post-Newtonian correction in the equations of motion of the source that changes sign under a time-reversal operation. Thus, the Burke-Thorne force is expected to yield irreversible effects in the dynamics of the source, which will dominate, over the long run, the effects associated with the lowest order, but time-symmetric, post-Newtonian corrections. Unfortunately, this argument can be fully justified only if one has a good control of the lowest-order time-symmetric corrections themselves (see Damour [20] for a discussion). On the other hand, the way the Burke-Thorne force was originally derived is incomplete because of the neglect of nonlinearities in the field [21], [22].

A systematic post-Newtonian (PN) approximation method, pursued by Chandrasekhar *et al.* [23]–[26] and by many subsequent authors [27]–[32], has shown that at the five halves order post-Newtonian (2.5 PN) approximation radiation reaction effects appear in the dynamics of the source and imply a secular decrease of its total energy and total angular momentum, which are defined to be the quantities which are conserved up to the 2 PN approximation level. The secular decrease of the energy agrees with the Einstein quadrupole formula. Note that the expression of the radiation reaction force in the work of Chandrasekhar, for instance, has a much more intricate form than the expression found by Burke and Thorne, but Miller [33] has shown how to transform one expression into the other by means of a suitable coordinate transformation (see Schäfer [34] for a comparison between various expressions of the radiation reaction force). The post-Newtonian approximation method is, however, no longer applicable at the higher 4 PN approximation level, because of the appearance in the metric of "hereditary" contributions depending on the dynamics of the source at all instants in the past (Blanchet and Damour [35]). As we shall see, this means that the post-Newtonian method is unable to reach radiation reaction effects of nonlinear origin.

In this paper we shall examine (using the post-Minkowskian method developed in a recent sequence of papers [35]–[40]) the component of the exterior gravitational field (generated by an isolated system) which is responsible for radiation reaction effects both of linear and nonlinear origin. More precisely we shall investigate a specific component of the field which is "antisymmetric" in the sense that it is a solution of the wave equation of the type retarded minus advanced. The possibility of defining such a component, notably in nonlinear iterations of the field, will be made precise below. Antisymmetric waves (retarded minus advanced) in the *exterior* metric are responsible for radiation reaction effects *inside* the source. Indeed these waves are in fact regular in a neighborhood of the origin where the source is located, and thus they are as well present in the inner metric of the source (this can be shown by matching) where they imply a small correction in the local equations of motion which can be interpreted in the usual way as a radiation reaction force.

The linearized coordinate system used in the work of Burke and Thorne yields the simplest form of the radiation reaction force. However, this coordinate system deals only with the dominant part of the radiation reaction force which appears at the first "odd" 2.5 PN approximation, where "odd" refers as usual to the half-integer PN approximations involving odd powers of c^{-1} in the equations of motion. Furthermore, the Burke-Thorne gauge is *a priori* only valid in the near zone of the source because it involves homogeneous solutions of Laplace's equation which are regular at the origin but blow up at infinity. In the first part of this paper (Sec. II), we shall prove the existence of a new *linearized* coordinate system having the following properties.

(1) The gauge transformation going from the linearized harmonic coordinates to the new coordinates is made of globally well behaved (in $\mathbb{R}^4$) "antisymmetric" waves of a retarded minus advanced type which are "odd" (in the usual PN sense).

(2) The gauge transformation reduces in the near zone, i.e., when $c \to +\infty$, to the usual Burke-Thorne gauge transformation.

(3) The antisymmetric part of the linearized metric in the new gauge is the "smallest" in the near zone, for all multipolarities of waves, in the sense that its order of magnitude in c^{-1} cannot be reduced by a further gauge transformation.

This linearized coordinate system will generalize the Burke-Thorne linearized coordinate system both in its space-time domain of validity (which will no longer be limited to the near zone of the source) and in the magnitude in the post-Newtonian sense of the "odd" part of the metric (which will be the smallest for all components of the metric and all multipolarities of waves). The smallest post-Newtonian order of magnitude of the odd part of the metric will be $O(c^{-2l-3})$, $O(c^{-2l-4})$, and $O(c^{-2l-5})$ in the components 00, $0i$, and ij of the metric density (i.e., $\mathcal{G}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}$), respectively, where l denotes the multipolarity of the wave. At the 2.5 PN approximation, we shall have only a quadrupolar wave and only the coefficient of c^{-7} in the 00 component of the metric which will yield the usual Burke-Thorne scalar potential. At the next 3.5 PN approximation, we shall have a scalar potential c^{-9} in the 00 component of the metric (involving the mass quadrupole and mass octupole moments), and a *vector* potential c^{-8} in the $0i$ component of the metric (involving the mass and current quadrupole moments). These scalar and vector potentials will parametrize, after due account of the nonlinearities and matching, the recoil force acting on the source. Higher-order "odd" PN approximations will be intrinsically tensorial in character. The antisymmetric part of this linearized metric will be referred to as a generalized radiation reaction *four-tensor* potential (whose validity will be limited to the domain of applicability of the linearized theory). This tensor potential is given by Eqs. (2.19) below. Everywhere in spacetime, including at the origin, it satisfies the homogeneous vacuum linearized equations. The interpretation of this potential as a radiation reaction potential will have to be fully justified, in a future work, by an explicit matching to the source.

The use of antisymmetric waves in the gauge transformation, and thus of advanced waves, has, however, several drawbacks. Indeed the metric no longer satisfies the no-incoming radiation conditions at past-null infinity, and its properties at future-null infinity depend on the dynamics of the source at future temporal infinity. For these reasons we shall use in fact a "regularized" gauge transformation which will act only in the near zone of the source, where it will agree with high accuracy with the unregularized gauge, and which will be zero elsewhere, preserving in particular the stationarity of the metric in the past (since this will be our initial assumption), and its properties near future-null infinity. The regularized linear metric so defined will then be a suitable first approximation of a full nonlinear metric that we shall construct, in the exterior region of the source, by the method of Blanchet and Damour [36]. Note that since the metric will not only be valid in the near zone but also in the regions at infinity from the source (where it will be possible to transform it into a "radiative" metric having good falloff properties at infinity [37]), the dynamics of the source as driven, for instance, by radiation reaction will be related to observable quantities at infinity, such as a shifted electromagnetic signal, and to well-defined notions of total (Bondi) energy and linear momentum.

The second part of this paper (Sec. III) will be devoted to the study of the reexpansion of the exterior metric in the near zone ($r/c \to 0$). To this end we shall use a very convenient decomposition of the retarded solution of the wave equation with some given source, into the sum of a particular solution which is an instantaneous functional of the source (in the sense that it depends, at time t, on the dynamics of the source at the same time t), and of a homogeneous solution of the antisymmetric type. The former particular solution (which shall be related to the usual symmetric solution) will be referred to as the solution of the *instantaneous potentials.* It is "even" in the sense that it explicitly contains only even powers of c^{-1}. The latter homogeneous solution defines a component of the exterior metric which is associated with radiation reaction effects of *nonlinear* origin (in particular it contains at quadratic order the "hereditary" contribution of the radiation reaction force obtained in Ref. [35]). Thanks to this decomposition we shall control, in any nonlinear iteration of the metric, the occurrence of the post-Newtonian "odd" terms and then compute the metric coefficients of the 3.5 PN approximation. The use of our generalized Burke-Thorne coordinate system will make this computation easy. These metric coefficients will permit the study in a forthcoming paper of the gravitational recoil of the source. Finally we shall recover and complete the result of Ref. [35] concerning the appearance of hereditary effects at the 4 PN approximation.

B. Recapitulation of the assumptions

For convenience we refer to the earlier papers, Refs. [36], [35], and [38], as paper I, paper II, and paper III, respectively. (See also Refs. [37], [39], and [40] for other papers of the same series.) Let us recall the assumptions underlying the method used in paper I. This method has its roots in the method of the "double series approximation," pioneered by Bonnor and Rotenberg [41], [42] and Hunter and Rotenberg [43]. It was later refined and clarified by Thorne [44]. Basically the method is to look for a metric in the form of a nonlinearity expansion, or expansion in powers of Newton's constant G (see Ref. [45] for our notation and conventions):

$$\mathcal{G}^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta} = \eta^{\alpha\beta} + Gh^{\alpha\beta}_{(1)} + \cdots + G^n h^{\alpha\beta}_{(n)} + \cdots , \tag{1.1}$$

satisfying Einstein's vacuum equations in the external weak-field region $D_e = \{(\mathbf{x},t)\,|\, r = |\mathbf{x}| > r_0\}$ around the source, where $r_0 \geq a$ and $r_0 >> GM/c^2$, a and M being the radius and mass of the source. Furthermore, it is postulated that each one of the coefficients of the series, $h^{\alpha\beta}_{(n)}(\mathbf{x},t)$, admits in D_e a multipolar expansion into symmetric and trace-free (STF) products $\hat{n}_L = (n_L)^{\rm STF} = (n_{i_1}n_{i_2}\cdots n_{i_l})^{\rm STF}$ of unit vectors $n_i = x^i/r$ (L denotes the multi-index $i_1 i_2 \cdots i_l$ of order l; see [45]):

$$h^{\alpha\beta}_{(n)}(\mathbf{x},t) = \sum_{l\geq 0} \hat{n}_L(\theta,\varphi) h^{\alpha\beta}_{(n)L}(r,t). \tag{1.2}$$

The expansion (1.2) is equivalent to an expansion in usual spherical harmonics $Y^m_l(\theta,\varphi)$. The two assumptions (1.1) and (1.2) are the basic ones, but in paper I we made also two supplementary assumptions, namely, that the multipole expansions are in fact finite (i.e., $l \leq l_{\max}$) and that the metric is stationary in time before some remote date $-T$ in the past, i.e.,

$$t \leq -T \quad \Rightarrow \quad (\partial/\partial t)\, h^{\alpha\beta}_{(n)}(\mathbf{x},t) = 0 \ . \tag{1.3}$$

Although the two supplementary assumptions allow the use of rigorous techniques, they do not seem to play a fundamental role, and presumably could be weakened without altering the results obtained by their means. For instance the stationarity in the past could be replaced by the assumption that the source contains freely moving masses in its initial state. We shall, however, maintain all these assumptions in this paper.

The construction of the nonlinear metric proceeds iteratively starting from the following "canonical" metric $h_{\text{can}(1)} \equiv (h_{(1)})_{\text{canonical}}$ of Thorne [44]:

$$h^{00}_{\text{can}(1)} = -\frac{4}{c^2}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left[\frac{1}{r}M_L\left(t-\frac{r}{c}\right)\right] \ , \tag{1.4a}$$

$$h^{0i}_{\text{can}(1)} = \frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l}{l!}\partial_{L-1}\left[\frac{1}{r}M^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)\right] + \frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l l}{(l+1)!}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}S_{bL-1}\left(t-\frac{r}{c}\right)\right], \tag{1.4b}$$

$$h^{ij}_{\text{can}(1)} = -\frac{4}{c^4}\sum_{l\geq 2}\frac{(-)^l}{l!}\partial_{L-2}\left[\frac{1}{r}M^{(2)}_{ijL-2}\left(t-\frac{r}{c}\right)\right] - \frac{8}{c^4}\sum_{l\geq 2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}\left[\frac{1}{r}\varepsilon_{ab(i}S^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right] \tag{1.4c}$$

(∂_L denotes the product of derivatives $\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$ where $\partial_i = \partial/\partial x^i$; see [45]). This linearized metric depends functionally on two sets of time-varying multipole moments $M_L(u)$, mass type or "electric" type moments having $l \geq 0$, and $S_L(u)$, current type or "magnetic" type moments having $l \geq 1$, and their derivatives, e.g., $M^{(1)}_L(u) = dM_L(u)/du$. The tensors M_L and S_L are STF in $L = i_1 i_2 \cdots i_l$. Thorne [44] has proved that the metric (1.4) represents the most general solution, modulo an arbitrary infinitesimal gauge transformation, of the Einstein linearized equations in D_e. This result was extended in paper I to a particular "canonical" construction of the nonlinear metric (1.1), having Eq. (1.4) as its first approximation, which was shown to represent the most general solution (within the multipolar and post-Minkowskian framework) of the Einstein equations in D_e modulo an arbitrary coordinate transformation (see Theorem 4.5 of paper I).

Thus the set of electric type and magnetic type multipole moments $\mathcal{M} = \{M_L, S_L\}$ is necessary and sufficient to parametrize the exterior gravitational field of the source. These moments have been shown to be related in a nontrivial manner (when going to higher post-Newtonian approximations) to the physical parameters of the source [38], [39] and to observable quantities at infinity [37], [40].

II. EXTERNAL GRAVITATIONAL FIELD

A. Symmetric versus antisymmetric multipolar waves

The "canonical" linearized metric (1.4) has a structure made of elementary multipolar retarded waves, solutions in the exterior region D_e of the d'Alembertian equation. Let us write this structure as

$$h^{\alpha\beta}_{\text{can}(1)} = \sum_{l\geq 0}\hat{\partial}_L\left[\frac{1}{r}F^{\alpha\beta}_L\left(t-\frac{r}{c}\right)\right], \tag{2.1}$$

where $F^{\alpha\beta}_L(u)$ denotes a function of u which is a contraction between some constant Cartesian tensor $K^{\alpha\beta}_{LL_1}$, made of Kronecker and Levi-Civita symbols, and some time derivative of a mass or current multipole moment:

$$F^{\alpha\beta}_L(u) = K^{\alpha\beta}_{LL_1}M^{(a)}_{L_1}(u) \quad\text{or}\quad K^{\alpha\beta}_{LL_1}S^{(a)}_{L_1}(u)\ . \tag{2.2}$$

The function $F^{\alpha\beta}_L(u)$ is constant when $u \leq -T$. For convenience we use in Eq. (2.1) the derivative operator $\hat{\partial}_L$ which is defined as the STF part of the product of l space derivatives $\partial_L \equiv \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$ (see Ref. [45]).

Each elementary retarded wave in Eq. (2.1) can be decomposed into the half-sum of the retarded wave and of the corresponding advanced wave, and the half-difference between the retarded and advanced waves. We write

$$\hat{\partial}_L\left(\frac{F(t-r/c)}{r}\right) = \hat{\partial}_L\left(\frac{F(t-r/c)+F(t+r/c)}{2r}\right) + \hat{\partial}_L\left(\frac{F(t-r/c)-F(t+r/c)}{2r}\right), \tag{2.3}$$

where for simplicity's sake we suppress the indices on the function $F(u)$. In this paper we shall refer to the half-sum of the retarded and advanced waves as a "symmetric" wave, and to the half-difference of the retarded and advanced waves as an "antisymmetric" wave. These waves, respectively, remain invariant and change sign if we reverse the time evolution of F: $F(u) \to F(-u)$ and if we afterwards evaluate the wave at the reversed time $-t$.

The antisymmetric wave [second term in (2.3)] represents, for arbitrary l and F, the most general solution of the d'Alembertian equation that is regular at the origin. The formal Taylor expansion of this wave near $r = 0$ (which can be viewed as a formal expansion when the speed of light $c \to +\infty$) can be written, with the help of Eq. (A32) of paper I, as

$$\hat{\partial}_L\left(\frac{F(t-r/c)-F(t+r/c)}{2r}\right) = -\frac{\hat{x}_L}{c^{2l+1}}\sum_{k=0}^{\infty}\frac{(r/c)^{2k}}{(2k)!!(2k+2l+1)!!}F^{(2k+2l+1)}(t), \tag{2.4}$$

where we denote by $\hat{x}_L$ the STF part of the product $x_L \equiv x^L \equiv x^{i_1}x^{i_2}\cdots x^{i_l}$ and where $p!! = p(p-2)\cdots(2\text{ or }1)$. The first two terms in the expansion (2.4) are

$$\hat{\partial}_L\left(\frac{F(t-r/c)-F(t+r/c)}{2r}\right) = -\frac{\hat{x}_L}{(2l+1)!!c^{2l+1}}\left\{F^{(2l+1)}(t) + \frac{(r/c)^2}{2(2l+3)}F^{(2l+3)}(t) + \cdots\right\}. \tag{2.5}$$

Note that the same Taylor expansion (2.4) appears in the computation of the multipole expansion of the solution of the wave equation with a source that is of compact support in space (see the Appendix B of paper III). Using Eqs. (B11)–(B14) of paper III, we can rewrite the antisymmetric wave (2.4) in the form

$$\hat{\partial}_L\left(\frac{F(t-r/c)-F(t+r/c)}{2r}\right)=-\frac{\hat{x}_L}{(2l+1)!!c^{2l+1}}\int_{-1}^{1}dz\,\delta_l(z)\,F^{(2l+1)}(t+zr/c),\tag{2.6a}$$

where we have posed

$$\delta_l(z)=\frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l \quad , \quad \int_{-1}^{1}dz\,\delta_l(z)=1\ .\tag{2.6b}$$

Because of the explicit factor $1/c^{2l+1}$ in front of the expansion (2.4)–(2.6), we see that the antisymmetric wave is of an order $O(1/c^{2l+1})$ smaller than the symmetric or retarded corresponding waves when $c\to+\infty$. For the sake of completeness, let us also mention the following alternative forms for the antisymmetric wave:

$$\hat{\partial}_L\left(\frac{F(t-r/c)-F(t+r/c)}{2r}\right)=-\frac{\hat{n}_L}{2c^{l+1}}\int_{-1}^{1}dz\,P_l(z)F^{(l+1)}(t+zr/c)=-\frac{1}{4\pi c^{l+1}}\int d\Omega'\hat{n}'_LF^{(l+1)}(t+\mathbf{n}'\cdot\mathbf{x}/c),\tag{2.7}$$

where $P_l(z)$ is the usual Legendre polynomial, and where the solid angle $d\Omega'$ is associated with the unit vector $\mathbf{n}'$. [The forms (2.7) exhibit a coefficient $1/c^{l+1}$ which is larger by a factor c^l than the real order of magnitude of the antisymmetric wave when $c\to+\infty$.]

The decomposition (2.3) of the retarded wave can also be viewed as the decomposition of the wave into an "even" wave and an "odd" wave because the expansions when $r/c\to0$ of the symmetric and antisymmetric waves involve, respectively, only even and odd powers of c^{-1}. Furthermore, taking into account the explicit powers of c^{-1} in front of the canonical linearized metric (1.4), one sees that the symmetric and antisymmetric parts of the retarded waves generate respectively the "even" and "odd" parts of the linearized metric, where now the parity of a term has its usual post-Newtonian meaning of being the parity of the power of c^{-1} for terms in the components 00 and ij of the metric, and the inverse parity for terms in the components $0i$. However, the use of the post-Newtonian terminology can be somewhat dangerous in the study of radiation reaction effects (which carry an idea of "time oddness"). Indeed the fundamental property which is of interest in this study is that the antisymmetric wave (2.4) is a solution of the d'Alembertian equation which is regular at the origin, and not that it is "odd" in the post-Newtonian sense. In fact, we shall see that in any *even* nonlinear iterations of the metric, antisymmetric waves appear which are associated with nonlinear radiation effects but which are "even" in the post-Newtonian sense. (Note that in nonlinear approximations the function $F(u)$ will be a complicated functional of the dynamics of the source and that our terminology of symmetric or antisymmetric waves will refer only to the structure of the wave [first or second term in Eq. (2.3)] and not to its real behavior under the time-reversal operation.)

Inserting the decomposition (2.3) into the linearized canonical metric (1.4) we get a corresponding decomposition

$$h^{\alpha\beta}_{\text{can}(1)}=\left(h^{\alpha\beta}_{\text{can}(1)}\right)_{\text{sym}}+\left(h^{\alpha\beta}_{\text{can}(1)}\right)_{\text{antisym}}\ .\tag{2.8}$$

For instance the antisymmetric part of the metric is given by

$$\left(h^{00}_{\text{can}(1)}\right)_{\text{antisym}}=-\frac{4}{c^2}\sum_{l\geq0}\frac{(-)^l}{l!}\partial_L\left\{\frac{M_L(t-r/c)-M_L(t+r/c)}{2r}\right\}\ ,\tag{2.9a}$$

$$\begin{aligned}\left(h^{0i}_{\text{can}(1)}\right)_{\text{antisym}}&=\frac{4}{c^3}\sum_{l\geq1}\frac{(-)^l}{l!}\partial_{L-1}\left\{\frac{M^{(1)}_{iL-1}(t-r/c)-M^{(1)}_{iL-1}(t+r/c)}{2r}\right\}\\&\quad+\frac{4}{c^3}\sum_{l\geq1}\frac{(-)^l l}{(l+1)!}\varepsilon_{iab}\partial_{aL-1}\left\{\frac{S_{bL-1}(t-r/c)-S_{bL-1}(t+r/c)}{2r}\right\}\ ,\end{aligned}\tag{2.9b}$$

$$\begin{aligned}\left(h^{ij}_{\text{can}(1)}\right)_{\text{antisym}}&=-\frac{4}{c^4}\sum_{l\geq2}\frac{(-)^l}{l!}\partial_{L-2}\left\{\frac{M^{(2)}_{ijL-2}(t-r/c)-M^{(2)}_{ijL-2}(t+r/c)}{2r}\right\}\\&\quad-\frac{8}{c^4}\sum_{l\geq2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}\left\{\varepsilon_{ab(i}\frac{S^{(1)}_{j)bL-2}(t-r/c)-S^{(1)}_{j)bL-2}(t+r/c)}{2r}\right\}\ .\end{aligned}\tag{2.9c}$$

When $c\to+\infty$ the components of the symmetric part of the linearized canonical metric have an order of magnitude given by the explicit powers of c^{-1} in Eqs. (1.4), namely,

$$\left(h^{\alpha\beta}_{\text{can}(1)}\right)_{\text{sym}}=O(c^{-2},c^{-3},c^{-4}),\tag{2.10}$$

where the symbol $O(c^{-2},c^{-3},c^{-4})$ means that the components $\alpha\beta = 00$, $0i$, and ij of the metric are, respectively, of order $O(c^{-2})$, $O(c^{-3})$, and $O(c^{-4})$. But because of the explicit factor $1/c^{2l+1}$ in front of the expansion of the antisymmetric wave (2.4), we find that the components of the antisymmetric part of the metric are much smaller in magnitude, being

$$\left(h^{\alpha\beta}_{\mathrm{can}(1)}\right)_{\mathrm{antisym}} = O(c^{-7}, c^{-6}, c^{-5}) \ . \tag{2.11}$$

With the help of Eq. (2.5) we can compute the first ("odd") terms arising at the level $O(c^{-7}, c^{-6}, c^{-5})$ in the linearized canonical metric. The result is [22]

$$ {}_7h^{00}_{\mathrm{can}(1)} + {}_7h^{ss}_{\mathrm{can}(1)} = \tfrac{2}{15}x^a x^b M^{(5)}_{ab}(t) \ , \tag{2.12a}$$

$$ {}_6h^{0i}_{\mathrm{can}(1)} = -\tfrac{2}{3}x^a M^{(4)}_{ai}(t) \ , \tag{2.12b}$$

$$ {}_5h^{ij}_{\mathrm{can}(1)} = 2M^{(3)}_{ij}(t) \ , \tag{2.12c}$$

where we denote by ${}_nh^{\alpha\beta}_{\mathrm{can}(1)}$ the coefficient of c^{-n} in the expansion of the metric density, and where we have added for convenience to ${}_7h^{00}_{\mathrm{can}(1)}$ the spatial trace ${}_7h^{ss}_{\mathrm{can}(1)} = \Sigma\delta_{ij}{}_7h^{ij}_{\mathrm{can}(1)}$ (which is in fact zero in this case because the space-space canonical linearized metric is trace free).

B. Multipolar antisymmetric gauge transformation

The metric coefficients (2.12) are associated with the lowest-order radiation reaction effects in the dynamics of the source. Indeed, since they are present in the exterior metric of the source, and since they satisfy the Einstein linearized vacuum equations everywhere, including in a neighborhood of the origin (modulo terms of higher order in c^{-1}), they will appear in, or they will have to be added by matching to, the inner metric of the source where they will imply a small correction to the local equations of motion which can be interpreted as a radiation reaction force density [indeed the terms (2.12) arise because of the use of retarded potentials for the emitted outgoing radiation]. More generally, we see that the whole "antisymmetric" linearized metric (2.9) is associated with radiation reaction effects in the source.

The scalar, vector, and tensor metric coefficients (2.12) play the role of scalar, vector, and tensor potentials in the expression of the radiation reaction force in harmonic coordinates (at the lowest-order post-Newtonian level c^{-5}). However, it has been shown by Burke [15] and Thorne [16] that by performing a suitable (linearized) gauge transformation in the near zone of the source, one can reduce to zero both the vector and tensor potentials so that the radiation reaction force appears, in the new gauge, as purely scalar (at lowest order c^{-5}). Indeed let us consider the linearized coordinate transformation $\delta x^\mu = (Gc^{-6}{}_6\xi^0,\ Gc^{-5}{}_5\xi^i)$, where

$$ {}_6\xi^0 = -\tfrac{1}{6}x^a x^b M^{(4)}_{ab}(t), \tag{2.13a}$$

$$ {}_5\xi^i = -x^a M^{(3)}_{ia}(t) \ . \tag{2.13b}$$

Then the new linearized metric $h^{\alpha\beta}_{(1)}$, say, transformed from $h^{\alpha\beta}_{\mathrm{can}(1)}$ by this linearized coordinate transformation, will have new metric coefficients at the level $O(c^{-7}, c^{-6}, c^{-5})$ (or at the 2.5 PN level) given by

$$\begin{aligned} {}_7h^{00}_{(1)} + {}_7h^{ss}_{(1)} &= {}_7h^{00}_{\mathrm{can}(1)} + {}_7h^{ss}_{\mathrm{can}(1)} - 4\partial_{t6}\xi^0, \\ {}_6h^{0i}_{(1)} &= {}_6h^{0i}_{\mathrm{can}(1)} + \partial_{i6}\xi^0 - \partial_{t5}\xi^i, \\ {}_5h^{ij}_{(1)} &= {}_5h^{ij}_{\mathrm{can}(1)} + 2\partial^{(i}{}_5\xi^{j)} - \delta^{ij}\partial_{k5}\xi^k \ . \end{aligned}$$

Using Eqs. (2.12) and (2.13) we then obtain

$$ {}_7h^{00}_{(1)} + {}_7h^{ss}_{(1)} = \tfrac{4}{5}x^a x^b M^{(5)}_{ab}(t), \tag{2.14a}$$

$$ {}_6h^{0i}_{(1)} = 0, \tag{2.14b}$$

$$ {}_5h^{ij}_{(1)} = 0 \ . \tag{2.14c}$$

The scalar metric coefficient Eq. (2.14a), which is now alone at the 2.5 PN level, yields (for instance, after matching) a purely scalar expression for the radiation reaction force density:

$$F_i(\mathbf{x}, t) = \rho\,\partial_i V_{\mathrm{react}}(\mathbf{x}, t), \tag{2.15a}$$

where V_{react} is the Burke-Thorne [15]–[19] radiation reaction potential,

$$V_{\mathrm{react}} = -\frac{G}{5c^5}x^a x^b M^{(5)}_{ab}(t) \ . \tag{2.15b}$$

In Eqs. (2.15a) and (2.15b), ρ is the mass density of the source, and the quadrupole moment $M_{ab}(t)$ (as determined, for instance, by matching) is

$$M_{ab}(t) = \int d^3\mathbf{x}\,\rho(\mathbf{x}, t)\hat{x}_{ab} + O(c^{-2}) \ . \tag{2.15c}$$

The total power extracted in the system by the Burke-Thorne reaction force (2.15) agrees with the standard Einstein quadrupole formula. However, the above derivation of this reaction force, based on a gauge transformation of the linearized (harmonic coordinates) metric (2.12), is not correct. This can be seen from the fact that the metric coefficients (2.12) do not by themselves yield an expression for the force that agrees with the quadrupole formula. Indeed one must also take into account (in harmonic coordinates) a contribution from the quadratic nonlinearity in the exterior field [22]. It is only afterwards that we must perform the coordinate transformation (2.13). The end results (2.15) of Burke and Thorne are then recovered (and proved).

We shall now generalize the Burke-Thorne gauge transformation (2.13) to a general *multipolar* (not only quadrupolar) gauge transformation such that the "odd" part of the transformed linearized metric, which is responsible for radiation reaction effects, is "the smallest" when $c \to +\infty$, in a sense made precise below, for arbitrary orders of multipolarity l. In this sense, our generalized gauge will constitute the "maximal" generalization of the Burke-Thorne gauge. On the other hand, a problem with the Burke-Thorne gauge is that it is valid only in the near zone of the source [because the ${}_n\xi^\alpha$'s in Eqs. (2.13) blow up at infinity]. Thus we shall also extend the validity in space of the gauge transformation by noting that terms such as the ${}_n\xi^\alpha$'s in Eqs. (2.13) constitute in fact the leading terms, valid only in the near zone, of the expansion when $c \to +\infty$ of some *antisymmetric* waves valid all over $\mathbb{R}^4$; see Eq. (2.4). (However, we shall see that the introduction in the gauge transformation of antisymmetric waves, and thus of advanced waves, can cause some problems; these problems will be dealt with in the next subsection.) Let us now define a linearized gauge transformation $\delta x^\mu = G\xi^\mu$, where ξ^μ is the antisymmetric multipolar series:

$$\xi^0 = \frac{2}{c}\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{2l+1}{l(l-1)}\partial_L\left\{\frac{M_L^{(-1)}(t-r/c)-M_L^{(-1)}(t+r/c)}{2r}\right\}, \tag{2.16a}$$

$$\begin{aligned}\xi^i = &-2\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{(2l+1)(2l+3)}{l(l-1)}\partial_{iL}\left\{\frac{M_L^{(-2)}(t-r/c)-M_L^{(-2)}(t+r/c)}{2r}\right\}\\ &+\frac{4}{c^2}\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{2l+1}{l-1}\partial_{L-1}\left\{\frac{M_{iL-1}(t-r/c)-M_{iL-1}(t+r/c)}{2r}\right\}\\ &+\frac{4}{c^2}\sum_{l\geq 2}\frac{(-)^l l}{(l+1)!}\frac{2l+1}{l-1}\varepsilon_{iab}\partial_{aL-1}\left\{\frac{S_{bL-1}^{(-1)}(t-r/c)-S_{bL-1}^{(-1)}(t+r/c)}{2r}\right\},\end{aligned} \tag{2.16b}$$

where we have introduced *antiderivatives* of the multipole moments defined as

$$M_L^{(-1)}(u) = \int_{-T}^{u} dv\, M_L(v)\ ,$$

$$M_L^{(-2)}(u) = \int_{-T}^{u} dv\, M_L^{(-1)}(v)\ .$$

We assume that the integration range starts at the date $-T$ (the date in the past before which the moments are constant but not necessarily zero), but in fact the expressions (2.16) do not depend on $-T$. This can be proven by noting that antiderivatives in (2.16) appear only in the two types of combinations

$$\frac{M_L^{(-1)}(t-r/c)-M_L^{(-1)}(t+r/c)}{2r} = -\frac{1}{2r}\int_{t-r/c}^{t+r/c} dv\, M_L(v)$$

and

$$\partial_i\left\{\frac{M_L^{(-2)}(t-r/c)-M_L^{(-2)}(t+r/c)}{2r}\right\} = -\partial_i\left(\frac{1}{2r}\right)\int_{t-r/c}^{t+r/c} dv(t-v)\, M_L(v),$$

which do not depend on $-T$. Using Eq. (2.4) it is easily checked that at first order when $c \to +\infty$ (or in the near zone where $r \to 0$) the gauge transformation associated with ξ^μ agrees with the Burke-Thorne gauge transformation (2.13):

$$\xi^0 = \frac{1}{c^6}\,{}_6\xi^0 + O\left(\frac{1}{c^8}\right), \tag{2.17a}$$

$$\xi^i = \frac{1}{c^5}\,{}_5\xi^i + O\left(\frac{1}{c^7}\right). \tag{2.17b}$$

Then we define a "generalized radiation reaction four-tensor potential" $V^{\alpha\beta}_{\rm react}$ (in the linear theory) to be the sum of the antisymmetric part of the canonical metric (2.9) augmented by the gauge terms associated with the antisymmetric gauge vector (2.16):

$$\left(h^{\alpha\beta}_{\rm can(1)}\right)_{\rm antisym} + \partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha - \eta^{\alpha\beta}\partial_\mu\xi^\mu = \begin{cases} -\frac{4}{c^2G}\,V^{00}_{\rm react},\\ -\frac{4}{c^3G}\,V^{0i}_{\rm react},\\ -\frac{4}{c^4G}\,V^{ij}_{\rm react},\end{cases} \tag{2.18}$$

where we have factorized out c^{-2}, c^{-3}, and c^{-4} in the 00, $0i$, and ij components of the tensor. The explicit expression of the tensor potential $V^{\alpha\beta}_{\rm react}$, as a functional of the moments $\mathcal{M} = \{M_L, S_L\}$, can be computed from the expression of the antisymmetric part of the canonical linearized metric (2.9) and from (2.16). We find

$$V^{00}_{\rm react}[\mathcal{M}] = G\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{(l+1)(l+2)}{l(l-1)}\hat{\partial}_L\left\{\frac{M_L(t-r/c)-M_L(t+r/c)}{2r}\right\}, \tag{2.19a}$$

$$\begin{aligned}V^{0i}_{\rm react}[\mathcal{M}] = &-c^2G\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{(l+2)(2l+1)}{l(l-1)}\hat{\partial}_{iL}\left\{\frac{M_L^{(-1)}(t-r/c)-M_L^{(-1)}(t+r/c)}{2r}\right\}\\ &+G\sum_{l\geq 2}\frac{(-)^l l}{(l+1)!}\frac{l+2}{l-1}\varepsilon_{iab}\hat{\partial}_{aL-1}\left\{\frac{S_{bL-1}(t-r/c)-S_{bL-1}(t+r/c)}{2r}\right\},\end{aligned} \tag{2.19b}$$

$$V^{ij}_{\rm react}[\mathcal{M}] = c^4 G \sum_{l\geq 2} \frac{(-)^l}{l!} \frac{(2l+1)(2l+3)}{l(l-1)} \hat{\partial}_{ijL} \left\{ \frac{M_L^{(-2)}(t-r/c) - M_L^{(-2)}(t+r/c)}{2r} \right\}$$
$$-2c^2 G \sum_{l\geq 2} \frac{(-)^l l}{(l+1)!} \frac{2l+1}{l-1} \varepsilon_{ab(i}\hat{\partial}_{j)aL-1} \left\{ \frac{S_{bL-1}^{(-1)}(t-r/c) - S_{bL-1}^{(-1)}(t+r/c)}{2r} \right\} . \quad (2.19c)$$

The tensor $V^{\alpha\beta}_{\rm react}(\mathbf{x},t)$ satisfies

$$\Box V^{\alpha\beta}_{\rm react} = 0 \quad (2.20a)$$

and

$$\partial_t V^{\alpha 0}_{\rm react} + \partial_i V^{\alpha i}_{\rm react} = 0 \quad (2.20b)$$

everywhere in $I\!R^4$. Note also that its space components satisfy

$$V^{ss}_{\rm react} = 0 \ . \quad (2.20c)$$

Furthermore, we note that the expressions (2.19) for the components of this tensor involve only multiderivative operators that are *trace free*, $\hat{\partial}_J = (\partial_{i_1}\partial_{i_2}\cdots\partial_{i_j})^{\rm STF}$, and that bear the spatial "spin" indices of $V^{\alpha\beta}_{\rm react}$ (none, i, ij; spin $s \leq 2$). Thus, we see that the multipolarities of the waves composing $V^{\alpha\beta}_{\rm react}$, i.e., the number of indices j on the trace-free derivative operators $\hat{\partial}_J$, take the maximum permitted values $j = l + s$ and $j = l + s - 1$ for a given number l of indices on M_L and S_L, respectively, where $s = 0, 1, 2$ according to $\alpha\beta = 00$, $0i$, ij. (These values can be viewed as the maximum values permitted by the law of addition of angular momenta.) Since we know by Eq. (2.4) that the magnitude in c^{-1} of an antisymmetric wave of multipolarity j is $O(1/c^{2j+1})$, we conclude that the waves composing $V^{\alpha\beta}_{\rm react}$ have the *smallest* permitted order of magnitude in the near zone (when $c \to +\infty$), for any number l of indices on the moments M_L and S_L. [This is true even though there are positive powers of c in front of Eqs. (2.19).] Thus a linearized metric related to $h^{\alpha\beta}_{{\rm can}(1)}$ by the gauge transformation $\delta x^\mu = G\xi^\mu$, and so having $V^{\alpha\beta}_{\rm react}$ as its antisymmetric part by Eq. (2.18), will have the property that, among all linearized metrics differing from each other by gauge transformations, it is the one whose antisymmetric ("odd") part is the smallest in the near zone. It is in that sense that we say that the coordinate system defined from the canonical (harmonic) coordinate system by $\delta x^\mu = G\xi^\mu$ constitutes the "maximal" generalization, at linear order, of the Burke-Thorne coordinate system. (Below we shall have to define a regularized coordinate system that will preserve this property of being "maximal.")

When $c \to +\infty$ the scalar part $V^{00}_{\rm react}$ of the four-tensor potential (2.19) is given by

$$V^{00}_{\rm react} = -\frac{G}{5c^5} x^a x^b M^{(5)}_{ab}(t) + O(c^{-7}), \quad (2.21)$$

which agrees in first approximation with the Burke-Thorne scalar potential (2.15b). Computing the components of $V^{\alpha\beta}_{\rm react}$ up to a higher level [using Eq. (2.5)] we find

$$V^{00}_{\rm react} = -\frac{G}{5c^5} x^a x^b M^{(5)}_{ab}(t) + \frac{G}{c^7}\left[\frac{1}{189} x^a x^b x^c M^{(7)}_{abc}(t) - \frac{1}{70} r^2 x^a x^b M^{(7)}_{ab}(t)\right] + O(c^{-9}) , \quad (2.22a)$$

$$V^{0i}_{\rm react} = \frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{iab} M^{(6)}_{ab}(t) - \frac{4}{45}\varepsilon_{iab} x^a x^c S^{(5)}_{bc}(t)\right] + O(c^{-7}) , \quad (2.22b)$$

$$V^{ij}_{\rm react} = \frac{G}{c^5}\left[-\frac{1}{108}\hat{x}^{ijab} M^{(7)}_{ab}(t) + \frac{2}{63}\varepsilon^{ab(i}\hat{x}^{j)ac} S^{(6)}_{bc}(t)\right] + O(c^{-7}) . \quad (2.22c)$$

C. Definition of a regularized linear metric

Let us tentatively consider the linearized metric obtained from the canonical linearized metric $h_{{\rm can}(1)}$ by the gauge transformation $\delta x^\mu = G\xi^\mu$ of Eqs. (2.16):

$$h^{\alpha\beta}_{(1)} = h^{\alpha\beta}_{{\rm can}(1)} + \partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha - \eta^{\alpha\beta}\partial_\mu \xi^\mu \ . \quad (2.23)$$

The antisymmetric part of this linearized metric is given by the radiation reaction tensor $V^{\alpha\beta}_{\rm react}$ defined in Eq. (2.18); its symmetric part is equal to the symmetric part of the canonical metric.

The linearized metric $h^{\alpha\beta}_{(1)}$ has good properties, which we qualified as "maximal," concerning the smallness of its antisymmetric part in the near zone, and it has also "normal" properties concerning its symmetric part which is of order $O(c^{-2}, c^{-3}, c^{-4})$ [see Eq. (2.10)]. Furthermore, this metric is, like the canonical metric, well behaved in all the exterior region D_e, including the regions at infinity (say, for a realistic behavior of the moments at late times, as in a scattering situation), because both symmetric and antisymmetric waves are so. However, since we have introduced advanced waves in the gauge transformation ξ^μ, the metric $h^{\alpha\beta}_{(1)}$ is not stationary in the past (i.e., not constant before the date $-T$) like the canonical metric, except in the interior region of the past-null cone $t + r/c \leq -T$. In particular, at past-null infinity, the metric does not satisfy the no-incoming-radiation condition (even though the incoming waves are pure gauge). Another drawback of the nonstationarity in the past is that it can prevent the construction of a full nonlinear metric.

For this reason we shall now define a new "regularized" gauge transformation which has been suggested to the author by T. Damour in a personal communication. This gauge transformation will be purely retarded, and in the near zone it will agree with the gauge transformation ξ^μ to a high degree of accuracy. Furthermore, we shall define this gauge transformation so that it is zero in a vicinity of infinity. Thus the good properties of the metric (2.23) in the near zone will be preserved, and this is the only thing that we shall need in a study of radiation reaction effects. Consider the single monopolar antisymmetric wave

$$\{F\} \equiv \frac{F(t-r/c)-F(t+r/c)}{2r} , \tag{2.24}$$

where the function $F(u)$ is constant when $u \leq -T$. This wave is *a priori* nonstationary (i.e., time varying) everywhere except in the interior region of the past-null cone $t + r/c \leq -T$ where it is zero. The expansion of $\{F\}$ in the near zone is given by Eq. (2.4) with $l = 0$ or, more generally, if we expand along some arbitrary cone $t - \theta r/c =$ const (where θ is some constant), by

$$\{F\} = \frac{1}{2r}\sum_{n=1}^{+\infty} \frac{(\theta-1)^n-(\theta+1)^n}{n!}\left(\frac{r}{c}\right)^n F^{(n)}(t-\theta r/c) . \tag{2.25}$$

Now, by keeping only a *finite* number of terms in the expansion (2.25), say, the K first terms, we define a new object depending on K and θ:

$$\{F\}_{K,\theta} = \frac{1}{2r}\sum_{n=1}^{K} \frac{(\theta-1)^n-(\theta+1)^n}{n!}\left(\frac{r}{c}\right)^n F^{(n)}(t-\theta r/c). \tag{2.26}$$

This new object has the following properties. First of all, we see that it differs from the original object $\{F\}$ by terms which, choosing K large enough, can be made of arbitrary small order in the near zone, namely,

$$\{F\}_{K,\theta} = \{F\} + S_{K,\theta}(r,t), \tag{2.27}$$

where we have

$$S_{K,\theta}(r,t) = O(r^K), \tag{2.28a}$$

when $r \to 0$, and

$$S_{K,\theta}(r,t) = O(1/c^{K+1}), \tag{2.28b}$$

when $c \to +\infty$. Second we see that, because of the constancy of $F(u)$ in the past, the object $\{F\}_{K,\theta}$ is zero in all the space-time region $t - \theta r/c \leq -T$ (indeed it is made of a finite number of terms having this property). The choice $\theta \geq 0$ is sufficient to ensure that $\{F\}_{K,\theta}$ is zero in the past (for $t \leq -T$). However, because we want to minimize the impact (notably in the nonlinear iterations of the metric) of the gauge transformation at infinity and in particular at future-null infinity, we shall make the choice $\theta > 1$ below. With this choice we see that $\{F\}_{K,\theta}$ is zero in all the region exterior to the future *timelike* cone $r \geq (c/\theta)(t+T)$ and thus is zero at large distances from the source when we recede from the source at any speed larger than c/θ and in particular at speed c. In a conformal (Minkowskian) diagram, $\{F\}_{K,\theta}$ is zero everywhere except in a timelike neighborhood of the time axis $r = 0$, starting at $t = -T$ and joining the timelike future point I^+.

The regularized linear metric, which will constitute the linear approximation of a full nonlinear metric below, is now simply defined by

$$h^{\alpha\beta}_{(1)K,\theta} = h^{\alpha\beta}_{\text{can}(1)} + \partial^\alpha \xi^\beta_{K,\theta} + \partial^\beta \xi^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu \xi^\mu_{K,\theta}, \tag{2.29}$$

where K is a large integer, where $\theta > 1$, and where the gauge vector $\xi^\alpha_{K,\theta}$ is obtained by replacing all monopolar waves $\{F\}$ by the $\{F\}_{K,\theta}$ we defined in Eq. (2.26) in the expression (2.16) for ξ^α, which gives

$$\xi^0_{K,\theta} = \frac{2}{c}\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{2l+1}{l(l-1)}\partial_L\left\{M_L^{(-1)}\right\}_{K,\theta} , \tag{2.30a}$$

$$\begin{aligned}\xi^i_{K,\theta} = &-2\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{(2l+1)(2l+3)}{l(l-1)}\partial_{iL}\left\{M_L^{(-2)}\right\}_{K,\theta}\\ &+\frac{4}{c^2}\sum_{l\geq 2}\frac{(-)^l}{l!}\frac{2l+1}{l-1}\partial_{L-1}\{M_{iL-1}\}_{K,\theta}\\ &+\frac{4}{c^2}\sum_{l\geq 2}\frac{(-)^l l}{(l+1)!}\frac{2l+1}{l-1}\varepsilon_{iab}\partial_{aL-1}\left\{S^{(-1)}_{bL-1}\right\}_{K,\theta} .\end{aligned} \tag{2.30b}$$

The linear metric (2.29) will be referred to as the "modified canonical" linear metric. (Note that the "K,θ" modification applied only to the gauge transformation and not to the linearized metric itself.) It differs from the unregularized metric (2.23) by the equation

$$h^{\alpha\beta}_{(1)K,\theta} = h^{\alpha\beta}_{(1)} + \partial^\alpha \varepsilon^\beta_{K,\theta} + \partial^\beta \varepsilon^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu \varepsilon^\mu_{K,\theta}, \tag{2.31}$$

where $\varepsilon^\alpha_{K,\theta}$ is a gauge vector which is by Eqs. (2.27) and (2.28) of very small magnitude in the near zone:

$$\varepsilon^\alpha_{K,\theta} = \xi^\alpha_{K,\theta} - \xi^\alpha = O(1/c^{K+2},\, 1/c^{K+1}) . \tag{2.32}$$

In particular we see from Eqs. (2.18) and (2.22) that the antisymmetric part of the linear metric is

$$\left(h^{\alpha\beta}_{(1)K,\theta}\right)_{\text{antisym}} = -\frac{4}{Gc^{2+s}}V^{\alpha\beta}_{\text{react}} + O\left(\frac{1}{c^{K+1}}, \frac{1}{c^{K+2}}, \frac{1}{c^{K+1}}\right) \tag{2.33a}$$

(where $s = 0,1,2$ according to $\alpha\beta = 00, 0i, ij$), which, when $c \to +\infty$, is of order

$$\left(h^{\alpha\beta}_{(1)K,\theta}\right)_{\text{antisym}} = O\left(\frac{1}{c^7}, \frac{1}{c^8}, \frac{1}{c^9}\right) . \tag{2.33b}$$

Equation (2.33b) is to be compared with Eq. (2.11). Second, the metric (2.29) reduces to the linearized canonical metric (1.4) in the region outside the timelike retarded cone $t - \theta r/c = -T$:

$$h^{\alpha\beta}_{(1)K,\theta} = h^{\alpha\beta}_{\text{can}(1)} \quad \text{when} \quad r \geq \frac{c}{\theta}(t+T) . \tag{2.34}$$

In particular the metric is stationary in the past, and its asymptotic properties at future-null infinity are the same as those of the linearized canonical metric. Finally let us write down that $h_{(1)K,\theta}$ satisfies the linearized Einstein vacuum equations (in D_e),

$$\Box h^{\alpha\beta}_{(1)K,\theta} - 2\partial^{(\alpha}\partial_\mu h^{\beta)\mu}_{(1)K,\theta} + \eta^{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu}_{(1)K,\theta} = 0 \ , \quad (2.35)$$

and the coordinate condition

$$\partial_\beta h^{\alpha\beta}_{(1)K,\theta} = \Box\varepsilon^\alpha_{K,\theta}, \quad (2.36)$$

which by Eq. (2.32) is nearly harmonic in the near zone, and by Eq. (2.34) is exactly harmonic in the far zone.

D. Nonlinear iterations of the exterior metric

We now construct a full "modified canonical" nonlinear metric

$$\mathcal{G}^{\alpha\beta}_{K,\theta} \equiv \sqrt{-g_{K,\theta}}\, g^{\alpha\beta}_{K,\theta} = \eta^{\alpha\beta} + \sum_{n=0}^{+\infty} G^n h^{\alpha\beta}_{(n)K,\theta}(\mathbf{x},t) \ , \quad (2.37)$$

which is a solution of Einstein's vacuum equations in D_e and which is based on the linearized metric of Eq. (2.29):

$$h^{\alpha\beta}_{(1)K,\theta} = h^{\alpha\beta}_{\mathrm{can}(1)} + \partial^\alpha\xi^\beta_{K,\theta} + \partial^\beta\xi^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu\xi^\mu_{K,\theta}, \quad (2.38)$$

where $h^{\alpha\beta}_{\mathrm{can}(1)}$ is the canonical metric (1.4) and where the gauge vector $\xi^\alpha_{K,\theta}$ is given by Eq. (2.30) (where K is some large integer, and where $\theta > 1$). By inserting the nonlinear metric (2.37) into Einstein's vacuum equations, we get a hierarchy of equations to be solved for each $h^{\alpha\beta}_{(n)K,\theta}$,

$$\Box h^{\alpha\beta}_{(n)K,\theta} - 2\partial^{(\alpha}\partial_\mu h^{\beta)\mu}_{(n)K,\theta} + \eta^{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu}_{(n)K,\theta} = N^{\alpha\beta}_{(n)K,\theta}, \quad (2.39)$$

in the right-hand side of which the nth nonlinear source is an algebraic functional of the previous $h_{(m)K,\theta}$ (for $1 \leq m \leq n-1$) and their first and second partial derivatives (with $N_{(1)K,\theta} \equiv 0$).

Let us now recall that the construction of nonlinear metrics in paper I uses a special class of functions of $\mathbb{R}^4$, the so-called L^n class of functions. A function $f(\mathbf{x},t)$ defined in $\mathbb{R}^4$, except at the spatial origin $r = |\mathbf{x}| = 0$, is said to belong to the L^n class of functions (for some $n \in \mathbb{N}$) if it admits a *finite* expansion of the type

$$f(\mathbf{x},t) = \sum_{p\leq n} F_{Lap}(t)\hat{n}_L r^a (\ln r)^p + R_N(\mathbf{x},t) \quad (2.40)$$

for *any* positive integer N. In this expansion, a is a positive or negative integer $a \in \mathbb{Z}$ (with $a \geq a_0$; a_0 independent of N), the powers of the logarithm are $p \in \mathbb{N}$ and satisfy $p \leq n$ (thus the integer n entering the definition of the class L^n is the maximal power of the logarithms), and the functions $F_{Lap}(t)$ are $C^\infty(\mathbb{R})$ and *constant* when $t \leq -T$. Note that in paper I we assumed that the functions $F_{Lap}(t)$ are zero when $t \leq -T$, but it is more convenient here to include in the definition of L^n constant functions of the type $\Sigma_{p\leq n} F_{Lap}\hat{n}_L r^a (\ln r)^p$ (where Σ is a finite sum) having constant coefficients F_{Lap}. Finally the remainder $R_N(\mathbf{x},t)$ is such that its partial time derivatives $R^{(q)}_N(\mathbf{x},t) = \partial^q R_N/\partial t^q$ of arbitrary order ($\forall q$) are (i) *zero* in the past (when $t \leq -T$), (ii) of class $C^N(\mathbb{R}^4)$, i.e., N times continuously differentiable in $\mathbb{R}^4$, and (iii) of order $O(r^N)$ when $r \to 0$, i.e.,

$$|R^{(q)}_N(\mathbf{x},t)| < M r^N, \quad (2.41)$$

when $r \to 0$ (with fixed t), where M is some constant. Given a function $f(\mathbf{x},t)$ belonging to the class of functions L^n (and thus being in general singular at the origin $r=0$) we can consider the retarded integral

$$\left[\Box_R^{-1}\left(r^B f\right)\right](\mathbf{x},t) = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}|\mathbf{x}'|^B f\left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right), \quad (2.42)$$

where we have introduced an analytic continuation factor $|\mathbf{x}'|^B$, B being a complex number. It has been shown in paper I that the integral (2.42) which is *a priori* defined only in some vertical strip of the complex plane can be analytically continued to all values of B in $\mathbb{C} - \mathbb{Z}$, and that at most multiple poles occur at integer values of B. Then we define the *finite part* of the retarded integral (2.42) at the value $B = 0$ to be the coefficient of the zeroth power of B (or constant coefficient) in the Laurent expansion of Eq. (2.42) near $B = 0$. As proven in paper I (see theorem 3.1 there) the latter finite part, denoted by finite $\mathrm{part}_{B=0}\Box_R^{-1}(r^B f)$ or more simply by $\mathrm{FP}\Box_R^{-1} f$, satisfies in $\mathbb{R}^4$ except the time axis the usual wave equation with source f,

$$\forall f \in L^n, \qquad \Box\left(\mathrm{FP}\,\Box_R^{-1} f\right) = f \ , \quad (2.43)$$

and furthermore it leaves the classes of functions L^n globally stable with respect to its action in the sense that

$$f \in L^n \Longrightarrow \mathrm{FP}\,\Box_R^{-1} f \in L^{n+1} \quad (2.44)$$

(note the increase by one unit of the superscript n).

In order to start the iteration process by means of the operator $\mathrm{FP}\Box_R^{-1}$ we need to check that the linearized metric (2.38) belongs to the class L^0. Since we know that the canonical metric $h_{\mathrm{can}(1)} \in L^0$ (see paper I; recall that we have enlarged the definition of L^n to include constant contributions) the only thing we need to check is that the gauge vector $\xi_{K,\theta} \in L^0$ (indeed the differentiation of a function of L^n is a function of L^n). This is proven by expanding the function $F(t - \theta r/c)$ in Eq. (2.26) around $r = 0$ by means of Taylor's formula with integral remainder, which shows that each elementary wave $\{F\}_{K,\theta}$ of which $\xi_{K,\theta}$ is composed, and hence also $\xi_{K,\theta}$ itself, belongs to L^0. Thus, we assume, as an induction hypothesis (following paper I), that the $(n-1)$ previous iterations $h_{(m)K,\theta}$ ($m \leq n-1$) have been constructed so as to satisfy Einstein's vacuum equations in D_e and to belong to the classes of L^{m-1} functions, respectively. Inserting the previous $h_{(m)K,\theta}$'s into the nth nonlinear source on the right-hand side of Eq. (2.39) we obtain

$$N^{\alpha\beta}_{(n)K,\theta} \in L^{n-2} \quad (2.45)$$

(by the structure of the source and the properties of the functions of L^n). Thus we can solve the wave equation with source $N_{(n)K,\theta}$ by means of the operator $\mathrm{FP}\,\Box_R^{-1}$. We first define

$$p^{\alpha\beta}_{(n)K,\theta} = \text{FP}\, \Box_R^{-1} N^{\alpha\beta}_{(n)K,\theta}, \tag{2.46}$$

which belongs to L^{n-1} by Eq. (2.44). [In our previous papers we introduced a constant P to make the analytic continuation factor dimensionless, i.e., $(r/cP)^B$, but we shall take $P = 1$ for convenience in this paper.] The tensor $p^{\alpha\beta}_{(n)K,\theta}$ is not yet a solution of Einstein's equations (2.39) and we must add to it a second tensor $q^{\alpha\beta}_{(n)K,\theta}$ defined as follows. The divergence of $p^{\alpha\beta}_{(n)K,\theta}$, namely, $r^{\alpha}_{(n)K,\theta} = \partial_\beta p^{\alpha\beta}_{(n)K,\theta}$, is computed from Eq. (2.46) and the fact that the source identically satisfies

$$\partial_\beta N^{\alpha\beta}_{(n)K,\theta} = 0 \tag{2.47}$$

by the Bianchi identities. It reads as

$$r^{\alpha}_{(n)K,\theta} = \text{finite part}_{B=0}\, \Box_R^{-1} \left[B r^{B-1} n_i N^{\alpha i}_{(n)K,\theta} \right], \tag{2.48}$$

where the factor B comes from the differentiations of the analytic-continuation factor r^B. It has been shown in paper I that $r^{\alpha}_{(n)K,\theta}$ given by Eq. (2.48) is a sum of retarded waves, solutions of the d'Alembertian equation in D_e, which can be written, in a unique manner, as

$$r^{0}_{(n)K,\theta} = \sum_{l\geq 0} \partial_L \left(\frac{1}{r} A_L(t-r/c) \right), \tag{2.49a}$$

$$r^{i}_{(n)K,\theta} = \sum_{l\geq 0} \left\{ \partial_{iL} \left(\frac{1}{r} B_L(t-r/c) \right) + \partial_L \left(\frac{1}{r} C_{iL}(t-r/c) \right) + \varepsilon_{iab}\partial_{aL} \left(\frac{1}{r} D_{bL}(t-r/c) \right) \right\}, \tag{2.49b}$$

where $A_L(u), \ldots, D_L(u)$ (labels n, K, and θ suppressed for simplicity) are some STF tensorial functions of the retarded time. Then the tensor $q^{\alpha\beta}_{(n)K,\theta}$ is defined by its components as

$$q^{00}_{(n)K,\theta} = -\frac{c}{r} A^{(-1)} - c\partial_a \left(\frac{1}{r} A_a^{(-1)} \right) + c^2 \partial_a \left(\frac{1}{r} C_a^{(-2)} \right), \tag{2.50a}$$

$$q^{0i}_{(n)K,\theta} = -\frac{c}{r} C_i^{(-1)} - c\varepsilon_{iab}\partial_a \left(\frac{1}{r} D_b^{(-1)} \right) - \sum_{l\geq 2} \partial_{L-1} \left(\frac{1}{r} A_{iL-1} \right), \tag{2.50b}$$

$$q^{ij}_{(n)K,\theta} = -\delta_{ij} \left[\frac{1}{r} B + \partial_a \left(\frac{1}{r} B_a \right) \right] + \sum_{l\geq 2} \left\{ \partial_{L-2} \left(\frac{1}{rc} A^{(1)}_{ijL-2} + \frac{3}{rc^2} B^{(2)}_{ijL-2} - \frac{1}{r} C_{ijL-2} \right) + 2\delta_{ij}\partial_L \left(\frac{1}{r} B_L \right) - 6\partial_{L-1(i} \left(\frac{1}{r} B_{j)L-1} \right) - 2\partial_{aL-2} \left(\varepsilon_{ab(i} \frac{1}{r} D_{j)bL-2} \right) \right\}. \tag{2.50c}$$

Note that the spatial trace of $q^{\alpha\beta}_{(n)K,\theta}$ is simply

$$q^{ii}_{(n)K,\theta} = -3 \left[\frac{1}{r} B + \partial_a \left(\frac{1}{r} B_a \right) \right]. \tag{2.50d}$$

In Eqs. (2.50) we have introduced antiderivatives: $A^{(-1)}(u) = \int_{-\infty}^{u} A(x)dx$, $A^{(-2)}(u) = \int_{-\infty}^{u} A^{(-1)}(x)dx$. [Note that it has been shown in Appendix C of paper I that for stationary metrics the tensors $A_L, \ldots, D_L$ are identically zero. Hence the $A_L(u), \ldots, D_L(u)$ in Eqs. (2.50) are zero in the past and the above antiderivatives are well defined.] The tensor $q^{\alpha\beta}_{(n)K,\theta}$ has been defined so that its divergence is the opposite of $r^{\alpha}_{(n)K,\theta}$. So we can now pose, as an nth-order nonlinear iteration of the metric,

$$h^{\alpha\beta}_{(n)K,\theta} = p^{\alpha\beta}_{(n)K,\theta} + q^{\alpha\beta}_{(n)K,\theta}. \tag{2.51}$$

This $h^{\alpha\beta}_{(n)K,\theta}$ satisfies the nth-order Einstein vacuum equations (2.39) with the harmonic coordinate condition

$$\partial_\beta h^{\alpha\beta}_{(n)K,\theta} = 0 \quad (n \geq 2), \tag{2.52}$$

and it belongs to the class L^{n-1} (because $p_{(n)K,\theta} \in L^{n-1}$ and $q_{(n)K,\theta} \in L^0$) so that the induction hypothesis are satisfied at nth order. This ends the construction of the "modified canonical" metric (2.37).

Finally note that, by the choice (2.52) of the harmonic coordinate condition for the nonlinear iterations, the exterior metric (2.37) satisfies

$$\partial_\beta \mathcal{G}^{\alpha\beta}_{K,\theta} = G\, \Box\, \varepsilon^{\alpha}_{K,\theta} \tag{2.53}$$

[see Eq. (2.36)]. Thus, the "nonharmonicity" of the metric (i.e., its divergence) comes only from its linear part, that we have seen to be nonzero only in the near zone and to have there the small magnitude given by Eq. (2.32).

III. NEAR-ZONE EXPANSION OF THE EXTERNAL GRAVITATIONAL FIELD

A. Near-zone expansion of retarded integrals

In this subsection we present formulas, which have been somewhat implicit in our previous papers (I and II) but have not yet received a complete proof, concerning the structure of the near-zone expansion, when $r/c \to 0$, of (the finite part of) the retarded integral of some source. Let $f(\mathbf{x},t)$ be a function belonging to some class L^n, i.e., admitting for any $N \in I\!N$ a near-zone expansion of the type (2.40) [with remainder $R_N(\mathbf{x},t) = O(r^N)$], and let $\bar{f}(\mathbf{x},t)$ be the *formal* infinite near-zone expansion of $f(\mathbf{x},t)$ (written "without remainder"), i.e.,

$$\bar{f}(\mathbf{x},t) = \sum_{p\leq n} F_{Lap}(t)\hat{n}_L r^a (\ln r)^p \ , \qquad (3.1)$$

in which the values of the (positive or negative) integer a range from a minimum value a_0 up to $+\infty$, and where we recall that the functions $F_{Lap}(t)$ are constant or zero in the past. Then we can state the following result.

Theorem. The finite part of the retarded integral of $\bar{f}(\mathbf{x},t)$ (i.e., $\mathrm{FP}\,\Box_R^{-1}\bar{f}$), defined to be the series made of the finite part of the retarded integrals of each separate terms in Eq. (3.1), can be written in the form

$$\mathrm{FP}\,\Box_R^{-1}\bar{f} = \mathrm{FP}\,I^{-1}\bar{f} + \sum_{l\geq 0} \frac{(-)^l}{l!}\hat{\partial}_L \left\{ \frac{R_L(t-r/c) - R_L(t+r/c)}{2r} \right\} . \qquad (3.2)$$

In this expression, the function $R_L(u)$, which parametrizes an "antisymmetric" wave in the sense of Sec. II, is related to the source $\bar{f}(\mathbf{x},t)$ by the equation

$$R_L(u) = \text{finite part}_{B=0} \left\{ -\frac{1}{4\pi}\int d^3\mathbf{y}\, \hat{y}_L |\mathbf{y}|^B g_l(\mathbf{y},u) \right\} , \qquad (3.3)$$

where $\hat{y}_L = (y_{i_1} y_{i_2} \cdots y_{i_l})^{\mathrm{STF}}$ and where the function $g_l(\mathbf{y},u)$ is defined by

$$g_l(\mathbf{y},u) = \int_1^{+\infty} dz\, \gamma_l(z) \bar{f}(\mathbf{y}, u - z|\mathbf{y}|/c), \qquad (3.4)$$

with

$$\gamma_l(z) = (-)^{l+1} \frac{(2l+1)!!}{2^l l!}(z^2-1)^l \ . \qquad (3.5a)$$

[Note that we can assume that $\bar{f}(\mathbf{x},t)$ in Eq. (3.4) is zero in the past because the constant part of $\bar{f}$ does not contribute to the second term in Eq. (3.2). So the integral Eq. (3.4) is convergent.] Thus, the function $R_L(u)$ appears to be the (finite part of the) lth order multipole moment of an effective l-dependent source $g_l(\mathbf{y},u)$ representing an average of the real source $\bar{f}(\mathbf{y},v)$ over the retarded values $v \leq u$, and weighted by the function $\gamma_l(z)$.

Note that one can easily give a sense to the (divergent) integral from 1 to infinity of the weighting function $\gamma_l(z)$. Indeed, by assuming that l is a complex number satisfying $-1 < \mathrm{Re}(l) < -1/2$, and by using in the expression of $\gamma_l(z)$, the Euler gamma function, we compute

$$\int_1^{+\infty} dz\, \gamma_l(z) = 2(-)^{l+1} \frac{\Gamma(2l+2)\Gamma(-2l-1)}{\Gamma(l+1)\Gamma(-l)} \ . \qquad (3.5b)$$

The right-hand side of this equation can be extended by analytic continuation to all values of $l \in \mathbb{C}$ (except half-integer values), and it is equal to one when l is an integer:

$$\int_1^{+\infty} dz\, \gamma_l(z) = 1 \qquad (l \in I\!N) \ . \qquad (3.5c)$$

[Equation (3.5c) is consistent with Eq. (2.6b) and the fact that $\int_{-\infty}^{+\infty} dz (z^2-1)^l$ is zero by analytic continuation when l is an integer.]

Second, the operator $\mathrm{FP}I^{-1}$ appearing in the right-hand side of Eq. (3.2) is defined to be the finite part at $B=0$ (or constant coefficient in the Laurent expansion) of the formal solution of the wave equation with source $r^B\bar{f}$ obtained by iterated use of the inverse Laplace operator Δ^{-1} defined, when acting on each separate term of the series Eq. (3.1), by

$$\Delta^{-1}\left[\hat{n}_L r^{B+a}(\ln r)^p\right] = \left(\frac{\partial}{\partial B}\right)^p \left[\frac{\hat{n}_L r^{B+a+2}}{(B+a+2-l)(B+a+3+l)} \right] \qquad (3.6)$$

[see Eq. (3.9) of paper I]. Thus, $\mathrm{FP}I^{-1}\bar{f}$ is defined by the infinite series

$$\mathrm{FP}\,I^{-1}\bar{f} = \text{finite part}_{B=0} \sum_{k=0}^{+\infty} \left(\frac{\partial}{c\partial t}\right)^{2k} \Delta^{-k-1}[r^B \bar{f}(\mathbf{x},t)], \qquad (3.7a)$$

where $\Delta^{-k-1} = (\Delta^{-1})^{k+1}$ is the $(k+1)$th iteration of the operator Eq. (3.6). Note that an alternative expression for the operator $\mathrm{FP}I^{-1}\bar{f}$, straightforwardly obtained from Eq. (3.7a), is

$$\mathrm{FP}\,I^{-1}\bar{f} = \text{finite part}_{B=0} \sum_{k=0}^{+\infty} \left(\frac{\partial}{c\partial t}\right)^{2k} \left\{ \frac{-1}{4\pi}\int d^3\mathbf{x}'\, \frac{|\mathbf{x}-\mathbf{x}'|^{2k-1}}{(2k)!} |\mathbf{x}'|^B \bar{f}(\mathbf{x}',t) \right\} , \qquad (3.7b)$$

in which the integrals are defined by analytic continuation in B. We have, in all $I\!R^4$ except the time axis,

$$\Box\,(\mathrm{FP}\,I^{-1}\bar{f}) = \bar{f} \ . \qquad (3.8)$$

The proof of the theorem (3.2) is given in Appendix A. Note that the structure of the near-zone expansion of the retarded integral is expressed in a very clear way by this theorem. The particular solution $\mathrm{FP}I^{-1}\bar{f}$ computes the near-zone expansion of the "solution" directly from the near-zone expansion of the "source" in the way we would like to proceed in practical computations performed in the near zone of the source (see Secs. III B and III C below). The second term in Eq. (3.2) adds to this "practical" solution a particular (homogeneous) solution of the antisymmetric type discussed in Sec. II. Recall that the order of magnitude in the near zone of an antisymmetric wave is small [it is damped by the factor c^{-2l-1} of Eqs. (2.4)–(2.6a) with respect to the corresponding retarded wave], and thus we shall often be able to neglect it in the computation of the lowest-order nonlinear metric coefficients in the near zone. On the other hand, this antisymmetric solution is associated with *nonlinear* radiation reaction effects. It contains in particular the hereditary radiation reaction effect found in paper II (see also Sec. III D below).

It is important to note that the particular solution $\mathrm{FP}I^{-1}\bar{f}$ is an instantaneous functional of the source in the sense that (following the terminology of paper II) its value at some time t depends on the values of the source at only one instant, the same time t. We shall refer to the operator $\mathrm{FP}I^{-1}$ as the operator of the *instantaneous potentials.* Note also that the operator $\mathrm{FP}I^{-1}$ is "even" in the sense that it explicitly involves only even powers of c^{-1}. As for the functions $R_L(u)$ parametrizing the antisymmetric wave in Eq. (3.2), they are *retarded* functionals of the source, as is clear from their expressions (3.3)–(3.4); in particular, the functions $R_L(u)$ are zero when $u \leq -T$.

In Appendix B, for the sake of completeness, we investigate the link between the operator of the instantaneous potentials $\mathrm{FP}I^{-1}$, given by Eq. (3.7a) or (3.7b), and the operator of the *symmetric* potentials defined, when it exists, as the (finite part of the) half-sum of the retarded and advanced integrals:

$$\mathrm{FP}\,\Box_S^{-1}\bar{f} = \mathrm{FP}_{B=0}\left\{-\frac{1}{8\pi}\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}|\mathbf{x}'|^B\left[\bar{f}\left(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)+\bar{f}\left(\mathbf{x}',t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\right]\right\}\ . \tag{3.9}$$

Let us suppose for a moment that the function $\bar{f}(\mathbf{x},t)$ is zero both in the past $t \leq -T$ and in the future $t \geq +T$, so that Eq. (3.9) is well defined. By comparing Eqs. (3.7b) and (3.9) we see that the instantaneous integral $\mathrm{FP}I^{-1}\bar{f}$ is equal to the *formal* Taylor expansion when $c \to +\infty$ of the symmetric integral $\mathrm{FP}\,\Box_S^{-1}\bar{f}$. The exact relation between $\mathrm{FP}I^{-1}\bar{f}$ and $\mathrm{FP}\,\Box_S^{-1}\bar{f}$ (which takes into account the remainder of the Taylor expansion) is, however, not an equality and is given by Eq. (B9) in Appendix B. It is shown also in Appendix B that the decomposition of the retarded integral analogous to Eq. (3.2) but with the symmetric operator $\Box_S^{-1}$ in place of the instantaneous one I^{-1} is given by

$$\mathrm{FP}\,\Box_R^{-1}\bar{f} = \mathrm{FP}\,\Box_S^{-1}\bar{f} + \sum_{l\geq 0}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{S_L(t-r/c)-S_L(t+r/c)}{2r}\right\}, \tag{3.10}$$

in which the functions $S_L(u)$ are the following lth-order moments of the source:

$$S_L(u) = \text{finite part}_{B=0}\left\{-\frac{1}{4\pi}\int d^3\mathbf{y}\,\hat{y}_L|\mathbf{y}|^B\int_{-1}^{1}dz\,\delta_l(z)\bar{f}(\mathbf{y},u-z|\mathbf{y}|/c)\right\}, \tag{3.11}$$

involving a weighting function $\delta_l(z)$ which is simply related to the weighting function $\gamma_l(z)$ of Eq. (3.5):

$$\delta_l(z) = -\tfrac{1}{2}\,\gamma_l(z),\quad \int_{-1}^{1}dz\,\delta_l(z) = 1\ . \tag{3.12}$$

This weighting function $\delta_l(z)$ is the one appearing in the computation of the multipole expansion of the solution of the wave equation outside a compact in space source; see Eqs. (B.11) and (B.12) of paper III. It can also be used to express the antisymmetric wave itself; see Eqs. (2.6) in Sec. II above.

Note that the function $S_L(u)$, parametrizing the antisymmetric wave in the decomposition of the retarded integral in terms of the symmetric integral [Eq. (3.10)], is a mixed retarded *and* advanced functional (and in fact a symmetric functional) of the source. Of course this is evident because the symmetric integral is itself such a functional. But this shows by contrast how convenient is the decomposition (3.2) of the retarded integral in terms of the instantaneous potentials I^{-1} and of the antisymmetric waves parametrized by the retarded functionals $R_L(u)$. On the other hand the computation of paper II has shown that nonlinear radiation reaction effects are contained in the antisymmetric waves involving the R_L's, and not in those involving the S_L's. We now come back to a situation in which $\bar{f}(\mathbf{x},t)$ is only zero in the past, and where the decomposition (3.10) in terms of the symmetric integral is *a priori* not valid.

B. Near-zone expansion of the exterior metric

In this subsection, with the help of the theorem stated in Eq. (3.2), we define an operational method for computing the near-zone expansion of an arbitrary nth-order nonlinear metric, $h^{\alpha\beta}_{(n)K,\theta}$, given that of its source, $N^{\alpha\beta}_{(n)K,\theta}$. Since the method will be valid for any construction of the nonlinear metric of the type (3.15) below, we shall suppress the indices K and θ in this subsection.

Let us first write down the structure of the near-zone expansion of the source which is of the type (3.1) since the source belongs to L^{n-2} [Eq. (2.45)]. Using the results of paper I we can obtain the dependence of the source on c and write

$$\bar{N}^{\alpha\beta}_{(n)} = \sum_{E_n} c^{-(3n+\Sigma_{i=1}^n \underline{l}_i+2)} \times \left\{ \sum_{p\leq n-2} F^{\alpha\beta}_{Lap}(t)\hat{n}_L \left(\frac{r}{c}\right)^a \left(\ln\frac{r}{c}\right)^p \right\}, \tag{3.13}$$

where E_n denotes a set of n mass-type or current-type multipole moments $M_{L_1}, M_{L_2}, \ldots, \varepsilon_{ai_{l_n+1}i_n} S_{aL_n-1}$ (where the current-type moments are endowed with their natural Levi-Civita symbols), where $\Sigma_{i=1}^n \underline{l}_i$ is the total number of indices on the moments composing the set E_n ($\Sigma_{i=1}^n \underline{l}_i = \Sigma_{i=1}^n l_i + j$, where j is the number of current-type moments) and where the functions $F^{\alpha\beta}_{Lap}(t)$ are some multilinear functionals (homogeneous of order n) of the moments and their time derivatives having the form

$$F^{\alpha\beta}_{Lap}(t) = \sum \int \cdots \int du_1 \cdots du_n \mathcal{K}^{\alpha\beta}_{L\underline{L}_1\cdots\underline{L}_n}(t, u_1, \ldots, u_n) \times M^{(q_1)}_{L_1}(u_1) \cdots \varepsilon_{ai_{l_n+1}i_n} S^{(q_n)}_{aL_n-1}(u_n)\,. \tag{3.14}$$

The kernel $\mathcal{K}$ in Eq. (3.14) has an index structure made out only of Kronecker δ's and it is a function only of quantities having the dimension of time: the time argument t, the n integration time arguments $u_1 \cdots u_n$ (which all satisfy $u_i \leq t$ because of the retarded nature of the metric), and also the constant P (chosen here to be one) in the factors $(r/cP)^B$.

The nth-order nonlinear iteration of the exterior metric is given by

$$h^{\alpha\beta}_{(n)} = p^{\alpha\beta}_{(n)} + q^{\alpha\beta}_{(n)}, \tag{3.15}$$

where the first term is the (finite part of) the retarded integral of the source [see Eq. (2.46)] and where the second term is defined in Eq. (2.50). We deal first with the near-zone expansion of the first term $p^{\alpha\beta}_{(n)}$. With the help of Eq. (3.2) we can write this expansion as

$$\bar{p}^{\alpha\beta}_{(n)} = \mathrm{FP}\, I^{-1}\bar{N}^{\alpha\beta}_{(n)} + \sum_{l\geq 0} \frac{(-)^l}{l!}\hat{\partial}_L \left\{ \frac{R^{\alpha\beta}_L(t-r/c) - R^{\alpha\beta}_L(t+r/c)}{2r} \right\}, \tag{3.16}$$

where $\mathrm{FP}\, I^{-1}$ is the operator defined in Eqs. (3.7) (acting on each separate term of the L^{n-2} expansion of the source), and where the functions $R^{\alpha\beta}_L(u)$ are the (averaged) moments

$$R^{\alpha\beta}_L(u) = \mathrm{FP}_{B=0}\left\{ -\frac{1}{4\pi}\int_1^{+\infty} dz\, \gamma_l(z) \int d^3\mathbf{y}\, \hat{y}_L |\mathbf{y}|^B \bar{N}^{\alpha\beta}_{(n)}(\mathbf{y}, u - z|\mathbf{y}|/c) \right\}. \tag{3.17}$$

Inserting Eq. (3.13) into Eq. (3.17), we easily see that the expression (3.16) can be written as

$$\bar{p}^{\alpha\beta}_{(n)} = \mathrm{FP}\, I^{-1}\bar{N}^{\alpha\beta}_{(n)} + \sum_{E_n} \frac{1}{c^{3n+\Sigma\underline{l}_i}} \sum_{l\geq 0} c^{l+1}\hat{\partial}_L \left\{ \frac{P^{\alpha\beta}_L(t-r/c) - P^{\alpha\beta}_L(t+r/c)}{2r} \right\}, \tag{3.18}$$

where now the functions $P^{\alpha\beta}_L(u)$ have the same structure as in Eq. (3.14), i.e., involving a kernel depending only on variables having the dimension of time (the $P^{\alpha\beta}_L$'s do not depend on c). The antisymmetric waves in Eq. (3.18) define a component of the metric which is associated with nth-order nonlinear radiation reaction effects.

We shall now (in the present Sec. III B and in the following one III C) view these waves as small remainders in the near-zone expansion of the metric. Inserting into Eq. (3.18) the expansion when $c \to +\infty$ of the antisymmetric waves [which involve an extra factor c^{-2l-1} where l is the multipolarity of the waves; see Eq. (2.4)] we obtain

$$\bar{p}^{\alpha\beta}_{(n)} = \mathrm{FP}\, I^{-1}\bar{N}^{\alpha\beta}_{(n)} + \sum_{E_n}\sum_{l} O\left(\frac{1}{c^{3n+\Sigma\underline{l}_i+l}}\right). \tag{3.19}$$

Then we use an equation relating the integer $\Sigma_{i=1}^n \underline{l}_i + l$ to the number s of (spin) spatial indices among the indices α and β: $s = 0, 1, 2$ when $\alpha\beta = 00$, $0i$, ij, respectively, and to the number k of contractions between the indices of the moments composing E_n. This equation is

$$\sum_{i=1}^n \underline{l}_i + l = 4 - s + 2k \tag{3.20}$$

and is valid whenever at least one of the moments composing E_n is time varying, which is the case in the second term of Eq. (3.18). The proof of Eq. (3.20) is given in paper III [see Eq. (3.14) there]. Equation (3.20) is very useful because it provides a uniform majoration of the magnitude in c^{-1} of the remainder in Eq. (3.19). We obtain

$$\bar{p}^{\alpha\beta}_{(n)} = \mathrm{FP}\, I^{-1}\bar{N}^{\alpha\beta}_{(n)} + O\left(\frac{1}{c^{3n+4-s}}\right). \tag{3.21}$$

This result is general (not depending on the particular algorithm used for the construction of the exterior metric), and also maximal in the sense that the majoration of the remainder is generally reached. For instance the quadratic order "hereditary" terms obtained in paper II were found to have the magnitude $O(c^{-10+s})$, consistent with Eq. (3.21) and $n=2$.

Note that by Eqs. (3.18) and (3.20) one sees that antisymmetric waves are "odd" (in the usual post-Newtonian sense) in odd nonlinear iterations of the metric (i.e., odd n), and on the contrary are "even" in even iterations of the metric (even n).

The formula (3.21) (which has already been found to be very useful in our previous papers [22], [35], [38]) permits the computation "in the near zone," using the operator FP I^{-1}, of the expansion of $p^{\alpha\beta}_{(n)}$ from the expansion of the source with the accuracy $O(c^{-3n-4+s})$. Furthermore, we see from the structure of the near-zone expansion of the antisymmetric wave [Eq. (2.4)] that the post-Newtonian terms (i) involving at least one power of the logarithm of c, or (ii) differing from the leading order $c^{-3n-4+s}$ of the remainder by an odd power of c^{-1} (e.g., the term c^{-3n-5} in $p^{00}_{(n)}$) can also be computed from the source by means of FP I^{-1} even though they are smaller than the remainder. [The computation of Eq. (3.44b) in the next subsection will use the previous remark (ii).] Note also that the remainder in Eq. (3.21) is of order $O(c^{-3n-4})$ whenever the left-hand side is a scalar ($s=0$). Thus one can write, for instance,

$$\bar{p}^{00}_{(n)}+\bar{p}^{ii}_{(n)} = \mathrm{FP}\, I^{-1}\left(\bar{N}^{00}_{(n)}+\bar{N}^{ii}_{(n)}\right)+O\left(\frac{1}{c^{3n+4}}\right) . \tag{3.22}$$

Next we consider the second part $q^{\alpha\beta}_{(n)}$ of the metric (3.15) which is a particular retarded solution of the wave equation defined by the multipolar series (2.50). Restoring the powers of c^{-1} we can write the structure of $q^{\alpha\beta}_{(n)}$ as

$$q^{\alpha\beta}_{(n)}=\sum_{E_n}\frac{1}{c^{3n+\Sigma l_i}}\sum_{l\geq 0}c^{l+1}\hat{\partial}_L\left(\frac{Q^{\alpha\beta}_L(t-r/c)}{r}\right), \tag{3.23}$$

where the functions $Q^{\alpha\beta}_L(u)$ have a structure analogous to Eq. (3.14). First we note that $q^{\alpha\beta}_{(n)}$ has been defined in such a way that its scalar components $q^{00}_{(n)}$ and $q^{ii}_{(n)}$ given, respectively, by Eqs. (2.50a) and (2.50d) involve only monopolar ($l=0$) or dipolar ($l=1$) waves. Using this fact in Eq. (3.23), and using the equality (3.20), we see that the expansions of these scalar components are at least of order

$$\bar{q}^{00}_{(n)} \quad \text{and} \quad \bar{q}^{ii}_{(n)} = O\left(\frac{1}{c^{3n+1}}\right) . \tag{3.24}$$

This result is useful in applications. Let us now proceed as follows. First of all the near-zone expansion of the divergence $r^{\alpha}_{(n)}=\partial_\beta p^{\alpha\beta}_{(n)}$ [which is given by Eq. (2.48)] can be written similarly to Eq. (3.21) by means of the operator I^{-1}:

$$\bar{r}^{\alpha}_{(n)}=\text{finite part}_{B=0}\, I^{-1}\left[Br^{B-1}n_i\bar{N}^{\alpha i}_{(n)}\right]+O\left(\frac{1}{c^{3n+5-s}}\right) \tag{3.25}$$

(in which $s=0$ if $\alpha=0$ and $s=1$ if $\alpha=i$). On the other hand we know that $r^{\alpha}_{(n)}$ admits the decomposition (2.49) into a series of retarded multipole waves. This fact shows that there must exist some STF tensors $\mathcal{A}_L(r,t),\dots,\mathcal{D}_L(r,t)$ such that the expansion (3.25) reads

$$\bar{r}^{0}_{(n)}=\sum_{l\geq 0}\partial_L\mathcal{A}_L(r,t) , \tag{3.26a}$$

$$\bar{r}^{i}_{(n)}=\sum_{l\geq 0}\{\partial_{iL}\mathcal{B}_L(r,t)+\partial_L\mathcal{C}_{iL}(r,t)+\varepsilon_{iab}\partial_{aL}\mathcal{D}_{bL}(r,t)\} . \tag{3.26b}$$

Now we associate to the decomposition (3.26) a new tensor defined in a manner analogous to that of Eqs. (2.50), i.e.,

$$\bar{t}^{00}_{(n)}=-c\mathcal{A}^{(-1)}-c\partial_a\mathcal{A}^{(-1)}_a+c^2\partial_a\mathcal{C}^{(-2)}_a , \tag{3.27a}$$

$$\bar{t}^{0i}_{(n)}=-c\mathcal{C}^{(-1)}_i-c\varepsilon_{iab}\partial_a\mathcal{D}^{(-1)}_b-\sum_{l\geq 2}\partial_{L-1}\mathcal{A}_{iL-1}, \tag{3.27b}$$

$$\bar{t}^{ij}_{(n)}=-\delta_{ij}\mathcal{B}-\delta_{ij}\partial_a\mathcal{B}_a+\sum_{l\geq 2}\left\{\partial_{L-2}\left(\frac{1}{c}\mathcal{A}^{(1)}_{ijL-2}+\frac{3}{c^2}\mathcal{B}^{(2)}_{ijL-2}-\mathcal{C}_{ijL-2}\right)+2\delta_{ij}\partial_L\mathcal{B}_L-6\partial_{L-1(i}\mathcal{B}_{j)L-1}-2\partial_{aL-2}\left(\varepsilon_{ab(i}\mathcal{D}_{j)bL-2}\right)\right\} , \tag{3.27c}$$

in which the antiderivatives refer to the variable t [for instance $\mathcal{A}^{(-1)}(r,t)=\int_{-\infty}^t\mathcal{A}(r,x)dx$]. The trace of Eq. (3.27c) is

$$\bar{t}^{ii}_{(n)}=-3\mathcal{B}-3\partial_a\mathcal{B}_a . \tag{3.27d}$$

The tensor $\bar{t}^{\alpha\beta}_{(n)}$ is computed directly from the near-zone expansion of the divergence (3.26) by formulas analogous to the defining formulas of $q^{\alpha\beta}_{(n)}$ [compare Eqs. (2.50) and (3.27)]. However, $\bar{t}^{\alpha\beta}_{(n)}$ is not exactly equal to the near-zone expansion $\bar{q}^{\alpha\beta}_{(n)}$ of $q^{\alpha\beta}_{(n)}$. This can be seen as follows. The tensors $\mathcal{A}_L(r,t),\dots,\mathcal{D}_L(r,t)$ are related to the functions $A_L(t-r/c),\dots,D_L(t-r/c)$ of Eqs. (2.49) by formulas such as

$$\mathcal{A}_L(r,t)=\frac{1}{r}\times\sum_{k\notin\{1,3,\dots,2l-1\}}\frac{1}{k!}\left(-\frac{r}{c}\right)^k A^{(k)}_L(t), \tag{3.28}$$

in which the sum ranges over all values of $k\in I\!N$ except the values $k=1,3,\dots,2l-1$. Indeed these values must be removed from the sum because the corresponding post-Newtonian terms, whose spatial dependence is $\sim r^{2j}$ with $j=0,1,\dots,l-1$, are killed by the STF operators $\hat{\partial}_L$ of Eq. (3.26) [see Eq. (A33) of paper I] and thus are in fact absent in the near-zone expansion (3.25). However, since the number of derivatives acting on some given tensor $\mathcal{A}_L, \ldots, \mathcal{D}_L$ differs in Eqs. (3.26) and (3.27), and in particular can decrease from Eqs. (3.26) to (3.27), we see that some of these absent terms will generate no terms in $\bar{t}^{\alpha\beta}_{(n)}$ while the corresponding terms really exist in $\bar{q}^{\alpha\beta}_{(n)}$. For instance we have $\partial_L \mathcal{A}_L$ in $\bar{r}^0_{(n)}$ [see Eq. (3.26a)] which produces $\partial_{L-1}\mathcal{A}_{iL-1}$ in $\bar{t}^{0i}_{(n)}$ [see Eq. (3.27b)], but

$$\partial_{L-1}\mathcal{A}_{iL-1} = \partial_{L-1}\left[\frac{1}{r}A_{iL-1}\left(t-\frac{r}{c}\right)\right] + \frac{x_{L-1}}{c^{2l-1}(2l-1)!!}A^{(2l-1)}_{iL-1}(t) \ . \quad (3.29)$$

The first term is the term which normally appears in $q^{0i}_{(n)}$ [see Eq. (2.50b)] but the second one is an additional unwanted term. Fortunately, this unwanted term has, as well as the other unwanted terms coming from the other functions $\mathcal{B}_L$, $\mathcal{C}_L$, and $\mathcal{D}_L$, a structure of the type $\hat{x}_L r^{2k} F(t)$ similar to the one of the near-zone expansion of the antisymmetric wave [Eq. (2.4)]. Thus, by the same reasoning as the one yielding Eq. (3.21), we can control the order of magnitude of these terms and we have

$$\bar{q}^{\alpha\beta}_{(n)} = \bar{t}^{\alpha\beta}_{(n)} + O\left(\frac{1}{c^{3n+4-s}}\right) \ . \quad (3.30)$$

Gathering Eqs. (3.21) and (3.30) we can now write the near-zone expansion of the nth nonlinear metric in the form

$$\bar{h}^{\alpha\beta}_{(n)} = \mathrm{FP}\, I^{-1}\bar{N}^{\alpha\beta}_{(n)} + \bar{t}^{\alpha\beta}_{(n)} + O\left(\frac{1}{c^{3n+4-s}}\right) \ . \quad (3.31)$$

This form defines our operational method for computing the metric directly in the near zone, with the precision indicated in Eq. (3.31), without having recourse to the full algorithm of paper I (or of Sec. II of this paper) computing the metric in all the exterior region of the system. Note that the method is applicable to any construction of the metric in which the nonlinear metrics are given by Eq. (3.15), independently of the linear metric, which needs only to belong to L^0. The method adequately completes the work of paper I.

As an application we now prove a theorem on the order of magnitude, when $c \to +\infty$, of the *odd* terms in the nonlinear metrics, where by odd term we mean a term whose dependence in c is $(\ln c)^p / c^{2k+1+s}$, where s is the number of spatial indices in $\alpha\beta$ and k, p are positive integers (the dependence on $\ln c$ was proved in paper I).

Theorem. If, when $c \to +\infty$, the odd part of the linear metric $h^{\alpha\beta}_{(1)}$ is of order $O(c^{-5}, c^{-4}, c^{-5})$ [using the notation of Eq. (2.10)], then the orders of the odd part of the subsequent iterations $h^{\alpha\beta}_{(n)}$ are given by

$$\left(\bar{h}^{\alpha\beta}_{(n)}\right)_{\rm odd} = O\left(\frac{1}{c^{2n+3}}, \frac{1}{c^{2n+2}}, \frac{1}{c^{2n+3}}\right) \ . \quad (3.32)$$

(For simplicity we suppress the indication of the dependence on $\ln c$.)

The condition on $h^{\alpha\beta}_{(1)}$ is largely satisfied in the cases of the canonical metric [see Eq. (2.11)] and of the modified canonical metric [see Eq. (2.33b)]. However, it turns out that the theorem is sharply valid for nonlinear metrics even in the case of the modified canonical metric (except for special peculiarities where an *a priori* allowed term is in fact absent). The proof of the theorem goes as follows. We insert into the source $\bar{N}^{\alpha\beta}_{(n)}$ the $(n-1)$th first iterations $\bar{h}^{\alpha\beta}_{(m)}$ (for $1 \le m \le n-1$), supposed by induction to satisfy Eq. (3.32), and (using the basic structure of $\bar{N}^{\alpha\beta}_{(n)}$) obtain that $\bar{N}^{\alpha\beta}_{(n)}$ itself satisfies

$$\left(\bar{N}^{\alpha\beta}_{(n)}\right)_{\rm odd} = O\left(\frac{1}{c^{2n+3}}, \frac{1}{c^{2n+2}}, \frac{1}{c^{2n+3}}\right) \ . \quad (3.33)$$

Then, using Eq. (3.21) and the fact that the operator $\mathrm{FP}I^{-1}$ involves only even powers of c^{-1} [see Eq. (3.7a)], we find that

$$\left(\bar{p}^{\alpha\beta}_{(n)}\right)_{\rm odd} = O\left(\frac{1}{c^{2n+3}}, \frac{1}{c^{2n+2}}, \frac{1}{c^{2n+3}}\right) \ . \quad (3.34)$$

This completes the first part of the proof. To prove the second part concerning the tensor $\bar{q}^{\alpha\beta}_{(n)}$ we insert Eq. (3.33) into Eq. (3.25) and get

$$\left(\bar{r}^{\alpha}_{(n)}\right)_{\rm odd} = O\left(\frac{1}{c^{2n+2}}, \frac{1}{c^{2n+3}}\right) \ . \quad (3.35)$$

Then we use Eqs. (3.27) and (3.30) to compute

$$\left(\bar{q}^{\alpha\beta}_{(n)}\right)_{\rm odd} = O\left(\frac{1}{c^{2n+1}}, \frac{1}{c^{2n+2}}, \frac{1}{c^{2n+3}}\right) \ . \quad (3.36)$$

Thus the time-time component $(\bar{q}^{00}_{(n)})_{\rm odd}$ is found to be $O(c^{-2n-1})$ instead of the desired result $O(c^{-2n-3})$. Fortunately we have already noted in Eq. (3.24) that $\bar{q}^{00}_{(n)}$ is in fact of order $O(c^{-3n-1})$, which is always smaller than $O(c^{-2n-3})$. The theorem is thereby proved by induction.

C. The 3.5 post-Newtonian approximation

As we recalled it in the Introduction, this approximation is associated with subdominant reaction effects including the gravitational recoil effect.

We consider the external metric in the modified canonical coordinate system of Sec. II (which generalizes the Burke-Thorne coordinate system in the exterior region). We note first that the regularization constants K and θ which have been introduced in the construction of the metric will not appear in the near-zone expansion of a given nonlinear metric $h^{\alpha\beta}_{(n)K,\theta}$ up to the order $O(c^{-3n-4+s})$ (taking K large enough). This can be checked from the formulas derived in the previous subsection, notably Eqs. (3.21), (3.27), and (3.30). Hence we can write

$$\bar{h}^{\alpha\beta}_{(n)K,\theta} = \bar{h}^{\alpha\beta}_{(n)} + O\left(\frac{1}{c^{3n+4-s}}\right) , \quad (3.37)$$

where $\bar{h}^{\alpha\beta}_{(n)}$ is the metric computed by formal iteration *in the near zone*, using the method expounded in the previous subsection, of the unregularized linear metric (2.23). In the linear case $n = 1$, the accuracy to which $h^{\alpha\beta}_{(1)K,\theta}$ agrees with $h^{\alpha\beta}_{(1)}$ is even higher and is given by Eqs. (2.31) and (2.32). Thus Eq. (3.37) tells us that the dependence on K and θ of the exterior metric in the near zone starts at 4 PN order only.

By the theorem (3.32) of Sec. III B we see that in order to control all contributions in the metric up to the 3.5 PN approximation level, i.e., up to the order $(c^{-1})^{9-s}$ in the components of the usual covariant metric, we must take into account the *quadratic* and *cubic* nonlinearities of Einstein's equations. (Recall that the quadratic nonlinearity is sufficient to compute the 2.5 PN approximation [22].)

At the linearized level the "odd" part of the metric up to 3.5 PN has been computed in Eqs. (2.18), (2.22a), and (2.22b). It reads as

$$\left(\bar{h}^{00}_{(1)} + \bar{h}^{ii}_{(1)}\right)_{\text{odd}} = \frac{4}{5c^7}x^a x^b M^{(5)}_{ab}(t) + \frac{1}{c^9}\left[-\frac{4}{189}x^a x^b x^c M^{(7)}_{abc}(t) + \frac{2}{35}x^a x^b r^2 M^{(7)}_{ab}(t)\right] + O\left(\frac{1}{c^{11}}\right) , \tag{3.38a}$$

$$\left(\bar{h}^{0i}_{(1)}\right)_{\text{odd}} = \frac{1}{c^8}\left[-\frac{4}{21}\hat{x}^{iab}M^{(6)}_{ab}(t) + \frac{16}{45}\varepsilon_{iab}x^a x^c S^{(5)}_{bc}(t)\right] + O\left(\frac{1}{c^{10}}\right) , \tag{3.38b}$$

$$\left(\bar{h}^{ij}_{(1)}\right)_{\text{odd}} = O\left(\frac{1}{c^9}\right) . \tag{3.38c}$$

Note that we are using the metric in the modified canonical gauge and that for instance the space-space components (3.38c) of the "odd" metric start at $O(c^{-9})$ instead of $O(c^{-5})$ in the harmonic gauge. This fact makes the nonlinear iterations of the "odd" metric easy to perform. At quadratic order the "odd" part of the source, which is generated by the interaction of the "odd" metric (3.38) and of the "even" one, is dominantly of order

$$\left(\bar{N}^{\alpha\beta}_{(2)}\right)_{\text{odd}} = O\left(\frac{1}{c^9}, \frac{1}{c^{10}}, \frac{1}{c^9}\right) . \tag{3.39}$$

To compute the first part $\bar{p}^{\alpha\beta}_{(2)}$ of the metric [see Eq. (3.15)] at 3.5 PN order we need the following easily computed source:

$$\left(\bar{N}^{00}_{(2)} + \bar{N}^{ii}_{(2)}\right)_{\text{odd}} = \frac{16}{5c^9}\Delta(Ux^a x^b)M^{(5)}_{ab}(t) + O\left(\frac{1}{c^{11}}\right), \tag{3.40}$$

in which $\Delta = \partial_i\partial_i$ and where U is such that

$$\bar{h}^{00}_{(1)} = -\frac{4}{c^2}U + O\left(\frac{1}{c^4}\right) . \tag{3.41a}$$

U is explicitly given [see Eq. (1.4a)] by

$$U = \sum_{l\geq 0}\frac{(-)^l}{l!}(\partial_L r^{-1})M_L(t) . \tag{3.41b}$$

Then Eq. (3.22) tells us that we can compute the solution by means of the instantaneous operator FPI^{-1} because the remainder $O(c^{-(3\times2+4)}) = O(c^{-10})$ in Eq. (3.22) is strictly smaller than the order c^{-9} we want to reach. The operator FPI^{-1} reduces at this order to FPΔ^{-1} and by standard computations [see, e.g., Eq. (3.20) of paper III and Eqs. (3.9), (3.10) of paper I] we arrive at

$$\left(\bar{p}^{00}_{(2)} + \bar{p}^{ii}_{(2)}\right)_{\text{odd}} = \frac{16}{5c^9}Ux^a x^b M^{(5)}_{ab}(t) - \frac{4}{c^9}\sum_{l\geq 0}\frac{(-)^l}{l!}(\partial_L r^{-1})T_L(t) + O\left(\frac{1}{c^{11}}\right) . \tag{3.42}$$

The functions $T_L(t)$ in this expression have been found to be given by

$$T_L(t) = \frac{4}{5}\frac{2l+1}{2l+5}M^{(5)}_{ab}(t)M_{Lab}(t) . \tag{3.43}$$

We now compute the second part $\bar{q}^{\alpha\beta}_{(2)}$ of the metric. First of all, the divergence $\bar{r}^{\alpha}_{(2)} = \partial_\beta\bar{p}^{\alpha\beta}_{(2)}$ can be computed by means of Eq. (3.25). We find that its "odd" part is dominantly $O(c^{-10}, c^{-9})$ and is given by the formulas

$$\left(\bar{r}^{0}_{(2)}\right)_{\text{odd}} = \text{FP}_{B=0}\Delta^{-1}\left[Br^{B-1}n_j\left(\bar{N}^{0j}_{(2)}\right)_{\text{odd}}\right] + O\left(\frac{1}{c^{12}}\right) , \tag{3.44a}$$

$$\left(\bar{r}^{i}_{(2)}\right)_{\text{odd}} = \text{FP}_{B=0}\left[\Delta^{-1} + \left(\frac{\partial}{c\partial t}\right)^2\Delta^{-2}\right] \times\left[Br^{B-1}n_j\left(\bar{N}^{ij}_{(2)}\right)_{\text{odd}}\right] + O\left(\frac{1}{c^{13}}\right) . \tag{3.44b}$$

Actually, the remainder in Eq. (3.25) is $O(c^{-(3\times2+5-s)}) = O(c^{-11}, c^{-10})$, and thus allows us *a priori* to write Eq. (3.44a) but not Eq. (3.44b). However, we have already noted [see the discussion after Eq. (3.21)] that we are in fact still entitled to use the operator $I^{-1} = \Delta^{-1} + (\partial/c\partial t)^2\Delta^{-2} + \cdots$ when the order in c^{-1} differs from the remainder [in this case $O(c^{-10})$] by an odd power of c^{-1}, which is indeed the case for the term c^{-11}

in Eq. (3.44b). Now we can compute the tensor $\bar{q}^{\alpha\beta}_{(2)}$ by means of Eqs. (3.26) and (3.27), which generate the intermediate tensor $\bar{t}^{\alpha\beta}_{(2)}$, and of Eq. (3.30), which shows that $\bar{q}^{\alpha\beta}_{(2)}$ exactly agrees with $\bar{t}^{\alpha\beta}_{(2)}$ at the 3.5 PN level [because the remainder $O(c^{-10+s})$ starts at the 4 PN level]. Inspection of Eqs. (3.26) and (3.27) shows furthermore that not all of the functions $\mathcal{A}_L$, $\mathcal{B}_L, \ldots$ are needed, but only the functions $\mathcal{A}$, $\mathcal{A}_i$, $\mathcal{B}$, $\mathcal{B}_i$, $\mathcal{C}_i$, and $\mathcal{D}_i$ in the decomposition of $\bar{r}^{\alpha}_{(2)}$. The explicit computation is done in Appendix C and yields

$$\left(\bar{q}^{00}_{(2)} + \bar{q}^{ii}_{(2)}\right)_{\text{odd}} = \frac{4}{c^7}\partial_i\left[r^{-1}m_i\left(t-\frac{r}{c}\right)\right] - \frac{4}{c^9}r^{-1}m\left(t-\frac{r}{c}\right) + O\left(\frac{1}{c^{11}}\right) , \tag{3.45a}$$

$$\left(\bar{q}^{0i}_{(2)}\right)_{\text{odd}} = -\frac{4}{c^8}r^{-1}m_i^{(1)}\left(t-\frac{r}{c}\right) - \frac{2}{c^8}\varepsilon_{iab}\partial_a\left[r^{-1}s_b\left(t-\frac{r}{c}\right)\right] + O\left(\frac{1}{c^{10}}\right) , \tag{3.45b}$$

$$\left(\bar{q}^{ij}_{(2)}\right)_{\text{odd}} = O\left(\frac{1}{c^9}\right) , \tag{3.45c}$$

in which the functions $m(u)$, $m_i(u)$, and $s_i(u)$ are given by

$$m(u) = -\frac{1}{5}\int_{-\infty}^{u} dv\, M^{(3)}_{ab}(v)M^{(3)}_{ab}(v) + F(u) , \tag{3.46a}$$

$$\begin{aligned} m_i(u) = &-\frac{2}{5}M_a M^{(3)}_{ia}(u) - \frac{2}{21c^2}\int_{-\infty}^{u} dv\, M^{(3)}_{iab}(v)M^{(3)}_{ab}(v) \\ &+\frac{1}{c^2}\int_{-\infty}^{u} dv\int_{-\infty}^{v} dw\left[-\frac{2}{63}M^{(4)}_{iab}(w)M^{(3)}_{ab}(w) - \frac{16}{45}\varepsilon_{iab}M^{(3)}_{ac}(w)S^{(3)}_{bc}(w)\right] + \frac{1}{c^2}G_i(u) , \end{aligned} \tag{3.46b}$$

$$s_i(u) = -\frac{2}{5}\varepsilon_{iab}\int_{-\infty}^{u} dv\, M^{(2)}_{ac}(v)M^{(3)}_{bc}(v) + H_i(u) . \tag{3.46c}$$

The functions $F(u)$, $G_i(u)$, and $H_i(u)$ in these expressions are functions of u which are some products of derivatives of the moments taken at the same time u (they are "instantaneous" functionals of the moments). In Appendix C we give the complete explicit expressions of the functions $m(u)$, $m_i(u)$, and $s_i(u)$.

We still have to find a contribution at 3.5 PN level due to the cubically nonlinear metric. The latter is readily computed: Its sole source is

$$\left(\bar{N}^{00}_{(3)} + \bar{N}^{ii}_{(3)}\right)_{\text{odd}} = \frac{16}{c^9}\Delta(U(\partial_i r^{-1}))m_i(t) + O\left(\frac{1}{c^{11}}\right), \tag{3.47}$$

from which [by Eq. (3.21)] we can deduce

$$\left(\bar{h}^{00}_{(3)} + \bar{h}^{ii}_{(3)}\right)_{\text{odd}} = \frac{16}{c^9}U(\partial_i r^{-1})m_i(t) + O\left(\frac{1}{c^{11}}\right) . \tag{3.48}$$

Indeed it is easily checked, using Eq. (3.20) of paper III and Eqs. (3.9), (3.10) of paper I, that no homogeneous solution of Laplace's equation has to be added to the expression (3.48).

Equations (3.38), (3.42), (3.45), and (3.48) give the complete "odd" part of the external metric up to the 3.5 PN approximation. Let us combine these equations with information concerning the "even" part of the metric. It has been shown in paper III that the exterior metric, in usual covariant form $g^{\text{ext}}_{\alpha\beta}$, up to the 1 PN approximation (to which it agrees with the present "modified canonical" metric) is given by

$$g^{\text{ext}}_{00} = -1 + \frac{2}{c^2}V[\mathcal{M}] - \frac{2}{c^4}(V[\mathcal{M}])^2 + O\left(\frac{1}{c^6}\right) , \tag{3.49a}$$

$$g^{\text{ext}}_{0i} = -\frac{4}{c^3}V^i[\mathcal{M}] + O\left(\frac{1}{c^5}\right) , \tag{3.49b}$$

$$g^{\text{ext}}_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}V[\mathcal{M}]\right) + O\left(\frac{1}{c^4}\right) . \tag{3.49c}$$

The scalar potential V and the tensor potential V^i are given in terms of the moments $\mathcal{M} = \{M_L, S_L\}$ by the symmetric (in the sense of Sec. II) multipolar expressions

$$V[\mathcal{M}] = G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left\{\frac{M_L\left(t-\frac{r}{c}\right) + M_L\left(t+\frac{r}{c}\right)}{2r}\right\} , \tag{3.50a}$$

$$\begin{aligned} V^i[\mathcal{M}] = &-G\sum_{l\geq 1}\frac{(-)^l}{l!}\partial_{L-1}\left\{\frac{M^{(1)}_{iL-1}\left(t-\frac{r}{c}\right) + M^{(1)}_{iL-1}\left(t+\frac{r}{c}\right)}{2r}\right\} \\ &-G\sum_{l\geq 1}\frac{(-)^l l}{(l+1)!}\varepsilon_{iab}\partial_{aL-1}\left\{\frac{S_{bL-1}\left(t-\frac{r}{c}\right) + S_{bL-1}\left(t+\frac{r}{c}\right)}{2r}\right\} . \end{aligned} \tag{3.50b}$$

[See Eqs. (3.22) and (3.3) of paper III in which, consistently with the accuracy of the metric in Eqs. (3.49), we have replaced the retarded waves by their corresponding symmetric waves; note that in Ref. [39] the "even" part of the external metric was investigated up to the more accurate level $O(c^{-6}, c^{-7}, c^{-6})$.] The scalar potential V agrees with GU, Eq. (3.41b), in the limit $c \to +\infty$. Now it is straighforward to show, from the nonlinear iteration we performed in Eqs. (3.38), (3.42), (3.45), and (3.48), that the external metric, accurate to both the 1 PN level as concerns its "even" part and the 3.5 PN level as concerns its "odd" part, is given by

$$g_{00}^{\text{ext}} = -1 + \frac{2}{c^2}\left(V[\widetilde{\mathcal{M}}] + V_{\text{react}}[\widetilde{\mathcal{M}}]\right) - \frac{2}{c^4}\left(V[\widetilde{\mathcal{M}}] + V_{\text{react}}[\widetilde{\mathcal{M}}]\right)^2 + \frac{1}{c^6}\,{}_6g_{00}^{\text{ext}} + \frac{1}{c^8}\,{}_8g_{00}^{\text{ext}} + O\left(\frac{1}{c^{10}}\right) , \tag{3.51a}$$

$$g_{0i}^{\text{ext}} = -\frac{4}{c^3}\left(V^i[\widetilde{\mathcal{M}}] + V^i_{\text{react}}[\widetilde{\mathcal{M}}]\right) + \frac{1}{c^5}\,{}_5g_{0i}^{\text{ext}} + \frac{1}{c^7}\,{}_7g_{0i}^{\text{ext}} + O\left(\frac{1}{c^9}\right) , \tag{3.51b}$$

$$g_{ij}^{\text{ext}} = \delta_{ij}\left[1 + \frac{2}{c^2}\left(V[\widetilde{\mathcal{M}}] + V_{\text{react}}[\widetilde{\mathcal{M}}]\right)\right] + \frac{1}{c^4}\,{}_4g_{ij}^{\text{ext}} + \frac{1}{c^6}\,{}_6g_{ij}^{\text{ext}} + O\left(\frac{1}{c^8}\right) , \tag{3.51c}$$

where in these equations we denote by $c^{-k}\,{}_kg_{\alpha\beta}^{\text{ext}}$ some (yet uncalculated) metric coefficients which are "even," where the set of multipole moments $\widetilde{\mathcal{M}} = \{\widetilde{M}_L, \widetilde{S}_L\}$ is defined from the original set $\mathcal{M} = \{M_L, S_L\}$ by the relations

$$\widetilde{M}_L(u) = M_L(u) + \left\{\begin{array}{ll} \frac{G}{c^7}m(u) & \text{for}\, l = 0 \\ \frac{G}{c^5}m_i(u) & \text{for}\, l = 1 \\ 0 & \text{for}\, l \geq 2 \end{array}\right\} + \frac{G}{c^7}T_L(u) + O\left(\frac{1}{c^8}\right), \tag{3.52a}$$

$$\widetilde{S}_L(u) = S_L(u) + \left\{\begin{array}{ll} \frac{G}{c^5}s_i(u) & \text{for}\, l = 1 \\ 0 & \text{for}\, l \geq 2 \end{array}\right\} + O\left(\frac{1}{c^6}\right) , \tag{3.52b}$$

and where the radiation reaction scalar and vector potentials V_{react} and V^i_{react} have at this approximation the same expressions as the ones of Eqs. (2.22a) and (2.22b), namely,

$$V_{\text{react}}[\widetilde{\mathcal{M}}] = -\frac{G}{5c^5}x^a x^b \widetilde{M}^{(5)}_{ab} + \frac{G}{c^7}\left[\frac{1}{189}x^{abc}\widetilde{M}^{(7)}_{abc} - \frac{1}{70}r^2 x^{ab}\widetilde{M}^{(7)}_{ab}\right] + O\left(\frac{1}{c^8}\right), \tag{3.53a}$$

$$V^i_{\text{react}}[\widetilde{\mathcal{M}}] = \frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{iab}\widetilde{M}^{(6)}_{ab} - \frac{4}{45}\varepsilon_{iab}x^{ac}\widetilde{S}^{(5)}_{bc}\right] + O\left(\frac{1}{c^6}\right) . \tag{3.53b}$$

The external metric (3.51), which appears as a functional of the moments (3.52), will be matched, in a forthcoming paper, to the inner metric of the source, and the effects associated with the scalar and vector reaction potentials (3.53) will be investigated.

D. The 4 post-Newtonian approximation

In paper II we proved that at the 4 PN approximation level, i.e., at the level $O(1/c^{10-s})$ in the components of the metric, terms appear in the near-zone metric which are "hereditary" in the sense that they depend on the full past history of the source. These terms were shown by matching to yield a hereditary modification of the local radiation reaction force acting within the source. In the present subsection we wish to recover, and to complete, the result of paper II by using the expressions (3.3) and (3.4) of the functions parametrizing the antisymmetric waves in the nonlinear iterations of the metric. This will permit to compute a numerical coefficient 11/12 in the radiation reaction force which has been left undetermined in paper II.

By Eqs. (3.20) and (3.21) we see that nonlinear antisymmetric waves can arise at the 4 PN level $O(1/c^{10-s})$ only in the quadratic metric ($n = 2$) and furthermore only if the two moments $\underline{l}_1$ and $\underline{l}_2$ composing the wave satisfy

$$\underline{l}_1 + \underline{l}_2 + l = 4 - s, \tag{3.54}$$

where l is the multipolarity of the wave (and s the number of spatial indices in the component of the metric). Since one of the moments must be time varying we have $\underline{l}_1 \geq 2$ (say) and thus $l \leq 2 - s$.

The waves having multipolarities $l \leq 1 - s$ (i.e., multipolarities $l = 0$ and $l = 1$ in the 00 component of the metric, and multipolarity $l = 0$ in the $0i$ components of the metric) yield, in the equations of motion of the source, a modification of the radiation reaction force which is of the type $\rho\varphi_i(t)$, where ρ is the density of the source and $\varphi_i(t)$ a function of time. In a mass centered frame, this modification does not change the total amount of energy radiated by the source. We do not compute these waves having $l \leq 1-s$ but will give their structure in Eqs. (3.68) and (3.69).

The only remaining possibility is then $l = 2 - s$, which corresponds to the interaction of the mass quadrupole moment M_{ij} (which has $\underline{l}_1 = 2$) and of the mass monopole moment M, or Arnowitt-Deser-Misner mass (which has $l_2 = 0$). We now explicitly compute the contribution of the monopole-quadrupole interaction (symbolized by "$M \times M_{ij}$") in the quadratic antisymmetric waves.

First of all, it is shown in Appendix D that the quadratic metrics for this interaction in the modified canonical and original canonical coordinates differ by

$$h^{\alpha\beta}_{(2)K,\theta}\Big|_{M\times M_{ij}} = \left(h^{\alpha\beta}_{\rm can(2)} + \partial^\alpha \Pi^\beta_{K,\theta} + \partial^\beta \Pi^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu \Pi^\mu_{K,\theta} + \Omega^{\alpha\beta}_{K,\theta}\right)\Big|_{M\times M_{ij}}, \tag{3.55}$$

where the vector $\Pi^\alpha_{K,\theta}$ is given by Eqs. (D6) and (D2), and where the tensor $\Omega^{\alpha\beta}_{K,\theta}$ is given by Eq. (D3). Now, in the case of the interaction $M \times M_{ij}$, we easily see [from Eqs. (D10)] that the tensor $\Omega^{\alpha\beta}_{K,\theta}$ is "odd" in the usual PN sense, and thus that it cannot contribute to the antisymmetric waves at quadratic order since these waves are "even" (recall that antisymmetric waves are "even" in even nonlinear iterations of the metric). Therefore the antisymmetric waves of type $M \times M_{ij}$ in both constructions of the metric differ from each other by an infinitesimal gauge transformation, and we can compute them in any coordinate system, for instance in the canonical coordinate system.

The quadratic source corresponding to the interaction $M \times M_{ij}$ in the canonical metric has been computed in Appendix B of Ref. [40]. It reads as

$$N^{00}_{\rm can(2)|_{M\times M_{ij}}} = \frac{\hat n_{ab}}{c^4 r^6} M \left\{-126 M_{ab} - 126\left(\frac{r}{c}\right) M^{(1)}_{ab} - 112\left(\frac{r}{c}\right)^2 M^{(2)}_{ab} - 46\left(\frac{r}{c}\right)^3 M^{(3)}_{ab} - 8\left(\frac{r}{c}\right)^4 M^{(4)}_{ab}\right\}\left(t-\frac{r}{c}\right), \tag{3.56a}$$

$$\begin{aligned} N^{0i}_{\rm can(2)|_{M\times M_{ij}}} &= \frac{\hat n_{iab}}{c^5 r^5} M \left\{6 M^{(1)}_{ab} + 6\left(\frac{r}{c}\right) M^{(2)}_{ab} + 2\left(\frac{r}{c}\right)^2 M^{(3)}_{ab}\right\}\left(t-\frac{r}{c}\right) \\ &+ \frac{n_a}{c^5 r^5} M \left\{-\frac{108}{5} M^{(1)}_{ai} - \frac{108}{5}\left(\frac{r}{c}\right) M^{(2)}_{ai} - \frac{116}{5}\left(\frac{r}{c}\right)^2 M^{(3)}_{ai} - 8\left(\frac{r}{c}\right)^3 M^{(4)}_{ai}\right\}\left(t-\frac{r}{c}\right), \end{aligned} \tag{3.56b}$$

$$\begin{aligned} N^{ij}_{\rm can(2)|_{M\times M_{ij}}} &= \frac{\hat n_{ijab}}{c^4 r^6} M \left\{60 M_{ab} + 60\left(\frac{r}{c}\right) M^{(1)}_{ab} + 24\left(\frac{r}{c}\right)^2 M^{(2)}_{ab} + 4\left(\frac{r}{c}\right)^3 M^{(3)}_{ab}\right\}\left(t-\frac{r}{c}\right) \\ &+ \frac{\hat n_{a(i}}{c^4 r^6} M \left\{\frac{72}{7} M_{j)a} + \frac{72}{7}\left(\frac{r}{c}\right) M^{(1)}_{j)a} - \frac{72}{7}\left(\frac{r}{c}\right)^2 M^{(2)}_{j)a} - \frac{96}{7}\left(\frac{r}{c}\right)^3 M^{(3)}_{j)a}\right\}\left(t-\frac{r}{c}\right) \\ &+ \frac{\delta_{ij} n_{ab}}{c^4 r^6} M \left\{-\frac{66}{7} M_{ab} - \frac{66}{7}\left(\frac{r}{c}\right) M^{(1)}_{ab} + \frac{24}{7}\left(\frac{r}{c}\right)^2 M^{(2)}_{ab} + \frac{46}{7}\left(\frac{r}{c}\right)^3 M^{(3)}_{ab}\right\}\left(t-\frac{r}{c}\right) \\ &+ \frac{M}{c^6 r^4} \left\{-\frac{24}{5} M^{(2)}_{ij} - \frac{24}{5}\left(\frac{r}{c}\right) M^{(3)}_{ij} - 8\left(\frac{r}{c}\right)^2 M^{(4)}_{ij}\right\}\left(t-\frac{r}{c}\right). \end{aligned} \tag{3.56c}$$

We need to substitute this source into the quadratically nonlinear antisymmetric waves

$$\mathcal{A}^{\alpha\beta}_{\rm can(2)} = \sum_{l\geq 0} \frac{(-)^l}{l!} \hat\partial_L \left\{\frac{R^{\alpha\beta}_L(t-r/c) - R^{\alpha\beta}_L(t+r/c)}{2r}\right\}, \tag{3.57}$$

where

$$R^{\alpha\beta}_L(t) = {\rm FP}_{B=0}\left\{-\frac{1}{4\pi}\int d^3\mathbf{y}\, \hat y_L |\mathbf{y}|^B \int_1^{+\infty} dz\, \gamma_l(z) \bar N^{\alpha\beta}_{\rm can(2)}(\mathbf{y}, t - z|\mathbf{y}|/c)\right\} \tag{3.58}$$

[see Eqs. (3.16) and (3.17)]. Let us recall that the quadratic metric corresponding to the interaction $M \times M_{ij}$ is equal to its part $p^{\alpha\beta}_{\rm can(2)}$ (see Appendix B of [40]) and therefore we consider only the antisymmetric waves in $p^{\alpha\beta}_{\rm can(2)}$. The problem of computing these waves is reduced to the problem of computing the functions $R^{\alpha\beta}_L(t)$ for each elementary source composing $N_{\rm can(2)|_{M\times M_{ij}}}$ [Eqs. (3.56)]. Let us denote such an elementary source with radial dependence r^{-k} (with $k \geq 2$) by

$$\frac{1}{r^k} F\left(t-\frac{r}{c}, \mathbf{n}\right) = \sum_{l\geq 0} \frac{\hat n_L}{r^k} F_L\left(t-\frac{r}{c}\right). \tag{3.59}$$

Then the corresponding functions

$$R_{kL}(t) = {\rm FP}_{B=0}\left\{-\frac{1}{4\pi}\int d^3\mathbf{y}\, \hat y_L |\mathbf{y}|^{B-k} \int_1^{+\infty} dz\, \gamma_l(z) F(t-(z+1)|\mathbf{y}|/c, \mathbf{y}/|\mathbf{y}|)\right\} \tag{3.60}$$

take three different forms depending on the relative values of the integers k and l.

(i) When $k = 2$ and l is arbitrary, we obtain (using the technical tools of our previous papers) the hereditary expression

$$R_{2L}(t) = l! \int_0^{+\infty} d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + 2\sum_{i=1}^{l} \frac{1}{i} \right] F_L^{(-l)}(t-\lambda), \tag{3.61a}$$

where $F_L(t)$ is the lth multipolar projection of the function $F(t, \mathbf{n})$ [see Eq. (3.59)], and where $F_L^{(-l)}(t)$ denotes the lth antiderivative of F_L which is zero when $t \leq -T$. Equation (3.61a) is the result of paper II [see Eq. (5.20) of paper II]. (Recall that $P = 1$ in this paper.)

(ii) When $3 \leq k \leq l+2$ we get

$$R_{kL}(t) = -l! \frac{2^{k-2}(l+2-k)!(k-3)!}{(l+k-2)!} F_L^{(-l-3+k)}(t), \tag{3.61b}$$

which involves simple antiderivatives of the function $F_L(t)$.

(iii) Finally when $k \geq l+3$ we again have a fully hereditary expression

$$R_{kL}(t) = l! \frac{(-)^{l+k}2^{k-2}(k-3)!}{(l+k-2)!(k-l-3)!} \int_0^{+\infty} d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \sum_{i=0}^{l} \frac{1}{k-2+i} + \sum_{i=1}^{k-l-3} \frac{1}{i} \right] F_L^{(k-l-2)}(t) \ . \tag{3.61c}$$

Thanks to the above expressions we now straightforwardly obtain the antisymmetric waves $\mathcal{A}^{\alpha\beta}_{\text{can}(2)}$ [Eq. (3.57)] corresponding to the interaction $M \times M_{ij}$. The result is

$$\mathcal{A}^{00}_{\text{can}(2)|M\times M_{ij}} = -\frac{8M}{c^5}\int_0^\infty d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{1}{2}\right] \partial_{ab}\left\{M^{(2)}_{ab}\right\}_{|t-\lambda} , \tag{3.62a}$$

$$\mathcal{A}^{0i}_{\text{can}(2)|M\times M_{ij}} = \frac{8M}{c^6}\int_0^\infty d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{5}{12}\right] \partial_{a}\left\{M^{(3)}_{ai}\right\}_{|t-\lambda} + \frac{2M}{3c^4}\partial_{iab}\{M_{ab}\} , \tag{3.62b}$$

$$\mathcal{A}^{ij}_{\text{can}(2)|M\times M_{ij}} = -\frac{8M}{c^7}\int_0^\infty d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{11}{12}\right] \left\{M^{(4)}_{ij}\right\}_{|t-\lambda} - \frac{M}{c^3}\partial_{ijab}\left\{M^{(-1)}_{ab}\right\} + \frac{8M}{c^5}\partial_{a(i}\left\{M^{(1)}_{j)a}\right\} - \frac{11M}{3c^5}\delta_{ij}\partial_{ab}\left\{M^{(1)}_{ab}\right\} , \tag{3.62c}$$

where we have used the notation (2.24) to denote monopolar antisymmetric waves:

$$\{F\} = \frac{F(t-r/c) - F(t+r/c)}{2r} , \tag{3.62d}$$

$$\{F\}_{|t-\lambda} = \frac{F(t-\lambda-r/c) - F(t-\lambda+r/c)}{2r} . \tag{3.62e}$$

Note that the tensor (3.62) is divergenceless:

$$\partial_\beta \mathcal{A}^{\alpha\beta}_{\text{can}(2)|M\times M_{ij}} = 0 . \tag{3.63}$$

It is then easy to extract its physical information from Eqs. (3.62). Using, for instance, Eqs. (2.26) of paper I we can rewrite these equations as

$$\mathcal{A}^{00}_{\text{can}(2)|M\times M_{ij}} = -\frac{8M}{c^5}\int_0^\infty d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{11}{12}\right] \times\partial_{ab}\left\{M^{(2)}_{ab}\right\}_{|t-\lambda} + \partial w^{00}, \tag{3.64a}$$

$$\mathcal{A}^{0i}_{\text{can}(2)|M\times M_{ij}} = \frac{8M}{c^6}\int_0^\infty d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{11}{12}\right] \times\partial_{a}\left\{M^{(3)}_{ai}\right\}_{|t-\lambda} + \partial w^{0i}, \tag{3.64b}$$

$$\mathcal{A}^{ij}_{\text{can}(2)|M\times M_{ij}} = -\frac{8M}{c^7}\int_0^\infty d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{11}{12}\right] \times\left\{M^{(4)}_{ij}\right\}_{|t-\lambda} + \partial w^{ij}, \tag{3.64c}$$

using the notation $\partial w^{\alpha\beta} = \partial^\alpha w^\beta + \partial^\beta w^\alpha - \eta^{\alpha\beta}\partial_\mu w^\mu$ for the gauge term associated with the vector

$$w^0 = \frac{M}{6c^4}\,\partial_{ab}\{M_{ab}\}, \tag{3.65a}$$

$$w^i = -\frac{M}{2c^3}\,\partial_{iab}\left\{M^{(-1)}_{ab}\right\} + \frac{4M}{c^5}\,\partial_a\left\{M^{(1)}_{ai}\right\} . \tag{3.65b}$$

From the form (3.64) of the antisymmetric waves, and from Sec. II of this paper, we can now conclude that the effect of these antisymmetric waves is to modify the radiation reaction tensor potential (2.19) by a linear correction

$$\delta V^{\alpha\beta}_{\text{react}} = V^{\alpha\beta}_{\text{react}}[\delta M_{ij}], \tag{3.66}$$

where δM_{ij} is a small hereditary correction to the mass quadrupole moment given by

$$\delta M_{ij}(t) = \frac{4GM}{c^3}\int_0^{+\infty} d\lambda \left[\ln\left(\frac{\lambda}{2}\right) + \frac{11}{12}\right] M^{(2)}_{ij}(t-\lambda) \ . \tag{3.67}$$

However, at the level of approximation we are using here (namely, the 4 PN level), only the hereditary modification of the 00 component of the radiation reaction potential makes sense, and thus we only write down the expressions of the scalar and vector components of the reaction potential complete up to 4 PN:

$$V_{\rm react} = -\frac{G}{5c^5}x^a x^b \widetilde{M}^{(5)}_{ab} + \frac{G}{c^7}\left\{\frac{1}{189}x^a x^b x^c \widetilde{M}^{(7)}_{abc} - \frac{1}{70}r^2 x^a x^b \widetilde{M}^{(7)}_{ab}\right\}$$
$$+\frac{G}{c^8}\left\{-\frac{4GM}{5}x^a x^b \int_0^{+\infty} d\lambda\left[\ln\left(\frac{\lambda}{2}\right)+\frac{11}{12}\right]\widetilde{M}^{(7)}_{ab}(t-\lambda) + x^a f_a(t) + g(t)\right\} + O\left(\frac{1}{c^9}\right) \tag{3.68a}$$

and

$$V^i_{\rm react} = \frac{G}{c^5}\left\{\frac{1}{21}\hat{x}^{iab}\widetilde{M}^{(6)}_{ab} - \frac{4}{45}\varepsilon_{iab}x^a x^c \widetilde{S}^{(5)}_{bc}\right\}$$
$$+\frac{G}{c^6}\{h_i(t)\} + O\left(\frac{1}{c^7}\right) \ . \tag{3.68b}$$

(Recall that the ij component of the reaction tensor belongs to a higher approximation.) The functions $f_i(t)$, $g(t)$ and $h_i(t)$, parametrizing monopolar and dipolar waves, are some instantaneous functionals of the moments. They have the form

$$f_i(t) = \alpha M_a M^{(6)}_{ai}(t), \tag{3.69a}$$

$$g(t) = \sum_{k=0}^{6} \beta_k M^{(k)}_{ab}(t) M^{(6-k)}_{ab}(t), \tag{3.69b}$$

$$h_i(t) = \gamma M_a M^{(5)}_{ai}(t), \tag{3.69c}$$

where α, β_k, γ are some uncalculated constants. Note that, consistently with the 4 PN approximation, the mass moments $\widetilde{M}_L$ [given by Eq. (3.52a)] involve a hereditary contribution at the level $O(c^{-8})$ coming from the *cubic* iteration of the metric [see Eqs. (5.27) and (6.35) of paper II].

The coefficient 11/12 completing the hereditary modification for the scalar reaction potential in Eq. (3.68a) is in agreement with the related coefficient in the hereditary modification of the radiative quadrupole moment computed at infinity from the source [see Eq. (3.10) of Ref. [40]].

Finally let us end up this paper by stressing that the expressions above for the scalar and vector radiation reaction potentials are for the moment disconnected from the actual dynamics of the source, and thus will have to be fully justified by an explicit matching to the inner field of the source, as was done in paper II for the hereditary term in Eq. (3.68a). This will be the subject of a future work.

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APPENDIX A: RELATION BETWEEN THE OPERATORS OF THE RETARDED POTENTIALS AND OF THE INSTANTANEOUS POTENTIALS (PROOF OF THE THEOREM IN SEC. III A)

We first decompose the (formal) near-zone expansion $\bar{f}(\mathbf{x},t)$ given by Eq. (3.1) into a sum of pieces with given multipolarities l:

$$\bar{f}(\mathbf{x},t) = \sum_{l\geq 0} \hat{n}_L(\theta,\varphi)\bar{f}_L(r,t), \tag{A1}$$

with

$$\bar{f}_L(r,t) = \sum F_L(t) r^a (\ln r)^p \ , \tag{A2}$$

where the functions $F_L(t)$ are STF in L and constant or zero in the past (indices suppressed for simplicity). Then, for the computation of the retarded integral of $r^B\bar{f}(\mathbf{x},t)$ [in fact of each separate term composing $r^B\bar{f}(\mathbf{x},t)$], we use the explicit expression which has been obtained in theorem 6.1 of paper I. With the structure (A1) and (A2) of $\bar{f}(\mathbf{x},t)$, we check that all the hypotheses of this theorem are satisfied if the real part of B is chosen large enough. Thus we can write

$$[\Box^{-1}_R(r^B\bar{f})](\mathbf{x},t) = \sum_{l\geq 0}\int_r^{+\infty} ds\, \hat{\partial}_L\left\{\frac{\mathcal{R}^B_L\left(\frac{s-r}{2}, t-s/c\right) - \mathcal{R}^B_L\left(\frac{s+r}{2}, t-s/c\right)}{2r}\right\}, \tag{A3}$$

where the function $\mathcal{R}_L^B(\rho,u)$ is related to the source (A2) by

$$\mathcal{R}_L^B(\rho,u) = (2\rho)^l \int_0^\rho dy\,\frac{(\rho-y)^l}{l!}y^{B-l+1}\bar{f}_L(y,u+y/c) \tag{A4}$$

[see Eqs. (6.3) and (6.4) in paper I]. This function satisfies

$$\left(\frac{\partial}{\partial\rho}\right)^{l+1}\left[\frac{1}{(2\rho)^l}\mathcal{R}_L^B(\rho,u)\right] = \rho^{B-l+1}\bar{f}_L(\rho,u+\rho/c)\ . \tag{A5}$$

Note that in Eq. (A4) we have explicitly chosen the lower bound of the integral to be zero. [Recall that Eq. (A3) is valid if the lower bound of the integral in Eq. (A4) is replaced by an arbitrary function of t and s; see paper I.] In the case where the bound is zero it is easy to see [using the structure (A2) of the source] that when $\rho\to 0$ the function $\mathcal{R}_L^B(\rho,u)$ admits an expansion of the type $\Sigma\rho^{B+k}(\ln\rho)^q F_L^{(n)}(u)$, which is proportional to the analytic continuation factor ρ^B, where k, q, n are integers.

Now we can proceed in a way similar to that of Eqs. (5.16)–(5.18) in paper II, and split the RHS of Eq. (A3) into three integrals. The first integral is the first term in Eq. (A3) which involves the argument $(s-r)/2$ and is purely retarded. The second integral is purely *advanced* and is equal to the second term in Eq. (A3), involving the argument $(s+r)/2$, but in which the lower bound $+r$ of the integral is replaced by $-r$. The third integral is what remains, namely, the second term in Eq. (A3) but with bounds $-r$ and $+r$. Hence we write

$$\begin{aligned}\Box_R^{-1}(r^B\bar{f}) &= \sum_{l\geq 0}\int_r^{+\infty} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s-r}{2},t-s/c\right)\right\}\\ &\quad-\sum_{l\geq 0}\int_{-r}^{+\infty} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2},t-s/c\right)\right\}\\ &\quad+\sum_{l\geq 0}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2},t-s/c\right)\right\}\ .\end{aligned} \tag{A6}$$

It is known from paper I that a permissible operation is to take the derivative operator $\hat{\partial}_L$ outside the first integral because the terms coming from the differentiation of the bound r of the integral are cancelled out by the factor $(2\rho)^l \equiv (s-r)^l$ in the expression of $\mathcal{R}_L^B$ [see Eq. (A4)] [this is true whatever be the value of the lower bound of the integral in Eq. (A4)]. The same reasoning applies also to the second term in Eq. (A6). Thus we obtain

$$\begin{aligned}\Box_R^{-1}(r^B\bar{f}) &= \sum_{l\geq 0}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{R_L^B\left(t-\frac{r}{c}\right)-R_L^B\left(t+\frac{r}{c}\right)}{2r}\right\}\\ &\quad+\sum_{l\geq 0}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2},t-s/c\right)\right\},\end{aligned} \tag{A7}$$

where the function $R_L^B(u)$ is given by

$$R_L^B(u) = 2(-)^l l!\int_0^{+\infty} d\rho\,\mathcal{R}_L^B(\rho,u-2\rho/c)\ . \tag{A8}$$

In view of Eq. (A7), the theorem in Sec. III A of the text will be proved if we show that the above expression of $R_L^B(u)$ yields Eqs. (3.3)–(3.5) in the text, and if the near-zone expansion of the last term in Eq. (A7) agrees with the operator I^{-1} defined in Eqs. (3.7) acting on the source.

Inserting the expression (A4) of the function $\mathcal{R}_L^B$ into Eq. (A8), and replacing the variable ρ by the variable $z=2(\rho/y)-1$, we readily obtain

$$R_L^B(u) = \frac{(-)^l}{2^l}\int_0^{+\infty} dy\,y^{B+l+2}\times\int_1^{+\infty} dz(z^2-1)^l\bar{f}_L(y,u-zy/c)\ . \tag{A9}$$

[Since the constant parts of the source cancel in the first term of Eq. (A7), we can assume that $\bar{f}_L(y,t)$ in Eq. (A9) is zero in the past so that the integral is convergent.] Then we use the inverse of Eq. (A1), namely,

$$\bar{f}_L(r,t) = \frac{(2l+1)!!}{4\pi l!}\int d\Omega\,\hat{n}_L\bar{f}(\mathbf{x},t) \tag{A10}$$

[Eq. (A9b) of paper I] and get

$$R_L^B(u) = \frac{-1}{4\pi}\frac{(-)^{l+1}(2l+1)!!}{2^l l!}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\times\int_1^{+\infty} dz(z^2-1)^l\bar{f}(\mathbf{y},u-z|\mathbf{y}|/c) \tag{A11}$$

(where $d^3\mathbf{y}=dy\,y^2d\Omega$; $y=|\mathbf{y}|$). Posing

$$\gamma_l(z) = (-)^{l+1}\frac{(2l+1)!!}{2^l l!}(z^2-1)^l, \tag{A12}$$

we obtain

$$R_L^B(u) = \frac{-1}{4\pi}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_1^{+\infty} dz\,\gamma_l(z)\bar{f}(\mathbf{y},u-z|\mathbf{y}|/c), \tag{A13}$$

which, after taking the finite part, gives the expressions (3.3) and (3.4) in the text.

We now prove that the near-zone expansion of the last term in Eq. (A7) is equal to the operator I^{-1} of the source $r^B\bar{f}(\mathbf{x},t)$. We first write the Taylor expansion of this term when $c\to+\infty$, namely,

$$\sum_{l\geq 0}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2},t-s/c\right)\right\} = \sum_{i=0}^{+\infty}(-)^i\left(\frac{\partial}{c\partial t}\right)^i\{K_i^B(\mathbf{x},t)\}\ , \tag{A14}$$

where we have set

$$K_i^B(\mathbf{x},t) = \int_{-r}^{r} ds\, \frac{s^i}{i!}\, \varphi^B(\mathbf{x},s,t)\ , \tag{A15a}$$

$$\varphi^B(\mathbf{x},s,t) = \sum_{l\geq 0} \hat{\partial}_L \left\{ \frac{1}{2r} \mathcal{R}_L^B \left(\frac{s+r}{2}, t \right) \right\}\ . \tag{A15b}$$

(Note that K_i^B and φ^B depend also on c through the functions $\bar{f}_L$ and $\mathcal{R}_L^B$.) Then we compute the usual Laplacian ($\Delta = \Sigma_a \partial_a \partial_a$) of K_i^B. The differentiation ∂_a will act both on φ^B in the integrand and on the bounds of the integral. However, it is easily seen that only the upper bound of the integral makes a contribution. Indeed the contribution coming from the differentiation of the lower bound $-r$ of the integral vanishes by analytic continuation since it vanishes when Re(B) is large enough, because of the behavior $\mathcal{R}_L^B(\rho,u) \sim \rho^{B+k}(\ln\rho)^q$ when $\rho \to 0$. [We see at this point that our choice zero for the lower bound of the integral in Eq. (A4) is important.] After an easy computation we find

$$\Delta K_i^B = \int_{-r}^{r} ds \frac{s^i}{i!} \Delta\varphi^B(\mathbf{x},s,t) + \frac{r^i}{i!}\frac{\partial}{\partial r}[\varphi^B(\mathbf{x},r,t)] + \frac{r^i}{i!}\frac{\partial\varphi^B}{\partial r}(\mathbf{x},r,t) + (i+2)\frac{r^{i-1}}{i!}\varphi^B(\mathbf{x},r,t), \tag{A16}$$

where in the second term the differentiation with respect to r is done after the replacement of s by r in $\varphi^B(\mathbf{x},s,t)$, while in the third term the differentiation is done first. Now, by the structure of φ^B, Eq. (A15b), we have $\Delta\varphi^B = \partial^2\varphi^B/\partial s^2$ and we can integrate by parts the first term in Eq. (A16), taking care of the all-integrated terms coming from the upper bound $+r$. As a result we get

$$\Delta K_i^B = K_{i-2}^B + 2\frac{r^{i-1}}{i!}\frac{\partial}{\partial r}[r\varphi^B(\mathbf{x},r,t)]\ . \tag{A17}$$

Using Eq. (A.35a) of paper I we can compute the value of $\varphi^B(\mathbf{x},r,t)$ and obtain

$$\varphi^B(\mathbf{x},r,t) = \sum_{l\geq 0} \frac{\hat{n}_L}{2^{l+1}} r^l \left(\frac{\partial}{\partial r}\right)^l \left[\frac{1}{r^{l+1}}\mathcal{R}_L^B(r,t)\right]\ . \tag{A18}$$

Some further manipulations now show that

$$\frac{\partial}{\partial r}[r\varphi^B(\mathbf{x},r,t)] = \sum_{l\geq 0} \frac{\hat{n}_L}{2^{l+1}} r^l \left(\frac{\partial}{\partial r}\right)^{l+1} \left[\frac{1}{r^l}\mathcal{R}_L^B(r,t)\right]\ . \tag{A19}$$

We recognize in the RHS of Eq. (A19) an expression which can be simplified by means of Eq. (A5). Hence we can write the following simple formula for the Laplacian of K_i^B in terms of K_{i-2}^B and the source:

$$\Delta K_i^B = K_{i-2}^B + \frac{r^{B+i}}{i!}\bar{f}\left(\mathbf{x}, t+\frac{r}{c}\right)\ . \tag{A20}$$

This formula is also valid in the cases $i = 0$ and 1 if we define $K_{-2}^B = K_{-1}^B = 0$. From Eq. (A20) we easily deduce the equation satisfied by K_i^B separately:

$$\Delta^{j+1} K_i^B = \sum_{p=0}^{j} \Delta^{j-p}\left[\frac{r^{B+i-2p}}{(i-2p)!}\bar{f}\left(\mathbf{x},t+\frac{r}{c}\right)\right]\ , \tag{A21}$$

where $j = [i/2]$ is the integer part of $i/2$.

Now the point is that by the structure of the expansion when $\rho \to 0$ of the function $\mathcal{R}_L^B(\rho,u)$ [which is of the type $\Sigma\rho^{B+k}(\ln\rho)^q F_L^{(n)}(u)$] we can check that the integral K_i^B given by Eq. (A15a) admits an expansion similar to that of the source, i.e., of the type of Eqs. (A1) and (A2). Thus we see that the *unique* solution of Eq. (A21) is the solution obtained by means of the inverse Laplace operator defined, when acting on terms of the type $\hat{n}_L r^{B+a}(\ln r)^q$, by

$$\Delta^{-1}[\hat{n}_L r^{B+a}(\ln r)^q] = \left(\frac{\partial}{\partial B}\right)^q \left[\frac{\hat{n}_L r^{B+a+2}}{(B+a+2-l)(B+a+3+l)}\right]\ . \tag{A22}$$

Namely, the solution of Eq. (A21) is

$$K_i^B = \sum_{p=0}^{[i/2]} \Delta^{-p-1}\left[\frac{r^{B+i-2p}}{(i-2p)!}\bar{f}\left(\mathbf{x},t+\frac{r}{c}\right)\right]\ , \tag{A23}$$

where it is understood that in the right-hand side we must replace $\bar{f}(\mathbf{x}, t+r/c)$ by its expansion (A1),(A2), expand the argument $t + r/c$ around $r = 0$, and then use repeatedly the operator Eq. (A22) on each term of the series. The result is then the expansion of K_i^B. Finally we replace the solution found for each K_i^B back into Eq. (A14) to obtain

$$\sum_{l\geq 0}\int_{-r}^{r} ds\, \hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2}, t-s/c\right)\right\} = \sum_{p=0}^{\infty}\left(\frac{\partial}{c\partial t}\right)^{2p}\Delta^{-p-1}\left[r^B\left(\sum_{j=0}^{\infty}\frac{(-r)^j}{j!}\left(\frac{\partial}{c\partial t}\right)^j \bar{f}\left(\mathbf{x},t+\frac{r}{c}\right)\right)\right]\ . \tag{A24}$$

After reconstruction of the Taylor series in the RHS we recognize

$$\sum_{l\geq 0}\int_{-r}^{r} ds\, \hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2}, t-s/c\right)\right\} = \sum_{p=0}^{\infty}\left(\frac{\partial}{c\partial t}\right)^{2p}\Delta^{-p-1}[r^B\bar{f}(\mathbf{x},t)] \equiv I^{-1}[r^B\bar{f}(\mathbf{x},t)], \tag{A25}$$

which is the equation we wanted to prove.

Note that the above proof has rested on the argument that the integrals K_i^B, and thus the LHS of Eq. (A25), admit an expansion when $r \to 0$ which is proportional to the analytic continuation factor r^B. It has been suggested to me by T. Damour that this fact alone is sufficient to ensure that Eq. (A25) holds true, the operator Δ^{-1} being defined by Eq. (A22). Nevertheless, we have found it interesting (and maybe useful for future work) to present a direct proof of this equation.

APPENDIX B: RELATION BETWEEN THE OPERATORS OF THE INSTANTANEOUS POTENTIALS AND OF THE SYMMETRIC POTENTIALS

We assume that the source $\bar{f}(\mathbf{x},t)$ admits the same near-zone expansion as in Eqs. (A1) and (A2) of Appendix A but with the functions $F_L(t)$ constant or zero not only in the past ($t \le -T$) but also in the future ($t \ge +T$). In this way retarded and advanced integrals are convergent at infinity. Then the formula analogous to Eq. (A3) but for the *advanced* potentials reads as

$$[\Box_A^{-1}(r^B\bar{f})](\mathbf{x},t) = \sum_{l\ge 0}\int_r^{+\infty} ds\, \hat{\partial}_L \left\{\frac{\mathcal{A}_L^B\left(\frac{s-r}{2}, t+s/c\right) - \mathcal{A}_L^B\left(\frac{s+r}{2}, t+s/c\right)}{2r}\right\}, \tag{B1}$$

where the function $\mathcal{A}_L^B(\rho,u)$ is given by

$$\mathcal{A}_L^B(\rho,u) = (2\rho)^l \int_0^\rho dy\, \frac{(\rho-y)^l}{l!} y^{B-l+1}\bar{f}_L(y,u-y/c)\ . \tag{B2}$$

Note the changes of sign between Eqs. (A3) and (A4) and Eqs. (B1) and (B2).

Following the same reasoning as in the retarded case, we can transform Eq. (B1) into a form analogous to Eq. (A7), namely,

$$\Box_A^{-1}(r^B\bar{f}) = \sum_{l\ge 0}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{A_L^B\left(t+\frac{r}{c}\right) - A_L^B\left(t-\frac{r}{c}\right)}{2r}\right\} + \sum_{l\ge 0}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{A}_L^B\left(\frac{s+r}{2}, t+s/c\right)\right\}, \tag{B3}$$

in which the function $A_L^B(u)$ is given by

$$A_L^B(u) = 2(-)^l l! \int_0^{+\infty} d\rho\, \mathcal{A}_L^B(\rho, u+2\rho/c)\ . \tag{B4}$$

This function can be expressed in the same form as in Eq. (A13), but with the retarded argument $u-z|\mathbf{y}|/c$ replaced by the corresponding advanced argument. Hence

$$A_L^B(u) = \frac{-1}{4\pi}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_1^{+\infty} dz\,\gamma_l(z)\bar{f}(\mathbf{y},u+z|\mathbf{y}|/c)\ . \tag{B5}$$

Now it is easy to show that the second term in Eq. (B3) is equal to the corresponding second term in Eq. (A7) of the retarded case, the near-zone expansion of both terms being equal by Eq. (A25) to I^{-1} of the source:

$$\sum_{l\ge 0}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2}, t-s/c\right)\right\} = \sum_{l\ge 0}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{A}_L^B\left(\frac{s+r}{2}, t+s/c\right)\right\} = I^{-1}[r^B\bar{f}(\mathbf{x},t)]\ . \tag{B6}$$

The equalities Eq. (B6) can be proved in two ways. First, it can be argued that both the retarded and advanced terms in Eq. (B6) are solutions of the same d'Alembertian equation (with source $r^B\bar{f}$) and admit expansions when $r \to 0$ which are proportional to r^B. Therefore the expansions of both terms must be equal to the expansion of I^{-1} of the source. However, a direct proof of the first equality in Eq. (B6) reads as

$$\begin{aligned}\int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{R}_L^B\left(\frac{s+r}{2}, t-s/c\right)\right\} &= \frac{1}{2^{l+1}l!}\int_{-r}^{r} ds\int_0^{\frac{s+r}{2}} dy\, y^{B-l+1}\bar{f}_L\left(y, t-\frac{s}{c}+\frac{y}{c}\right)\hat{\partial}_L\left\{\frac{(s+r)^l(s+r-2y)^l}{r}\right\}\\ &= \frac{1}{2^{l+1}l!}\int_0^{r} dy\, y^{B-l+1}\int_{-(r-y)}^{r-y} d\lambda\,\bar{f}_L(y,t-\lambda/c)\hat{\partial}_L\left\{\frac{(\lambda+r+y)^l(\lambda+r-y)^l}{r}\right\}\\ &= \int_{-r}^{r} ds\,\hat{\partial}_L\left\{\frac{1}{2r}\mathcal{A}_L^B\left(\frac{s+r}{2}, t+s/c\right)\right\}\ .\end{aligned} \tag{B7}$$

(The last equality results from the fact that one can change λ into $-\lambda$ in the integrand.)

The integral of the symmetric potential, defined to be the half-sum of the retarded and advanced potentials,

$$\Box_S^{-1}(r^B\bar{f}) = \tfrac{1}{2}\left\{\Box_R^{-1}(r^B\bar{f}) + \Box_A^{-1}(r^B\bar{f})\right\}, \tag{B8}$$

can now easily be related to the operator I^{-1}. By summing Eqs. (A7) and (B3) and by using Eq. (B6) we obtain

$$\Box_S^{-1}(r^B\bar{f}) = \sum_{l\geq 0}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{T_L^B\left(t-\frac{r}{c}\right) - T_L^B\left(t+\frac{r}{c}\right)}{2r}\right\} + I^{-1}(r^B\bar{f}), \tag{B9}$$

where the function $T_L^B(u)$ is the half-difference between the functions $R_L^B(u)$ [Eq. (A13)] and $A_L^B(u)$ [Eq. (B5)]:

$$T_L^B(u) = \frac{R_L^B(u) - A_L^B(u)}{2}\ . \tag{B10}$$

We can also write down the relation between the retarded integral and the symmetric integral. It reads

$$\Box_R^{-1}(r^B\bar{f}) = \sum_{l\geq 0}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{S_L^B\left(t-\frac{r}{c}\right) - S_L^B\left(t+\frac{r}{c}\right)}{2r}\right\} + \Box_S^{-1}(r^B\bar{f}), \tag{B11}$$

where the function $S_L^B(u)$ is given by

$$S_L^B(u) = \frac{R_L^B(u) + A_L^B(u)}{2}\ . \tag{B12}$$

Let us obtain a more convenient expression for the function $S_L^B(u)$. By the explicit expressions (A13) and (B5) of R_L^B and A_L^B we get

$$S_L^B(u) = +\frac{1}{8\pi}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_{-1}^{1}dz\,\gamma_l(z)\bar{f}(\mathbf{y},u-z|\mathbf{y}|/c) - \frac{1}{8\pi}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_{-\infty}^{+\infty}dz\,\gamma_l(z)\bar{f}(\mathbf{y},u-z|\mathbf{y}|/c)\ . \tag{B13}$$

Now, by using $z|\mathbf{y}|$ as a new variable in place of z in the second integral of Eq. (B13), and by using the structure Eqs. (A1) and (A2) of the source, we see that this second integral is in fact identically *zero* since it involves integrals of the type $\int_{-\infty}^{+\infty} d|\mathbf{y}||\mathbf{y}|^{B+k}(\ln|\mathbf{y}|)^p$ which are zero by analytic continuation. Thus the correct expression of the function $S_L^B(u)$ is

$$S_L^B(u) = -\frac{1}{4\pi}\int d^3\mathbf{y}|\mathbf{y}|^B\hat{y}_L\int_{-1}^{1}dz\,\delta_l(z)\bar{f}(\mathbf{y},u-z|\mathbf{y}|/c), \tag{B14}$$

where we have set

$$\delta_l(z) = -\frac{1}{2}\gamma_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l\ . \tag{B15}$$

This function $\delta_l(z)$ is the one which appears in the expression (2.6) of the antisymmetric wave.

Finally we end up this appendix by noting an alternative form of the symmetric integral (B8) [or (B9)]. It reads as

$$\Box_S^{-1}(r^B\bar{f}) = \sum_{l\geq 0}\int_{-r}^{r}ds\,\hat{\partial}_L\left\{\frac{1}{4r}\left[\mathcal{R}_L'^B\left(\frac{s+r}{2},t-s/c\right) + \mathcal{A}_L'^B\left(\frac{s+r}{2},t+s/c\right)\right]\right\}, \tag{B16}$$

where the functions $\mathcal{R}_L'^B$ and $\mathcal{A}_L'^B$ are defined by the same expressions as the functions $\mathcal{R}_L^B$ and $\mathcal{A}_L^B$ but with the lower bound of the integral taking the value $+\infty$ instead of the value zero, i.e.,

$$\mathcal{R}_L'^B(\rho,u) = -(2\rho)^l\int_\rho^{+\infty}dy\,\frac{(\rho-y)^l}{l!}\,y^{B-l+1}\bar{f}_L(y,u+y/c) \tag{B17a}$$

and

$$\mathcal{A}_L'^B(\rho,u) = -(2\rho)^l\int_\rho^{+\infty}dy\,\frac{(\rho-y)^l}{l!}\,y^{B-l+1}\bar{f}_L(y,u-y/c). \tag{B17b}$$

To prove Eq. (B16), we use the expressions (A7) and (B3) of the retarded and advanced integrals in which the functions $\mathcal{R}_L^B$ and $\mathcal{A}_L^B$ are replaced by $\mathcal{R}'^B_L$ and $\mathcal{A}'^B_L$. Then, from the easily checked fact that

$$\int_0^{+\infty}d\rho\,\mathcal{R}'^B_L(\rho,u-2\rho/c) = \int_0^{+\infty}d\rho\,\mathcal{A}'^B_L(\rho,u+2\rho/c)\ , \tag{B18}$$

one sees that the antisymmetric waves in these expressions involve functions $R_L'^B$ and $A_L'^B$ which are equal and thus cancel each other in the symmetric integral.

APPENDIX C: COMPUTATION OF THE FUNCTIONS $m(t)$, $m_i(t)$, AND $s_i(t)$

In this appendix, we denote, e.g., by ${}_kq^{\alpha\beta}$ or ${}_k\mathcal{A}$ the coefficients of c^{-k} in the near-zone expansions of $q^{\alpha\beta}$ or $\mathcal{A}$.

The decomposition of the coefficents ${}_{10}r^0_{(2)}$, ${}_9r^i_{(2)}$, and ${}_{11}r^i_{(2)}$ in the divergence $\bar{r}^\alpha_{(2)} = \partial_\beta\bar{p}^{\alpha\beta}_{(2)}$ of Eqs. (3.44) is of the type

$$ {}_{10}r^0_{(2)} = {}_{10}\mathcal{A} + \partial_a\, {}_{10}\mathcal{A}_a + \cdots , \quad \text{(C1a)} $$

$$ {}_9r^i_{(2)} = \partial_i\, {}_9\mathcal{B} + \partial_{ia}\, {}_9\mathcal{B}_a + {}_9\mathcal{C}_i + \varepsilon_{iab}\partial_a\, {}_9\mathcal{D}_b + \cdots , \quad \text{(C1b)} $$

$$ {}_{11}r^i_{(2)} = {}_{11}\mathcal{C}_i + \cdots , \quad \text{(C1c)} $$

from which, by Eqs. (3.27) and (3.30), we have

$$ {}_7q^{00}_{(2)} = \partial_a\, {}_9\mathcal{C}^{(-2)}_a , \quad \text{(C2a)} $$

$$ {}_9q^{00}_{(2)} + {}_9q^{ss}_{(2)} = -\, {}_{10}\mathcal{A}^{(-1)} - \partial_a\, {}_{10}\mathcal{A}^{(-1)}_a + \partial_a\, {}_{11}\mathcal{C}^{(-2)}_a - 3\, {}_9\mathcal{B} - 3\partial_a\, {}_9\mathcal{B}_a , \quad \text{(C2b)} $$

$$ {}_8q^{0i}_{(2)} = -{}_9\mathcal{C}^{(-1)}_i - \varepsilon_{iab}\partial_a\, {}_9\mathcal{D}^{(-1)}_b . \quad \text{(C2c)} $$

Inserting into the sources ${}_{10}N^{0i}_{(2)}$, ${}_9N^{ij}_{(2)}$, and ${}_{11}N^{ij}_{(2)}$ the linear "even" contributions ${}_2h^{00}_{(1)}$, ${}_3h^{0i}_{(1)}$, ${}_4h^{00}_{(1)}$ and "odd" contributions ${}_7h^{00}_{(1)}$, ${}_8h^{0i}_{(1)}$, ${}_9h^{00}_{(1)}$, and computing ${}_{10}r^0_{(2)}$, ${}_9r^i_{(2)}$, and ${}_{11}r^i_{(2)}$ according to Eqs. (3.46), we find

$$ {}_{10}\mathcal{A} = -\tfrac{4}{25}r^{-1}M_{ab}M^{(6)}_{ab} - \tfrac{24}{25}r^{-1}M^{(1)}_{ab}M^{(5)}_{ab} , \quad \text{(C3a)} $$

$$ {}_{10}\mathcal{A}_i = \tfrac{16}{105}r^{-1}M_{iab}M^{(6)}_{ab} + \tfrac{8}{15}r^{-1}M^{(1)}_{iab}M^{(5)}_{ab} + \tfrac{64}{75}\varepsilon_{iab}r^{-1}M_{ac}S^{(5)}_{bc} + \tfrac{64}{75}\varepsilon_{iab}r^{-1}M^{(5)}_{ac}S_{bc} , \quad \text{(C3b)} $$

$$ {}_9\mathcal{B} = \tfrac{16}{25}r^{-1}M_{ab}M^{(5)}_{ab} , \quad \text{(C3c)} $$

$$ {}_9\mathcal{B}_i = -\tfrac{96}{175}r^{-1}M_{iab}M^{(5)}_{ab} , \quad \text{(C3d)} $$

$$ {}_9\mathcal{C}_i = -\tfrac{8}{5}r^{-1}M_aM^{(5)}_{ai} , \quad \text{(C3e)} $$

$$ \begin{aligned} {}_{11}\mathcal{C}_i = {} & -\tfrac{4}{5}rM_aM^{(7)}_{ai} \\ & + \tfrac{4}{63}r^{-1}M^{(1)}_{iab}M^{(6)}_{ab} + \tfrac{4}{63}r^{-1}M^{(7)}_{iab}M_{ab} \\ & - \tfrac{128}{225}\varepsilon_{iab}r^{-1}M_{ac}S^{(6)}_{bc} - \tfrac{128}{225}\varepsilon_{iab}r^{-1}M^{(6)}_{ac}S_{bc} \\ & - \tfrac{32}{25}\varepsilon_{iab}r^{-1}M^{(5)}_{ac}S^{(1)}_{bc} - \tfrac{32}{25}\varepsilon_{iab}r^{-1}M^{(1)}_{ac}S^{(5)}_{bc} , \end{aligned} \quad \text{(C3f)} $$

$$ {}_9\mathcal{D}_i = \tfrac{4}{5}\varepsilon_{iab}r^{-1}M^{(5)}_{ac}M_{bc} . \quad \text{(C3g)} $$

The functions $m(t)$, $m_i(t)$, and $s_i(t)$ of Eqs. (3.45) are then

$$ m(t) = \int_{-\infty}^t du \left[\tfrac{11}{25}M_{ab}M^{(6)}_{ab} + \tfrac{6}{25}M^{(1)}_{ab}M^{(5)}_{ab} \right] , \quad \text{(C4a)} $$

$$ \begin{aligned} m_i(t) = {} & -\tfrac{2}{5}M_aM^{(3)}_{ai} + \frac{1}{c^2}\int_{-\infty}^t du \int_{-\infty}^u dv \Big[\tfrac{1}{63}M^{(1)}_{iab}M^{(6)}_{ab} + \tfrac{1}{63}M^{(7)}_{iab}M_{ab} - \tfrac{32}{225}\varepsilon_{iab}M_{ac}S^{(6)}_{bc} - \tfrac{8}{25}\varepsilon_{iab}M^{(1)}_{ac}S^{(5)}_{bc} \\ & \qquad - \tfrac{32}{225}\varepsilon_{iab}M^{(6)}_{ac}S_{bc} - \tfrac{8}{25}\varepsilon_{iab}M^{(5)}_{ac}S^{(1)}_{bc} \Big] \\ & + \frac{1}{c^2}\int_{-\infty}^t du \Big[\tfrac{28}{75}M_{iab}M^{(6)}_{ab} + \tfrac{146}{525}M^{(1)}_{iab}M^{(5)}_{ab} - \tfrac{16}{75}\varepsilon_{iab}M_{ac}S^{(5)}_{bc} - \tfrac{16}{75}\varepsilon_{iab}M^{(5)}_{ac}S_{bc} \Big] , \end{aligned} \quad \text{(C4b)} $$

$$ s_i(t) = \tfrac{2}{5}\varepsilon_{iab}\int_{-\infty}^t du\, M^{(5)}_{ac}M_{bc} . \quad \text{(C4c)} $$

By integrating by parts Eqs. (C4) we recover Eqs. (3.46) in the text.

APPENDIX D: RELATION BETWEEN THE QUADRATIC CANONICAL AND MODIFIED CANONICAL METRICS

This appendix presents and uses an identity relating the quadratic source $N_{(2)K,\theta} = N_{(2)}(h_{(1)K,\theta})$ of the modified canonical metric to the corresponding source $N_{\mathrm{can}(2)} = N_{(2)}(h_{\mathrm{can}(1)})$ of the original canonical metric. This identity, which follows from the fact that the linearized metrics $h_{(1)K,\theta}$ and $h_{\mathrm{can}(1)}$ differ by the gauge transformation associated with the gauge vector $\xi^\alpha_{K,\theta}$ [see Eq. (2.29)], reads

$$ N^{\alpha\beta}_{(2)K,\theta} = N^{\alpha\beta}_{\mathrm{can}(2)} + \partial^\alpha\sigma^\beta_{K,\theta} + \partial^\beta\sigma^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu\sigma^\mu_{K,\theta} + \Box\Omega^{\alpha\beta}_{K,\theta} , \quad \text{(D1)} $$

where $\Box = \partial_\mu\partial^\mu$ and where the vector $\sigma^\alpha_{K,\theta}$ and the tensor $\Omega^{\alpha\beta}_{K,\theta}$ are given by

$$ \sigma^\alpha_{K,\theta} = -h^{\mu\nu}_{\mathrm{can}(1)}\partial_\mu\partial_\nu\xi^\alpha_{K,\theta} + \partial_\mu(\xi^\mu_{K,\theta}\Box\xi^\alpha_{K,\theta}) \quad \text{(D2)} $$

and

$$ \begin{aligned} \Omega^{\alpha\beta}_{K,\theta} = {} & 2\, h^{\mu(\alpha}_{\mathrm{can}(1)}\partial_\mu\xi^{\beta)}_{K,\theta} - \partial_\mu\Big[\xi^\mu_{K,\theta}\big(h^{\alpha\beta}_{\mathrm{can}(1)} + 2\partial^{(\alpha}\xi^{\beta)}_{K,\theta} - \eta^{\alpha\beta}\partial_\nu\xi^\nu_{K,\theta}\big)\Big] \\ & + \partial_\mu\xi^\alpha_{K,\theta}\partial^\mu\xi^\beta_{K,\theta} \\ & + \tfrac{1}{2}\eta^{\alpha\beta}(\partial_\mu\xi^\nu_{K,\theta}\partial_\nu\xi^\mu_{K,\theta} - \partial_\mu\xi^\mu_{K,\theta}\partial_\nu\xi^\nu_{K,\theta}) . \end{aligned} \quad \text{(D3)} $$

Note that these expressions identically satisfy

$$ \partial_\beta\Omega^{\alpha\beta}_{K,\theta} + \sigma^\alpha_{K,\theta} \equiv 0 . \quad \text{(D4)} $$

Now we apply the operator $\mathrm{FP}\Box^{-1}_R$ on both sides of Eq. (D1) to obtain a relation between $p^{\alpha\beta}_{(2)K,\theta} = \mathrm{FP}\,\Box^{-1}_R N^{\alpha\beta}_{(2)K,\theta}$ of the modified canonical metric and the corresponding $p^{\alpha\beta}_{\mathrm{can}(2)} = \mathrm{FP}\,\Box_R^{-1} N^{\alpha\beta}_{\mathrm{can}(2)}$ of the canonical metric. Using the tools of our previous papers we obtain

$$p^{\alpha\beta}_{(2)K,\theta} = p^{\alpha\beta}_{\mathrm{can}(2)} + \partial^\alpha \Pi^\beta_{K,\theta} + \partial^\beta \Pi^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu \Pi^\mu_{K,\theta} + \Omega^{\alpha\beta}_{K,\theta} + u^{\alpha\beta}_{K,\theta}, \tag{D5}$$

where the vector $\Pi^\alpha_{K,\theta}$ is given by

$$\Pi^\alpha_{K,\theta} = \mathrm{FP}\,\Box_R^{-1}\sigma^\alpha_{K,\theta} \tag{D6}$$

and where the new tensor $u^{\alpha\beta}_{K,\theta}$, which arises because of the differentiations of the analytic continuation factor r^B, reads

$$u^{\alpha\beta}_{K,\theta} = \mathrm{FP}_{B=0}\,\Box_R^{-1}\Big\{ -2Br^{B-1}\partial_r \Omega^{\alpha\beta}_{K,\theta} - B(B+1)r^{B-2}\Omega^{\alpha\beta}_{K,\theta} - 2Bn_i r^{B-1}\eta^{i(\alpha}\sigma^{\beta)}_{K,\theta} + Bn_i r^{B-1}\eta^{\alpha\beta}\sigma^i_{K,\theta}\Big\}. \tag{D7}$$

Because of the explicit factor B in the integrand of the retarded integral (D7), the tensor $u^{\alpha\beta}_{K,\theta}$ is made of retarded waves, solutions of the d'Alembertian equation in D_e (see paper I). The divergence of $u^{\alpha\beta}_{K,\theta}$ will have also the same structure, and we can thus associate to $u^{\alpha\beta}_{K,\theta}$ a new tensor $v^{\alpha\beta}_{K,\theta}$ by formulas analogous to Eqs. (2.49) and (2.50). Then the relation between the quadratic modified canonical metric $h^{\alpha\beta}_{(2)K,\theta}$ [Eq. (2.51)] and the original canonical one $h^{\alpha\beta}_{\mathrm{can}(2)}$ is given by

$$h^{\alpha\beta}_{(2)K,\theta} = h^{\alpha\beta}_{\mathrm{can}(2)} + \partial^\alpha \Pi^\beta_{K,\theta} + \partial^\beta \Pi^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu \Pi^\mu_{K,\theta} + \Omega^{\alpha\beta}_{K,\theta} + u^{\alpha\beta}_{K,\theta} + v^{\alpha\beta}_{K,\theta}\,. \tag{D8}$$

Note that the combination $u^{\alpha\beta}_{K,\theta} + v^{\alpha\beta}_{K,\theta}$ is a retarded solution of the Einstein linearized equations (in harmonic coordinates). Therefore it can be written as $h^{\alpha\beta}_{\mathrm{can}(1)}$ of some quadratically nonlinear moments plus terms corresponding to an infinitesimal gauge transformation [44], [36]. These moments represent a nonlinear correction in the relation linking the moments used in the construction of the canonical and modified canonical metrics.

Finally let us consider the case of the quadratic monopole-quadrupole metric, corresponding to the interaction of the mass M and of the quadrupole M_{ij} (symbolized by "$M \times M_{ij}$"). In this case one has

$$\sigma^\alpha_{K,\theta}|_{M\times M_{ij}} = \frac{4M}{c^4 r}\,\partial_t^2 \xi^\alpha_{K,\theta} \tag{D9}$$

and

$$\Omega^{00}_{K,\theta}|_{M\times M_{ij}} = -\frac{4M}{c^3 r}\,\partial_t \xi^0_{K,\theta} + \partial_i\left(\frac{4M}{rc^2}\xi^i_{K,\theta}\right), \tag{D10a}$$

$$\Omega^{0i}_{K,\theta}|_{M\times M_{ij}} = -\frac{4M}{c^3 r}\,\partial_t \xi^i_{K,\theta}, \tag{D10b}$$

$$\Omega^{ij}_{K,\theta}|_{M\times M_{ij}} = 0, \tag{D10c}$$

where in the vector $\xi^\alpha_{K,\theta}$ [Eqs. (2.30)] we keep only the part corresponding to the quadrupole M_{ij}. Substituting the equations (D9) and (D10) into the tensor $u^{\alpha\beta}_{K,\theta}$ [Eq. (D7)] we find that $u^{\alpha\beta}_{K,\theta}$ is the finite part at $B = 0$ of a source whose near-zone expansion has a structure of the type $B\hat{n}_L r^{B+l+p-3}F(t)$, where p is a positive integer with $p \geq 2$ when $l = 0$. Since we know (paper I) that the finite part of the retarded integral of $B\hat{n}_L r^{B+a}F(t)$ is nonzero only if a is of the type $a = -l-3, -l-5, \ldots$, we easily conclude that

$$u^{\alpha\beta}_{K,\theta}|_{M\times M_{ij}} = 0 \tag{D11a}$$

and thus also

$$v^{\alpha\beta}_{K,\theta}|_{M\times M_{ij}} = 0\,. \tag{D11b}$$

Thus, for the interaction $M \times M_{ij}$, the relation between the canonical and modified canonical metrics becomes

$$h^{\alpha\beta}_{(2)K,\theta}|_{M\times M_{ij}} = \left(h^{\alpha\beta}_{\mathrm{can}(2)} + \partial^\alpha \Pi^\beta_{K,\theta} + \partial^\beta \Pi^\alpha_{K,\theta} - \eta^{\alpha\beta}\partial_\mu \Pi^\mu_{K,\theta} + \Omega^{\alpha\beta}_{K,\theta}\right)\Big|_{M\times M_{ij}}. \tag{D12}$$

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[45] Our conventions and notation are the following: signature $-+++$; greek indices = 0,1,2,3; latin indices =1,2,3; $g = \det(g_{\mu\nu})$; $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ = flat metric = diag $(-1,1,1,1)$; $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are the usual sets of non-negative integers, integers, real numbers, and complex numbers; $C^p(U)$ is the set of p-times continuously differentiable functions in U; $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$; $n^i = n_i = x^i/r$; $\partial_i = \partial/\partial x^i$; $n^L = n_L = n_{i_1} n_{i_2} \cdots n_{i_l}$ and $\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$, where $L = i_1 i_2 \cdots i_l$ is a multi-index with l indices; $n_{L-1} = n_{i_1} \cdots n_{i_{l-1}}$, $n_{aL-1} = n_a n_{i_1} \cdots n_{i_{l-1}}$, etc.; $\hat{n}_L$ and $\hat{\partial}_L$ are the (symmetric) and trace-free parts of n_L and ∂_L; $T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})$ and $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$.