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Spherically symmetric singularities and strong cosmic censorship

Peter Szekeres
Department of Physics and Mathematical Physics, University of Adelaide, G.P.O. Box 498, Adelaide, South Australia 5001, Australia

Vivek Iyer
Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637
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In this paper we consider spherically symmetric metrics having singularities of a form designated as being "of the power-law type." Included in this class are all Tolman-Bondi dust collapses, but the form postulated is very much more general. We calculate the asymptotic behavior of the energy-stress tensor near the singularity, and show that it is possible to satisfy the dominant energy condition, even when the singularity is locally naked. This seems to contradict the strong cosmic censorship hypothesis. However it is shown that if the singularity is not a shell cross, then the energy-stress tensor is asymptotically extreme ($|P_r| \approx \rho$ or $|P_\perp| \approx \rho$) or one of the pressures is negative ($P_r < 0$ or $P_\perp < 0$).

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I. INTRODUCTION

The cosmic censorship hypothesis, first proposed by Roger Penrose in 1969 [1], makes a claim along the following lines.

Cosmic censorship hypothesis (CCH). No physically realistic collapse (e.g., one evolving from a well posed initial data set and satisfying the dominant energy condition) results in a naked singularity—i.e., a singularity in the causal past of future infinity.

Over the years, however, there has been considerable disagreement on this subject. For example, it is certainly true that exact dust collapses can exhibit naked singularities [2–4], as can the Vaidya solution representing a null dust collapse [5–7]. Also, collapse of a scalar field does not in general lead to an event horizon [8, 9]. All this seems to mitigate against the CCH.

Nevertheless, these examples are not totally convincing. The naked singularities appearing in spherical and quasispherical dust collapses are frequently of the "shell-crossing" variety (although exactly what this term means has never been fully clarified), although Christodoulou [4] claims to have found a dust collapse possessing a naked singularity which is "shell focussing." Scalar fields, on the other hand, have an energy-stress tensor which generally satisfies what we will term an "extreme dominant energy condition." Such an energy-stress tensor cannot be discarded out of hand, but is possibly only marginally physically realistic. It certainly implies the possibility of (locally) naked singularities, as we shall see.

Some general results have been obtained by Krolak [10] and Newman [11] using global topological arguments. These tend to support the cosmic censorship hypothesis. However the significance of some of their conclusions, particularly those restricting the "strength" of a singularity, are difficult to assess. These approaches concentrate entirely on the global aspects of the problem and say nothing about the existence of locally naked singularities, i.e., singularities which are visible from regular points of the space-time, but possibly not visible at infinity. Yet from the point of view of infalling particles, such singularities must be as worrying as those visible at infinity, since they are likely to upset the physical conditions in their space-time neighborhood.

Such consideration have led Penrose [12] to propose a stronger hypothesis:

Strong Cosmic Censorship Hypothesis (SCCH). No physically realistic collapse leads to a locally naked (i.e., timelike) singularity.

In this paper we shall concentrate attention entirely on the validity of SCCH, and only in the context of spherical symmetry. Despite this specialization we believe that useful information can be gained. Furthermore SCCH is an easier question to investigate than CCH, since it is not necessary to integrate null geodesics globally, and only the behavior of the space-time in the neighborhood of the singularity need be considered.

Analyses of dust collapses have been popular since they are simply and explicitly exhibited in the Tolman-Bondi solutions [13, 14]. However any conclusions arrived at concerning dust may not be significant unless they can be shown to be stable with respect to physically interesting modifications of the equation of state. In particular, as the density approaches infinity, one would expect on physical grounds that the pressure should do likewise. In the limit, possibly a relativistic equation of state such as $P = \frac{1}{3}\rho$ should apply.

However, a perfect fluid energy-stress tensor appears to be too demanding a requirement, since it is physically reasonable to include a certain amount of radial streaming, shearing stresses, viscosity, etc., in the fluid. Also, exact perfect fluid solutions with nontrivial equations of state are hard to find. A more promising line of approach is to undertake an asymptotic investigation of the energy-stress tensor in the neighborhood of spherically symmetric singularities. This will be the method adopted in this paper.

In Sec. II we review the singularity behavior of Tolman-Bondi dust solutions. This leads us to consider the algebraic form of a much more general spherically symmetric singularity which we say to be of "power-law type." In Sec. III we transform such singularities to double null coordinates. These have the dual advantage of possessing much greater rigidity than (r, t) coordinates, while at the same time making null geodesics trivial to integrate. In Sec. IV we impose the dominant energy condition, and discover what further restrictions this imposes on the metric form. In the final two sections the asymptotic behavior of the energy-stress tensor at a singularity of power-law type is discussed. The analysis falls into two categories, which we term the "generic" case and the "nongeneric" case. Both cases seem to be important, since Friedmann dust models fall in the first case, while the general Tolman-Bondi singularity falls in the second. It is shown that timelike (locally naked) singularities can occur in both cases, such that the energy-stress tensor satisfies the dominant energy condition, but the tangential or the radial pressure must in general be either negative or equal in magnitude to the density.

II. SINGULARITY BEHAVIOR OF COLLAPSING DUST

The general spherically symmetric space-time metric can be expressed in Gaussian normal coordinates

$$ds^2 = -dt^2 + F^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $F = F(r, t)$, $R = R(r, t)$. This is particularly suitable for dust solutions

$$G_{\mu\nu} = \kappa T_{\mu\nu} = \rho u_\mu u_\nu \quad (2)$$

where $u^\mu = (1, 0, 0, 0)$. In this case the exact solutions can all be found and are known as the Tolman-Bondi metrics [13, 14]. The simplest case is that of zero energy (particles of fluid coming exactly from rest at infinity) and is given by

$$ds^2 = -dt^2 + (t_0 - t)^{-2/3}(t_1 - t)^2 dr^2 + r^2(t_0 - t)^{4/3} d\Omega^2, \quad (3)$$

where

$$t_0 = t_0(r), \quad t_1 = t_1(r) = t_0(r) + \frac{2}{3} r t_0'(r) \quad (4)$$

and the density is

$$\rho = \frac{4}{3(t_0 - t)(t_1 - t)} \quad (5)$$

(units are chosen such that $\kappa = c = 1$, and the standard shorthand $d\Omega^2 = d\theta^2 + r^2 d\phi^2$ is adopted). For the purpose of this paper, the metric is taken to represent a collapse, with time coordinate t increasing towards the singular epoch at t_0 or t_1. By reversing the arrow of time these space-times may be thought of as inhomogeneous cosmologies.

Although Eq. (3) is only a particular case of the Tolman-Bondi metric, the singularity behavior is similar for all members of this family. However, the types of singularity occurring at $t = t_0$ and $t = t_1$ are quite distinct. At $t = t_0(r)$ the singularity has zero area ($= 4\pi \times$ the coefficient of $d\Omega^2$) and can be considered as a central "focus." At $t = t_1(r)$ the area is finite and the metric component $g_{rr} \to 0$. This is usually regarded as being a "shell cross." Shell crosses can arise even in Minkowski space, by aiming shells of dust at each other in such a way that the outer shells overtake inner shells. They therefore have nothing whatsoever to do with *gravitational* collapse. It should however be noted that, by Eq. (5), the density does become infinite at a shell cross and with exactly the same time rate as at $t = t_0$. This also means that the curvature and its associated tidal effects become infinite there. It is, however, a "softer" singularity, since there are special coordinates (such as the double null coordinates adopted below) in which the metric becomes C^1 but not C^2 at $t = t_1(r)$. It seems to us a matter of some importance to provide a clear and rigorous distinction between these two types of singularity, which can be applied in general (nonspherical) situations.

The t_1 singularity is locally naked. This is easily seen, since radial null geodesics are horizontal in the (r, t) plane at $t = t_1(r)$ [i.e., $dt/dr \to 0$ as $t \to t_1$] and therefore intersect neighboring world lines of infalling particles. On the other hand, the singularity at $t = t_0(r)$ is censored provided $t_0'(r) \neq 0$, since null geodesics have $dr/dt = 0$ there. Places where $t_0'(r) = 0$ can have either character, which is not really surprising since at such points the two singular surfaces coincide, $t_1(r) = t_0(r)$. However, Newman [15] has shown that for a central ($r = 0$) singularity of this type to be naked one must have the central density be a local minimum [$\rho''(0) < 0$]. This is possibly an unrealistic physical requirement. For the most part, this paper will avoid discussion of this category of singular points.

Our analysis of Tolman-Bondi singularities is purely local, and does not refer to the visibility or otherwise of the singularity at infinity. For the rest of this paper it will be understood that whenever we refer to singularities as being "naked" or "censored," we are in fact only referring to local properties. Thus a singularity will be called *naked* if both ingoing and outgoing null geodesics enter it, while it will be *censored* if only ingoing null geodesics enter it. For those who feel our approach to singularities as "boundary points" is somewhat cavalier, we refer the reader to a recent approach to singularity theory [16] where it is shown how this kind of discussion can be made rigorous.

We will also make reference to the degenerate case arising when $t_0 =$ const, $t_1 = t_0$. In this case the metric reduces to the Einstein–de Sitter universe (in collapsing form)

$$ds^2 = -dt^2 + (t_0 - t)^{4/3}(dr^2 + d\Omega^2). \quad (6)$$

It is well known that the singularity at $t = t_0$ is censored.

The metrics discussed above are all special cases of a metric of the form

$$ds^2 = -dt^2 + [\tau(r)-t]^{2a} f^2(r,t)dr^2 + [\tau(r)-t]^{2b} g^2(r,t)d\Omega^2 \tag{7}$$

where f and g are functions of r and t which are regular and nonvanishing at the singularity surface $t = \tau(r)$. For example, the t_0 dust singularity arises on setting $\tau = t_0(r)$ and

$$a = -1/3,\ b = 2/3,\ f(r,t) = \tau + \frac{2}{3}\tau'(r) - t,\ g(r,t) = r.$$

Similarly, the t_1 singularity has the behavior $a = 1$, $b = 0$, while the Friedman-type behavior (Einstein–de Sitter) occurs if $\tau' = 0$, $a = b = 2/3$. Of course we do not mean to imply that Eq. (7) is the most general possible singularity behavior in spherical symmetry, but in view of the dust experience it seems reasonable to assume that it will apply to a fairly general class of fluid collapses. We will say that such spherically symmetric singularities are of the *power-law type.*

We propose to analyse the general case given by Eq. (7), and to determine for which exponents a and b the behavior of the metric is physically reasonable. Although this can be done quite straightforwardly, a major problem arises in how to distinguish coordinate singularities from genuine ones (for example, Minkowski space can be made to look singular if one chooses coordinates based on a radially infalling family of timelike geodesics and sets r to be a comoving radial coordinate and t to be the proper time parameter along the curves). Another problem is that, except in the dust case, there is no way of preferentially choosing one set of timelike geodesics over any other. This amounts to having arbitrary freedom in the choice of radial coordinate $r' = r'(r,t)$, provided it is accompanied by an appropriate transformation to a new time coordinate t'.

In other words, Gaussian normal coordinates such as those in Eq. (7) are too flexible—the powers a and b can in no way be regarded as being characteristic of the space-time.

Much greater rigidity is achieved if we go to *double null coordinates* u and v (both functions of r and t) such that the metric takes the form

$$ds^2 = -2e^U du\, dv + e^V d\Omega^2. \tag{8}$$

While the transformation to such coordinates may be nontrivial to find, the coordinates u and v are almost completely determined by the metric, since now the only available coordinate freedom is one of the form

$$u' = \mu(u),\quad v' = \nu(v). \tag{9}$$

The functions U and V are then determined up to essentially trivial transformations, and are truly characteristic of the space-time. We will see that the coordinates u and v can be further tied down by the singularity surface so that even this freedom is unavailable. Furthermore, the behavior of the null geodesics, trapped surfaces, horizons, etc., are simple to discuss in these coordinates, since the radial null lines ($\theta = $ const, $\phi = $ const) are exactly the curves $u = $ const or $v = $ const.

III. TRANSFORMATION TO DOUBLE NULL COORDINATES

In order to convert the metric (7) into double null coordinates (8) one can try a series transformation in a neighborhood of $r = r_0$, $t = \tau(r_0)$ of the form

$$r = r_0 + u + f_1(u)x^{a_1} + f_2(u)x^{a_2} + \cdots, \tag{10}$$

$$t = \tau(r_0 + u) + g_1(u)x^{b_1} + g_2(u)x^{b_2} + \cdots, \tag{11}$$

where $0 < a_1 < a_2 < \cdots$, $0 < b_1 < b_2 < \cdots$. The function $x(u,v)$, which characterizes the singularity at $t = \tau(r)$ as occurring at $x = 0$, may be assumed to have the form $x = \mu(u) - \nu(v)$. By using the coordinate freedom (9) there is no loss of generality in assuming this function to be linear:

$$x = -u + kv, \qquad \text{where } k = \pm 1, \tag{12}$$

the sign of k being chosen such that $x > 0$ for $t < \tau(r)$

The double null form of the metric is achieved if r and t satisfy the coupled equations

$$\frac{\partial t}{\partial u} = [\tau(r) - t]^a f \frac{\partial r}{\partial u}, \tag{13}$$

$$\frac{\partial t}{\partial v} = -[\tau(r) - t]^a f \frac{\partial r}{\partial v}. \tag{14}$$

Substituting the trial series (10) and (11) in these equations, and expanding f as a power series

$$f(r,t) = \varphi_0(u) + \varphi_1(u)x^{a_1} + \cdots \quad [\varphi_0(u) > 0]$$

one finds

$$\tau' - b_1 g_1 x^{b_1 - 1} + \cdots = (f_1 \tau' x^{a_1} - g_1 x^{b_1} + \cdots)^a (\varphi_0 + \cdots)(1 - a_1 f_1 x^{a_1 - 1} + \cdots), \tag{15}$$

$$b_1 g_1 x^{b_1 - 1} + \cdots = -(f_1 \tau' x^{a_1} - g_1 x^{b_1} + \cdots)^a (\varphi_0 + \cdots)(a_1 f_1 x^{a_1 - 1} + \cdots). \tag{16}$$

Assuming $\tau'(r_0) \neq 0$ we have the following two equations for the leading exponents

$$\min(0, b_1 - 1) = a \min(a_1, b_1) + \min(0, a_1 - 1), \tag{17}$$

$$b_1 - 1 = a \min(a_1, b_1) + a_1 - 1. \tag{18}$$

It is best to discuss these conditions under two separate cases.

(i) a > 0. It is easy to see that Eqs. (17) and (18) can only be satisfied if $0 < b_1 \leq 1$ and $a_1 = b_1/(1 + a)$. However, if $b_1 < 1$, then the coefficients of the leading terms in (15) and (16) result in

$$b_1 g_1 = (f_1 \tau')^a \varphi_0 a_1 f_1,$$

$$b_1 g_1 = -(f_1 \tau')^a \varphi_0 a_1 f_1,$$

which can only be satisfied if $f_1 = g_1 = 0$. But since these are the coefficients of the leading terms in (10) and (11), they are by definition nonvanishing. Hence we must have

$$b_1 = 1, \qquad a_1 = \frac{1}{1+a}.$$

The equations for the leading terms now become

$$\tau' - g_1 = -(f_1 \tau')^a \varphi_0 f_1/(1+a),$$

$$g_1 = -(f_1 \tau')^a \varphi_0 f_1/(1+a),$$

whence

$$g_1 = \frac{1}{2}\tau', \qquad f_1 = \frac{1}{\tau'}\left(\frac{(1+a)\tau'^2}{2\varphi_0}\right)^{1/1+a}.$$

N.B. $f_1 \tau' > 0$, as required for $\tau(r) - t > 0$.

(ii) a< 0. This case is handled in a very similar way. It results in

$$a_1 = 1, \qquad b_1 = \frac{1}{1-a},$$

$$f_1 = \frac{1}{2}, \qquad g_1 = -\left(\frac{(1-a)\varphi_0}{2}\right)^{1/1-a} < 0.$$

We do not propose to discuss the case $\tau'(r_0) = 0$, as it leads to various complications (e.g., see the discussion of $t_0' = 0$ in the previous section). However the Friedmann-like case $\tau =$ const is worth describing.

(iii) τ = const. If $a < 1$ then identical conditions to case (ii) result

$$a_1 = 1, \quad b_1 = \frac{1}{1-a}, \quad f_1 = \frac{1}{2},$$

$$g_1 = -\left(\frac{(1-a)\varphi_0}{2}\right)^{1/1-a} < 0.$$

The case $a \geq 1$ is somewhat special and will not be considered here. The higher exponents a_2, b_2, ... and coefficients f_2, g_2, ... can be worked out in particular cases (e.g., Tolman-Bondi dust, see below) but it is difficult to write down general formulas for them.

Finally we evaluate the functions U and V,

$$e^U = 2t_u t_v = x^p e^\alpha, \qquad e^V = (\tau - t)^{2b} g^2 = x^q e^\beta \tag{19}$$

where

$$\alpha = \alpha_0(u) + \alpha_1(u) x^{p_1} + \cdots, \tag{20}$$

$$\beta = \beta_0(u) + \beta_1(u) x^{q_1} + \cdots. \tag{21}$$

The exponents p and q depend on a and b as follows:

$$\text{in case (i)} \qquad p = 0, \;\; q = \frac{2b}{1+a}; \tag{22}$$

$$\text{in cases (ii) and (iii)} \qquad p = \frac{2a}{1-a}, \;\; q = \frac{2b}{1-a}. \tag{23}$$

Higher exponents p_1, q_1, etc., depend in a rather delicate way on the exponents $a_1, a_2, \ldots, b_1, b_2, \ldots$ occurring in (10) and (11). It is not easy to give a general form for them.

A detailed investigation of the Tolman-Bondi dust metrics discussed in Sec. II results in the following. A t_1 singularity ($t_1' \neq 0$) is of type (i) with $a = 1$, $b = 0$. The power series solutions begin with $a_1 = 1/2$, $b_1 = 1$ and the double null form (19)–(21) will have exponents

$$p = q = 0, \quad p_1 = q_1 = 1, \quad p_2 = q_2 = \frac{3}{2}.$$

A t_0 singularity ($t_0' \neq 0$) is of type (ii) with $a = -1/3$, $b = 2/3$, and results in power series solutions with $a_1 = 1$, $a_2 = 5/4, \ldots$ and $b_1 = 3/4$, $b_2 = 1, \ldots$ in (10) and (11). The exponents of the double null form are

$$p = -\frac{1}{2}, \quad q = 1, \quad p_1 = q_1 = \frac{1}{2}.$$

Finally the Friedmann–Einstein–de Sitter-type singularity will be of type (iii) with $\tau =$ const, $a = b = 2/3$. It gives $a_1 = 1$, $b_1 = 3$ and

$$p = q = 4, \quad p_1 = q_1 = 1.$$

IV. ENERGY CONDITIONS

From the previous section it is clear that a fairly general form of the metric is one which, when expressed in double null coordinates, has the form

$$ds^2 = -2e^U du\, dv + e^V d\Omega^2 \tag{24}$$

where

$$U = p \ln x + \alpha_0(u) + \alpha_1 x^{p_1} + \alpha_2 x^{p_2} + \cdots \tag{25}$$

$$V = q \ln x + \beta_0(u) + \beta_1 x^{q_1} + \beta_2 x^{q_2} + \cdots \tag{26}$$

where $0 < p_1 < p_2 < \cdots$, $0 < q_1 < q_2 < \cdots$, and

$$x = lu + kv, \qquad \text{where } l,\; k = \pm 1 \tag{27}$$

This form certainly encompasses all metrics originally postulated in Eq. (7) as being of power-law type at the singularity. Assuming that u and v are both future increasing null coordinates the case $lk = -1$ will be a (locally) naked singularity. This follows because $x = 0$ is a timelike curve in the (conformally regularized) u-v plane, so that null geodesics can both enter and leave it. Similarly $lk = 1$ will imply that $x = 0$ is a spacelike curve and therefore must be a censored singularity. In the latter case we shall only consider the case $l = k = -1$ so that the region $x > 0$ corresponds to $u+v < 0$. The other possibility, $l = k = 1$, will correspond to an inhomogeneous cosmology (rather than a collapse) and therefore will not be discussed in this paper.

It is also possible to consider the cases $k = 0$ or $l = 0$, which amount to a *null singularity*. Such cases occur in the null dust solutions of Vaidya [17], and can possibly occur in ordinary dust collapses [4]. Again, this situation will not be considered here.

The Einstein tensor can be calculated for the form (24) with the substitution of (19), giving the following nonvanishing components (here we set $x^0 = u$, $x^1 = v$, $x^2 = \theta$, $x^3 = \phi$):

$$G^0_0 = G^1_1 = -e^{-V} - e^{-U}(V_{01} + V_0 V_1) = -x^{-q}e^{-\beta} - e^{-\alpha}x^{-p}\left(\frac{kl(q^2-q)}{x^2} + \frac{k\beta_u + l\beta_v}{x} + \beta_{uv} + \beta_u\beta_v\right), \tag{28}$$

$$G^0_1 = e^{-U}(V_{11} - U_1V_1 + \tfrac{1}{2}{V_1}^2) = -e^{-\alpha}x^{-p}\left(\frac{k^2(q+pq-\frac{1}{2}q^2)}{x^2} + \frac{k((p-q)\beta_v + q\alpha_v)}{x} - \beta_{vv} + \alpha_v\beta_v - \frac{1}{2}{\beta_v}^2\right), \tag{29}$$

$$G^1_0 = e^{-U}(V_{00} - U_0V_0 + \tfrac{1}{2}{V_0}^2) = -e^{-\alpha}x^{-p}\left(\frac{l^2(q+pq-\frac{1}{2}q^2)}{x^2} + \frac{l((p-q)\beta_u + q\alpha_u)}{x} - \beta_{uu} + \alpha_u\beta_u - \frac{1}{2}{\beta_u}^2\right), \tag{30}$$

$$G^2_2 = G^3_3 = -e^{-U}(U_{01} + V_{01} + \tfrac{1}{2}V_0V_1) = -e^{-\alpha}x^{-p}\left(\frac{lk(\frac{1}{2}q^2-p-q)}{x^2} + \frac{q(k\beta_u + l\beta_v)}{2x} + \alpha_{uv} + \beta_{uv} + \frac{1}{2}\beta_u\beta_v\right). \tag{31}$$

For convenience we shall make the following assumptions: (1) The energy-stress tensor is of type I (i.e., $G_{\mu\nu}$ has a timelike eigenvector); (2) the dominant energy condition holds (i.e., $G_{\mu\nu}v^\nu$ is nonspacelike for all timelike vectors v^ν and $G_{\mu\nu}v^\mu v^\nu \geq 0$); (3) the singularity at $x = 0$ is a density singularity, i.e., $\rho \to \infty$ as $x \to 0$.

Condition 1 might be considered to be too restrictive, since it eliminates solutions such as Vaidya's which correspond to null fluids. However, these solutions are, in any case, overidealized, since the slightest contamination with a perfect fluid or ordinary dust will lead to a type I energy-stress tensor. They also lead to singularities of null character ($k = 0$ or $l = 0$) which we do not propose to discuss in this paper.

As shown in [18] conditions 1 and 2 imply that the eigenvalues λ of $G^\mu{}_\nu$,

$$G^\mu{}_\nu u^\nu = \lambda u^\mu, \tag{32}$$

are $-\rho$ and P_i (ρ is the density and P_i are the principal pressures), and that these satisfy $0 \leq |P_i| \leq \rho$. More specifically, in our case the eigenvectors lying in the u^0-u^1 plane have the form

$$u^1 = \pm\sqrt{\frac{G^1_0}{G^0_1}}u^0,$$

and since $u_\mu u^\mu = -2e^U u^0 u^1$, the plus sign belongs to the timelike eigenvector, the minus sign to the radial spacelike eigenvector. These are real only if either (a) both G^0_1 and G^1_0 are > 0, or (b) both G^0_1 and G^1_0 are < 0. In case (a) the timelike eigenvalue is

$$\lambda = -\rho = G^0_0 + \sqrt{G^0_1 G^1_0},$$

while the spacelike eigenvalue corresponding to radial pressure is

$$\lambda = P_r = G^0_0 - \sqrt{G^0_1 G^1_0}.$$

Hence this case leads to $\rho + P_r = -2\sqrt{G^0_1 G^1_0} < 0$, violating the dominant energy condition 2. We therefore restrict attention to case (b):

$$G^0_1 < 0, \quad G^1_0 < 0. \tag{33}$$

In this case the density and radial pressure are given by

$$\rho = -G^0_0 + \sqrt{G^0_1 G^1_0}, \tag{34}$$

$$P_r = G^0_0 + \sqrt{G^0_1 G^1_0}. \tag{35}$$

The dominant energy condition 2 now imposes the further inequality

$$G^0_0 \leq 0 \tag{36}$$

which follows from $\rho \geq P_r$. The final inequality $\rho > 0$ now follows automatically.

The tangential pressure is given by

$$P_\perp = G^2_2 = G^3_3 \tag{37}$$

and the dominant energy condition implies that we require $|P_\perp| < \rho$.

We shall, however, exclude from consideration fluids which only marginally satisfy the dominant energy condition, $\rho = \pm P_r$ or $\rho = \pm P_\perp$. Such cases, although not totally ruled out by physical considerations, will be regarded as *extreme.* Such cases tend to arise rather *too* easily near singularities of the type we are considering (see below), while at the same time not corresponding to "nice" fluids (e.g., perfect fluids with $P = a\rho$, $0 < a < 1$). We therefore shall impose strict inequality in Eq. (36), $G^0_0 < 0$.

From Eqs. (28), (29), and (34) it is clear that a necessary condition for $x = 0$ to represent a singularity of the density one must have that either $q > 0$ or $p > -2$. However if $q > p + 2$ then the first term of G^0_0 dominates all other terms in the energy-stress tensor, leading to limiting behavior $\rho \approx -P_r$, as $x \to 0$. This is just the kind of extreme behavior referred to above. The only way of achieving an infinite density singularity *without* an extreme equation of state in the limit is to impose the conditions

$$q \le p+2, \qquad p > -2. \tag{38}$$

V. ENERGY-STRESS BEHAVIOR AT THE SINGULARITY—THE GENERIC CASE

By the *generic* case we shall mean $q \neq 0$ or 1. Then the leading term in the square brackets in Eqs. (28)–(31) for $G^{\mu}{}_{\nu}$ dominates the behavior as $x \to 0$:

$$G^0_0 \approx -e^{-\alpha} x^{-p-2} \varepsilon (q^2 - q), \tag{39}$$

$$G^0_1 \approx G^0_1 \approx -e^{-\alpha} x^{-p-2} (q + pq - \tfrac{1}{2} q^2), \tag{40}$$

$$G^2_2 = G^3_3 \approx -e^{-\alpha} x^{-p-2} \varepsilon (\tfrac{1}{2} q^2 - p - q) \tag{41}$$

where $\varepsilon = kl = \pm 1$.

As seen in the previous section, the assumption of a type I energy-stress tensor satisfying the dominant energy conditions and obeying a nonextreme equation of state, $\rho \neq \pm P_r$ implies

$$G^0_0 < 0, \quad G^0_1 < 0, \quad G^1_0 < 0, \tag{42}$$

whence we see that

$$\varepsilon = 1 \Rightarrow q > 1 \text{ or } q < 0, \qquad \varepsilon = -1 \Rightarrow 0 < q < 1,$$

and

$$p > -1 \Rightarrow 0 < q < 2(1+p),$$

$$p < -1 \Rightarrow 2(1+p) < q < 0.$$

Combining with condition (38) one sees that the regions of interest are shown in Fig. 1. The censored case $\epsilon = 1$ divides into two regions, marked A_1 and A_2. A_1 is the triangular region defined by $p > -2$, $q < 0$, $q > 2(p+1)$, while A_2 is the open region defined by $1 < q < 2(1+p)$ for $-\frac{1}{2} < p \le 0$, and $1 < q < p+2$ for $p \ge 0$. On the other hand, the naked case $\epsilon = -1$ consists of the horizontal strip marked B defined by $0 < q < 1$ and bounded on the left by the line $q = 2(1+p)$.

In order to discuss the physical conditions holding in these regions, it is useful to define a pair of parameters

$$\kappa = \frac{P_r}{\rho}, \qquad \lambda = \frac{P_\perp}{\rho}. \tag{43}$$

For the censored case $\varepsilon = +1$, one finds on using (34), (35), and (37) that

$$\kappa = \frac{4 + 2p - 3q}{2p + q}, \qquad \lambda = \frac{2q + 2p - q^2}{q(2p+q)}, \tag{44}$$

which may be readily solved for p and q:

$$p = \frac{4(1 - 2\lambda + \kappa)}{(1 + 4\lambda - \kappa)(1 - \kappa)}, \qquad q = \frac{4}{1 + 4\lambda - \kappa}. \tag{45}$$

The triangular region A_1 is readily seen to correspond to values of κ and λ restricted by $-1 < \kappa < 1$,

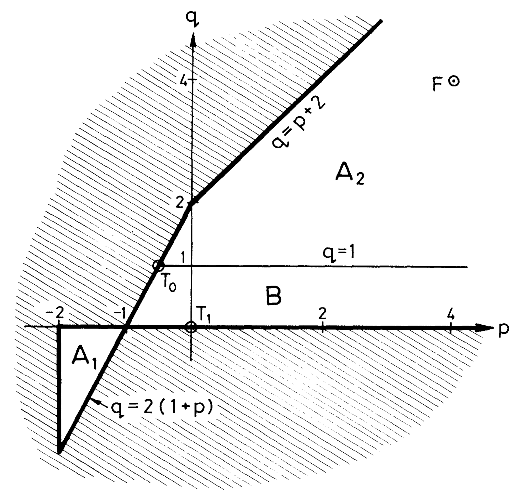

FIG. 1. Physical regions of the p-q plane. The unshaded area is where the energy-stress tensor satisfies the dominant energy condition near the singularity. In A_1 and A_2 the singularity is censored ($\varepsilon = +1$), in B it is naked ($\varepsilon = -1$). T_0 and T_1 are respectively the t_0 and t_1 singularities of the Tolman-Bondi dust solutions. F is the Friedmann dust singularity.

$\lambda < (\kappa - 1)/4$ and $\kappa^2 + 3 < 4\lambda\kappa$. However for $\kappa > 0$ these conditions imply $\kappa - 1 > (\kappa^2 + 3)/\kappa$ which is a contradiction since it gives immediately that $\kappa < -3$. Hence $-1 < \kappa < 0$. But in this range $\lambda < (\kappa^2 + 3)/4\kappa < -1$, whence the dominant energy condition is violated. Thus nothing of physical interest arises in region A_1.

The other censored region A_2, where $q > 1$, corresponds to a subset of the parallelogram bounded by

$$-1 < \kappa < 1, \qquad \frac{\kappa - 1}{4} < \lambda < \frac{\kappa + 3}{4},$$

shown in Fig. 2. The curved section of the boundary is a

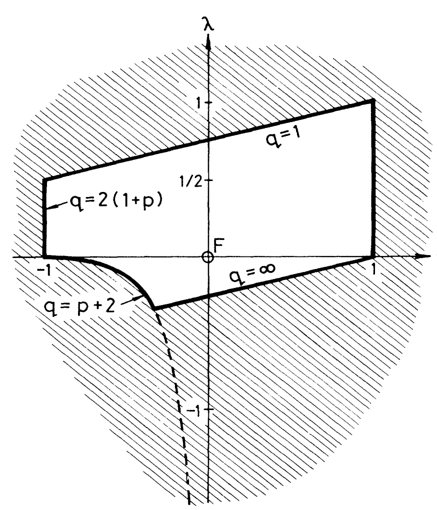

FIG. 2. The censored region A_2, shown as part of the κ-λ plane. The Friedmann singularity is at F. Boundary points such as T_0 of Fig. 1 cannot be properly represented on this plot, since the equations for κ and λ degenerate there.

portion of the curve $\lambda = (1+\kappa)^2/4\kappa$, which corresponds to the line segment $q = p + 2$, $q > 2$ of Fig. 1. The point $\kappa = \lambda = 0$ represented by a circle corresponds to the Friedman-like case $p = q = 4$. The entire parallelogram satisfies the dominant energy condition, whence this is automatically satisfied throughout region A_2.

More interesting from the point of view of the cosmic censorship hypothesis is region B of Fig. 1, where the naked singularity conditions $\varepsilon = -1$ holds. In this case one has

$$\kappa = \frac{2p+q}{4+2p-3q}, \qquad \lambda = \frac{q^2-2q-2p}{q(4+2p-3q)} \tag{46}$$

which solves for q, p

$$p = \frac{4\kappa(1+\kappa+2\lambda)}{(1-\kappa)(1-\kappa+4\lambda)}, \qquad q = \frac{-4\kappa}{1-\kappa+4\lambda}. \tag{47}$$

Since throughout this region $0 < q < 1$, one finds that κ, λ are confined to the regions

$$1 > \kappa > 0, \qquad \lambda < -\frac{1}{4}(1+3\kappa),$$

$$-1 < \kappa < 0, \qquad \lambda > -\frac{1}{4}(1+3\kappa).$$

These regions are depicted in Fig. 3, where the extra constraint implied by the dominant energy condition, $-1 < \lambda < 1$, is also imposed. As can be seen, the dominant energy condition can readily be satisfied and the corresponding region of B where the dominant energy condition holds is depicted in Fig. 4. However while values of p and q lying in this region can be regarded as representing naked singularities which violate the SCCH, it should be noted that for all such values one has $\kappa\lambda < 0$. That is, in the vicinity of such a naked singularity, negative pressures always arise either in the radial or the tangential direction. We shall see in the next section that this conclusion also hold under more general conditions.

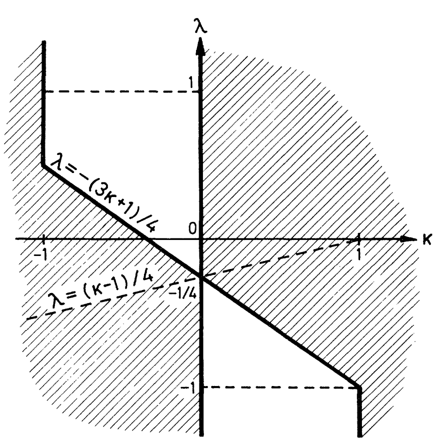

FIG. 3. The naked region B seen in the κ-λ plane as the unshaded region. The regions above and below the dashed lines $\lambda = \pm 1$ are shaded out when the dominant energy condition is applied to the tangential pressures.

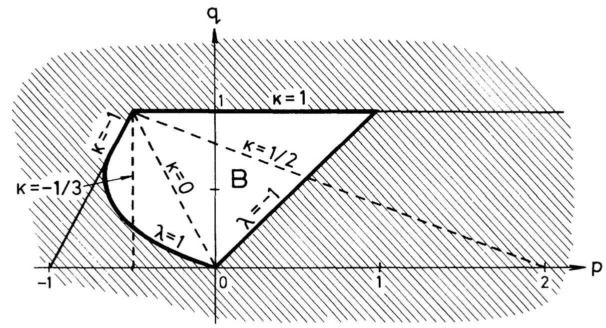

FIG. 4. The regions of the naked zone B where the dominant energy conditions $|\kappa| < 1$, $|\lambda| < 1$ hold.

VI. NONGENERIC CASES

The analysis of the previous section might be thought to be representative of the most general class of fluid collapses. However, it is interesting to note that while the Friedmann models fall right in the heart of this generic region, the general dust solution (Tolman-Bondi) is not encompassed by it. This leads us to suspect that the nongeneric cases $q = 0$ and $q = 1$ are not so exceptional as to be totally discounted, i.e., in some sense they do not form a "set of measure zero" among all spherical collapses. Accordingly, we proceed to discuss them in this section.

A. $q = 1$

Substituting the expansions (20) and (21) into (28) we find

$$G^0_0 = -x^{-1}e^{-\beta_0(u)} + O(x^{-p+q_1-2}) \tag{48}$$

When $p \neq -\frac{1}{2}$ one obtains, from (29) and (30)

$$G^0_1 \approx G^1_0 \approx -x^{-p-2}e^{-\alpha_0}(p+\tfrac{1}{2}).$$

Hence the requirement (42) implies that $p > -\frac{1}{2}$, whence

$$-p-2 < -\frac{3}{2} < -1,$$

and since $q_1 > 0$ we see that G^0_1, G^1_0 dominate G^0_0 as $x \to 0$. Hence $\rho \approx P_r$, which we discount as representing extreme energy-stress behavior. Thus the only case of interest is $p = -\frac{1}{2}$, precisely the condition prevailing at a t_0 singularity of the Tolman-Bondi solution (see Sec. II).

Setting $p = -\frac{1}{2}$ we have then

$$G^0_0 \approx -x^{-1}e^{-\beta_0} - x^{q_1-\frac{3}{2}}e^{-\alpha_0}\varepsilon q_1(q_1+1)\beta_1, \tag{49}$$

$$G^0_1 \approx -e^{-\alpha_0}[x^{p_1-\frac{3}{2}}\alpha_1 p_1 - x^{q_1-\frac{3}{2}}\beta_1 q_1(q_1+\tfrac{1}{2})], \tag{50}$$

$$G^1_0 \approx -e^{-\alpha_0}[\,lx^{-\frac{1}{2}}(-\tfrac{3}{2}\beta'_0 + \alpha'_0) + x^{p_1-\frac{3}{2}}\alpha_1 p_1 - x^{q_1-\frac{3}{2}}\beta_1 q_1(q_1+\tfrac{1}{2})], \tag{51}$$

$$G^2_2 \approx -e^{-\alpha_0}[\,\tfrac{1}{2}kx^{-\frac{1}{2}}\beta'_0 + \varepsilon x^{p_1-\frac{3}{2}}\alpha_1 p_1(p_1-1) + \varepsilon x^{q_1-\frac{3}{2}}\beta_1 {q_1}^2)]. \tag{52}$$

The number of cases to analyze at this stage is quite large.

(i) $\mathbf{0 < q_1 < \frac{1}{2}}$. In this case $G^0_0 < 0$ implies $\varepsilon\beta_1 > 0$. We divide the discussion into three subcases.

(a) $p_1 > q_1$. $G^0_1 < 0$, $G^1_0 < 0 \Rightarrow \beta_1 < 0$, whence $\varepsilon = -1$ and the singularity is necessarily naked. One finds

$$\kappa = \frac{-1}{3+4q_1}, \qquad \lambda = \frac{-2q_1}{3+4q_1}.$$

These ratios lie in the ranges

$$-\frac{1}{3} < \kappa < -\frac{1}{5}, \qquad -\frac{1}{5} < \lambda < 0$$

well within the requirements of the dominant energy condition, but definitely implying negative pressures.

(b) $p_1 = q_1$. The condition $G^0_1 < 0$ now results in

$$\beta_1(q_1 + \tfrac{1}{2}) < \alpha_1.$$

Since we are principally interested in the possible occurrence of naked singularities, let us set $\varepsilon = -1$, $\beta_1 < 0$ and we find that

$$\kappa = \frac{-1-2A}{3+4q_1-2A},$$

$$\lambda = \frac{-2q_1 + 2A(1-q_1)}{3+4q_1-2A}$$

where $A = \alpha_1/\beta_1 < q_1 + \frac{1}{2}$. Again there is no region such that both $\kappa > 0$ and $\lambda > 0$. A perfect fluid ($\kappa = \lambda$) occurs if $A = (2q_1-1)/(4-2q_1)$, in which case $\kappa = \lambda = 1/(4q_1 - 7)$, having range $-\frac{1}{5} < \kappa < -\frac{1}{7}$.

(c) $p_1 < q_1$. This leads to an extreme equation of state, $P_r \approx \rho$.

(ii) $\mathbf{q_1 = \frac{1}{2}}$. In this case

$$G^0_0 < 0 \implies e^{-\beta_0} + e^{-\alpha_0}\frac{3}{4}\varepsilon\beta_1 > 0,$$

and dividing again into subcases, we have the following.

(a) $p_1 > q_1$. In this case $\beta_1 < 0$ (independent of the sign of ε), and setting $\varepsilon = -1$, $b = -e^{-\beta_0+\alpha_0}/\beta_1 > 0$ we find

$$\kappa = \frac{-4b-1}{4b+5}, \qquad \lambda = \frac{-1}{4b+5}.$$

These values lie in the range $-\frac{1}{5} > \kappa > -1$, $-\frac{1}{5} < \lambda < 0$.

(b) $p_1 = \frac{1}{2}$. In this case $\alpha_1 > \beta_1$ and setting

$$a = \alpha_1 - \beta_1 > 0, \qquad B = e^{\alpha_0-\beta_0} + \frac{3}{4}\varepsilon\beta_1 > 0$$

one finds

$$\kappa = \frac{-2B+a}{2B+a}, \quad \lambda = \frac{\varepsilon a}{4B+a}.$$

It is interesting to note a perfect fluid arising for $\varepsilon = 1$, $B = a/4$, resulting in $\kappa = \lambda = \frac{1}{3}$ which represents a radiation collapse. The naked case $\varepsilon = -1$ results in $-1 < \kappa < 1$, $\lambda \le 0$.

(c) $p_1 < q_1$. The equation of state is extreme, $P_r \approx \rho$.

(iii) $\boldsymbol{q_1 > \frac{1}{2}}$. In this case the only nonextreme case $P_r \neq \pm\rho$ arises when $p_1 = \frac{1}{2}$. It then follows that $\alpha_1 > 0$ and

$$\kappa = \frac{-2D+\alpha_1}{2D+\alpha_1}, \qquad \lambda = \frac{\varepsilon\alpha_1}{4D+\alpha_1}.$$

where $D = e^{\alpha_0-\beta_0}$.

The treatment is the same as for case (ii)(b).

B. Ultradegenerate case

A highly nongeneric case appears as a subcase of (ii)(b) in the above analysis when $q_1 = p_1 = \frac{1}{2}$, $a = B = 0$. The last condition implies $\alpha_1 = \beta_1$ and $e^{-\beta_0} = -\frac{3}{4}\varepsilon\beta_1 e^{-\alpha_0}$, whence $\varepsilon\beta_1 < 0$. This is by no means as specialized as might at first appear, since the entire class of t_0-type singularities in Tolman-Bondi dust metrics fall in this category. In this case second-order terms $\alpha_2 x^{p_2}$, $\beta_2 x^{q_2}$ must be considered and we find

$$G^0_0 \approx -e^{\alpha_0}[x^{-1/2}(k\beta_0' + \tfrac{1}{4}\varepsilon\beta_1{}^2) + x^{q_2-\frac{3}{2}}q_2(q_2+1)\beta_2], \tag{53}$$

$$G^0_1 \approx -e^{-\alpha_0}[x^{-\frac{1}{2}}\tfrac{1}{8}\beta_1{}^2 + x^{p_2-\frac{3}{7}2}\alpha_2 p_2 - x^{q_2-\frac{3}{2}}\beta_2 q_2(q_2+\tfrac{1}{2})], \tag{54}$$

$$G^1_0 \approx -e^{-\alpha_0}[x^{-\frac{1}{2}}(\tfrac{1}{8}\beta_1{}^2 + l(\alpha_0' - \tfrac{3}{2}\beta_0')) + x^{p_2-\frac{3}{2}}\alpha_2 p_2 - x^{q_2-\frac{3}{2}}\beta_2 q_2(q_2+\tfrac{1}{2})], \tag{55}$$

$$G^2_2 \approx -e^{-\alpha_0}[x^{-\frac{1}{2}}(\frac{1}{2}k\beta_0' + \tfrac{1}{8}\varepsilon\beta_1{}^2) + x^{p_2-\frac{3}{2}}\alpha_2 p_2(p_2-1) + x^{q_2-\frac{3}{2}}\beta_2 q_2{}^2)]. \tag{56}$$

Concentrating attention on the naked case, $\varepsilon = -1$, we find the only cases with non-extreme equation of state are the following.

(i) $\boldsymbol{\frac{1}{2} < q_2 < 1,\ p_2 > q_2}$. $\varepsilon = -1$ implies $\beta_2 < 0$, and

$$\kappa = \frac{-1}{3+4q_2}, \qquad \lambda = \frac{-2q_2}{3+4q_2},$$

which lie in the ranges

$$-\frac{1}{5} < \kappa < -\frac{1}{7}, \qquad -\frac{1}{5} > \lambda > -\frac{2}{7}.$$

(ii) $\frac{1}{2} < q_2 < 1,\ p_2 = q_2$. In this case $\beta_2 < 0$ and $A = \alpha_2/\beta_2 > q_2 + \frac{1}{2}$,

$$\kappa = \frac{1+2A}{2A - 4q_2 - 3}, \qquad \lambda = \frac{2q_2 + 2A(q_2-1)}{2A - 4q_2 - 3}$$

whence κ lies in the range $-1 < \kappa < 1$, and λ lies between $-\frac{1}{2} - q_2$ and $q_2 - 1$. This results in $-\frac{1}{2} < \lambda < 0$.

(iii) $q_2 = 1,\ p_2 \geq 1$. Many possibilities arise, but always one finds

$$G_2^2 \approx \frac{1}{2} G_0^0 \approx -e^{-\alpha_0} x^{-1/2} \left(\frac{1}{2} k \beta_0' + \frac{1}{8} \varepsilon {\beta_1}^2 + \varepsilon \beta_2 \right) < 0$$

Thus $P_\perp < 0$ near such a naked singularity.

(iv) $q_2 > 1,\ p_2 \geq 1$. This is very similar to the previous case,

$$G_2^2 \approx \frac{1}{2} G_0^0 \approx -e^{-\alpha_0} x^{-1/2} \left(\frac{1}{2} k \beta_0' + \frac{1}{8} \varepsilon {\beta_1}^2 \right) < 0$$

Again $P_\perp < 0$ near such a naked singularity.

C. $q = 0$

This case is mentioned briefly for the sake of completeness. It appears, however, to correspond unmistakably to a shell-crossing singularity, since the area of the singularity is finite and, as seen below, the radial pressure will always vanish and be dustlike.

Substituting $q = 0$ into (28)–(31) we see that a singularity occurs at $x = 0$ only if $p > q_1 - 2 > -2$. However if $p \neq 0$ in this range one finds that

$$\lambda = \frac{G_2^2}{\rho} = O(x^{-q_1}) \longrightarrow \infty.$$

Thus we may concentrate attention on $p = 0,\ 0 < q_1 < 2$. The limiting behavior of the Einstein tensor as $x \to 0$ is then

$$G_0^0 \approx -e^{-\alpha_0} x^{q_1 - 2} \varepsilon \beta_1 q_1 (q_1 - 1), \tag{57}$$

$$G_1^0 = G_0^1 \approx e^{-\alpha_0} x^{q_1 - 2} \beta_1 q_1 (q_1 - 1), \tag{58}$$

$$G_2^2 \approx -e^{-\alpha_0} \varepsilon [x^{p_1 - 2} \alpha_1 p_1 (p_1 - 1) + x^{q_1 - 2} \beta_1 q_1 (q_1 - 1)]. \tag{59}$$

The inequalities (42) give at once that the singularity is locally naked, $\varepsilon = -1$, and $\beta_1(q_1 - 1) < 0$. We find then that

$$\kappa \approx 0$$

and

$$\lambda = -\frac{1}{2} \qquad \text{for} \qquad p_1 > q_1$$

while

$$\lambda = \frac{\beta_1 + \alpha_1}{-2\beta_1} \qquad \text{for} \qquad p_1 = q_1.$$

The solutions are dustlike in the latter case if $\alpha_1 = -\beta_1$. A detailed analysis of the Tolman-Bondi dust solution at the t_1 singularity shows that it corresponds to

$$p_1 = q_1 = 1, \qquad p_2 = q_2 = \frac{3}{2}, \qquad \alpha_2 = -\beta_2.$$

In general all metrics with $p_1 = q_1 = 1$ must be analyzed to second order. It is clear that all metrics of the form (24) with these leading coefficients are C^1, though possibly not C^2 at $x = 0$. This seems to be the main characterization of a shell cross.

VII. CONCLUSIONS

In this paper we have discussed the validity of the strong cosmic censorship hypothesis, by considering spherically symmetric metrics whose singularities are of the power-law type. Such metrics are not the most general conceivable, but have enough flexibility of form to permit a large number of possibilities to be investigated. In fact it is quite difficult to construct a singularity which would not be of this form, and there are good reasons to believe that the collapse of a physically realistic fluid starting with regular initial data will terminate in a singularity of this type.

By calculating the asymptotic behavior of the energy-stress tensor near the singularity, it has been possible to discuss the physical properties of the space-time near locally naked and locally censored singularities separately. While the dominant energy condition can readily be satisfied in the neighbourhood of a naked singularity, there appear to be greater restrictions than in the censored case. The main conclusion is the following.

In the neighborhood of a naked spherically symmetric singularity of power-law type which is not a shell cross, the energy-stress tensor can only satisfy the dominant energy condition if it is asymptotically extreme ($|P_r| \approx \rho$ *or* $|P_\perp| \approx \rho$) *or if one of the pressures is negative* ($P_r < 0$ *or* $P_\perp < 0$).

Whether this result holds for general spherically symmetric singularities, or indeed for nonspherical situations is at this stage an open question.

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